THE ENGLISH AND FRENCH TRANSLATIONS OF HILBERT'S GRUNDLAGEN DER GEOMETRIE.

1. *Les Principes fondamentaux de la Géométrie.* Par D. HILBERT. Traduit par L. LAUGEL. Paris, Gauthier-Villars, 1900. 4to. 111 pp.
2. *The Foundations of Geometry.* By DAVID HILBERT. Translated by E. J. TOWNSEND. Chicago, The Open Court Publishing Company, 1902. 8vo. 132 pp.

IT is indeed a matter for congratulation that Professor Hilbert's masterly discussion of the foundations of geometry has become so well known and so widely circulated. This circumstance is undoubtedly due to its clearness and force; for, while some of the previous studies along similar lines are difficult even for the advanced student to understand, Hilbert's style is so deceivingly clear as to lead certain minds to predict the use of this book in elementary instruction.

An excellent review of the original,* rendered into English by Professor Ziwet, was given in the BULLETIN (volume 6, 1900, pages 287–299) by Dr. Sommer, of Göttingen; the present review will therefore not deal with the German edition, except for purposes of comparison.

If a minute criticism of the language of the French translation were the main purpose, the reviewer would certainly feel great hesitation in undertaking the task. But there are certain additions to the French translation which are most noteworthy, and with these we shall occupy ourselves chiefly.

On page 291 of his review, mentioned above, Dr. Sommer explains that Hilbert's axiom V is not sufficient to furnish a satisfactory foundation for the complete discussion of "the continuity of the straight line in the ordinary sense." This lack is supplied in the French edition by an additional axiom (by Hilbert) entitled "Axiome d'intégrité" (Vollständigkeitsaxiom), which practically requires that the system already set up shall

*In this connection, a review of Hilbert's Grundlagen der Geometrie by H. Poincaré, *Bull. des Sciences Math.* (2), vol. 26 (Sept., 1902), p. 249, should be mentioned. This review is, as would be expected, of the greatest importance. The present writer regrets that his review was in type before that of Poincaré was seen. Footnotes have been added occasionally at points where this review would have been most influenced by Poincaré's.

be considered as closed, *i. e.*, incapable of extension by the adjunction of additional elements. This axiom corresponds to the one introduced by Hilbert (*Jahresbericht der Deutschen Mathematiker Vereinigung*, 1900) for arithmetic; and is certainly radically different, in the character of the assumption made, from the other axioms proposed by Hilbert. Its introduction enables us to prove the fundamental theorem of Bolzano, but the reviewer feels that its form is scarcely final, and that it is not wholly satisfactory to Hilbert himself.*

Another addition to the French edition merits more attention than it seems to have received; the original conclusion of the German edition is replaced by a new appendix (pages 106–111), written by Hilbert, which shows clearly the importance of his work for non-euclidean geometry in its broadest sense, a point upon which Hilbert dwelt to a considerable extent in his lectures preceding the publication of the original Festschrift, but which is scarcely mentioned in the Festschrift itself. The apparent reason for the tendency of non-euclidean geometers in the past to cast out the axiom of parallels alone, is that this was thought to be the only axiom *explicitly* assumed by Euclid which could not be directly tested by physical experiment.† Another axiom, or set of axioms, namely those regarding continuity, assumed *implicitly* by Euclid, is, however, equally open to philosophical exception.‡ The effect of the omission of the axiom of Archimedes (Hilbert, V) was therefore of peculiar interest to Hilbert, and the subject was discussed considerably in his lectures. A student, Dr. Dehn, completed these inquiries, and a review (by Hilbert) of Dr. Dehn's thesis forms the major portion of this addition to the French translation. The one striking fact that *a geometry is possible in which the sum of the angles of any triangle is two right angles and in which similar non-congruent triangles exist, while an infinity of parallels to any straight line may be drawn through any point*, will sufficiently indicate the remarkable nature of Dr. Dehn's results.

This geometry is called "semi-euclidean," and is evidently non-archimedean, the axioms III and V being omitted; but in the whole discussion the axioms I, IV and II (in modified form) are assumed to hold. To the reviewer it would seem, from the theorem just mentioned, that the assumption that the

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* See also Poincaré, l. c., pp. 256, 271, 272.

† The reviewer disclaims any intention of saying that this is a logically valid reason; it is merely the *actual* reason.

‡ See Poincaré, l. c., pp. 250, 258.

sum of the angles of a triangle is two right angles would be more advisable than the form of the parallel axiom given by Euclid (Hilbert III), since it is demonstrated that it is a "weaker" axiom, in the sense of Professor Moore. For while it can be proved from the axiom III, without the use of axiom V, we must include V in order to prove III from it. At any rate, as Dr. Dehn remarks, this substitute for the axiom of parallels is conclusively shown not to be equivalent to Euclid's form; and, while Euclid probably deserves no praise in the matter, his was certainly a happy choice in that he possessed Archimedes's axiom only by implicit assumption. While this subject merits considerable discussion, we cannot enter into further details without passing from the book in hand. Still, Dr. Dehn's thesis (*Mathematische Annalen*, volume 53 (1900)), as also his subsequent papers in the *Annalen*, form a direct and important addition to Hilbert's memoir.

Aside from these additions, the French edition is simply a remarkably good translation. The translator has happily given himself just enough liberty to render the spirit of the German into idiomatic French; but the reviewer feels confident that the sense has always been perfectly preserved. It is especially noteworthy that M. Laugel had the direct and enthusiastic assistance of Hilbert himself in the preparation of the French edition.

In spite of the hesitation expressed above, the reviewer feels moved to enquire whether better translations for "Axiome der Verknüpfung" (axiomes d'association), and for "Axiome der Anordnung" (axiomes de distribution), could not have been found, especially in view of the necessity of using these same terms in describing the corresponding sets of axioms of arithmetic in Chapter III, where the associative, distributive, and commutative laws of arithmetic are all placed under the head of "associative" axioms, whereas the "distributive" axioms are of an entirely different nature.* Finally the translations of "flächengleich" (égaux par addition), and of "inhaltsgleich" (égaux par soustraction), are certainly widely divergent from the original, and might possibly have been improved upon; but it may be contended that they are justifiable as a closer description of the ideas represented.

Of the typography nothing need be said, further than that the book is printed by the firm of Gauthier-Villars.

*Poincaré, l. c., uses "axiomes projectifs" and "axiomes de l'ordre." He expressly discards "axiomes de connection" as a translation of "Axiome der Verknüpfung."

On the whole, on account of the additions mentioned above, and in view of the faithful reproduction of the spirit of the original, the French edition compares very favorably with the German, for those who understand both languages equally well. The reviewer had excellent success in the use of the French edition by a portion of his class last winter, the students having been advised to procure either the French or the German editions at their discretion.

The English translation is the work of one who had the advantage of hearing Hilbert's preparatory lectures. It contains the additions to the French translation, but there are no new additions. Some unimportant omissions are made, including the original conclusion (also omitted in the French edition). In a short preface Professor Townsend expresses his own views upon the important features of the work. It is unfortunate that he does not emphasize the radical departures of Hilbert's *method* of presentation — especially the power obtained by considering the axioms as fundamental, the conceptions of point, line, and plane as trivial.* Notice is taken in footnotes of some recent memoirs commenting on Hilbert's work.

Some features of the translation are particularly happy. Especially does the rendering of "Widerspruchslosigkeit" by "compatibility" appeal to the reviewer. The translations of "flächengleich" (of equal area), and of "inhaltsgleich" (of equal content), §18, etc., are certainly nearer the original than the expressions used by the French translator. Where custom has established a translation of a standard technical word, Professor Townsend generally appears to have used it, as, for example, "set" for "Menge," "enumerable" for "abzählbar," "(number) field" for "(Zahl) körper," to all of which excellent alternate expressions are, however, in common use. The expression "numerical multiplicity," translated on page 126 from the French "multiplicité numérique," suffers on the other hand by comparison with Ziwet's† "manifold of numbers" as an equivalent for the German word "Zahlenmannigfaltigkeit."

*To illustrate: Hilbert, in his lectures, used effectively a geometry in which "points" are the ordinary positive integers, and "lines" are the ordinary negative rational numbers. The principle involved was, of course, used before Hilbert by others, but it has taken a far more tangible form in Hilbert's hands. See Poincaré, l. c., pp. 250, 252, etc.

† In his translation of Sommer's review above mentioned.

Professor Halsted * informs us that the translation of "Anordnung" *must* be "arrangement," but the reviewer feels that several other words are also suitable translations of "Anordnung," in its technical sense, just as "ausgezeichnet," when used technically, need not necessarily be rendered "remarkable." † The translations "order," "arrangement," and "order of sequence" are to be found in Professor Townsend's edition, and any one of them, used consistently, would have been acceptable. Professor Ziwet, in the review mentioned above, uses "order." Perhaps even a still better word might be found, but Professor Townsend is to be congratulated in not having followed the French translator in his unhappy choice. The omission of the heading "Erklärung" can be defended, although the reviewer personally inclines to its retention.

The work is unfortunately somewhat marred by several errors, some of which, occurring at critical points, are apt to be misleading. On page 24, line 11, the word "all" is wholly wrong, as the translator might have known from Hilbert's lectures or at least from Dr. Sommer's review. It may be admitted that the original is not entirely clear on this point, but no such word as "all" occurs in the German edition. The translation of "einzig" by "definite" in line 8, page 53, may possibly be confusing; *any* single point is to determine a segment 0. The word "only" must be inserted between "which" and "such" on line 11 from the bottom of page 98. On page 107, line 1, we are not to understand that Pascal's theorem is *identical* with the commutative law, but merely that the truth of the commutative law insures that Pascal's theorem holds, under the given conditions.‡ On page 124, line 11, we understand that the extraction of the square root of f_3 is the third in order, *not* that it is now to be performed "a third time." The statement on page 125, that "every regular polygon can be

* In a brief review of Professor Townsend's edition, *Science*, August 22, 1902. The review is not especially appreciative and dwells on several mistakes and misprints. Some of the rather harsh criticisms are quite unfounded, as, for instance, those regarding axiom IV, 1, p. 12; theorem 16, p. 22; and a sentence on page 25, lines 26–28; while others seem trivial, for example, that of "emanating," p. 13, etc. The errors on pp. 24 and 125, noticed by Professor Halsted, and the omission of the heading "Erklärung," are considered later in the present review. The remainder of Professor Halsted's criticisms seem unimportant, in the sense that the passages in question are at least never misleading as they stand, and certainly do not call for separate reconsideration here.

† An accepted translation for "ausgezeichnete Untergruppe" is "self-conjugate subgroup."

‡ See Poincaré, l. c., p. 267.

constructed by the drawing of straight lines and the laying off of segments" will scarcely mislead any one. It is conceivable that this curious mistranslation was caused by some oversight, but it seems that "jene" has been mistaken for "jede(s)": "Every regular polygon" should, of course, be replaced by "the above-mentioned regular polygons."

But the most dangerous errors occur in the translation of the important new appendix of the French edition. On page 127, lines 17, 19, 20, 21, "par" is translated "from" instead of "by," and on page 130, line 11, "vis-a-vis de" is also rendered "from" instead of "with respect to," while on page 128, line 9, "imposée" is changed into "confined." A reading of the paragraphs in question will disclose the fact that at least the first two of these mistakes totally destroy all meaning at critical points. These errors, and some minor ones, make this conclusion the poorest part of the English edition.

The translator's English sometimes lacks the true idiomatic ring, the worst instances being, perhaps, the footnote on page 85, and the sentence which begins on line 7, page 124. There are also several minor faults of translation, which will fortunately not mislead the student. Among these are changes from singular to plural nouns and *vice versa*, for instance, "square root(s)," line 23, page 121; the insertion and omission of words, as "consequently" line 21, page 127; and mistranslations of words. Of such minor errors the reviewer has noticed, not counting duplicates, nineteen.

Among the misprints found only two are serious: the reference to "§ 30" on page 98, line 24, should be "§ 27"; and "and" three lines from the bottom of page 123, should be "any." The misprint (or error) in the original in formula (1), § 33, corrected in the Errata (page 111) of the French edition, is corrected properly on page 104 of Professor Townsend's translation.

The word "all" on page 74, line 22, while not strictly a mistranslation of language, gives rise to an impression which is faulty, and which is not intended in the original. For it is nowhere proved that the assumption of each of the axioms IV, 1–5, is necessary for the proof of Desargues's theorem. The proper impression would possibly be conveyed by substituting the phrase "or the entire set of axioms of congruence" for the corresponding phrase in the text; the idea being that, whereas we have assumed axioms IV, 1–5, it is *necessary* to *add* to these axiom IV, 6, but it is not shown that we need *retain* each of the axioms IV, 1–5.

The notation for a segment joining two points, *e. g.*, 0 and $\frac{1}{2}$, has been changed in § 17, perhaps for the better, from $0\frac{1}{2}$ to $(0, \frac{1}{2})$. There are also other slight changes of notation, which are generally commendable. But a great deal is lost by not following Hilbert's italicization of words and of some of the theorems, which enabled him to emphasize what he considered most important. In particular the words "NON PAS" of the principal theorem of the new conclusion to the French edition, which Hilbert and Laugel emphasized not only by extraordinary type, but also by extraordinary position in the sentence, are rendered on the last line of page 128, by an ordinary "not," in ordinary type.

The Open Court Publishing Company deserves praise for continuing to publish translations of foreign scientific classics into English. The book is well printed, except for a few bad figures, of which the worst are figures 18 (page 41) and 38 (page 76), the latter being actually misleading. The type is good, excepting the identity sign. With regret we are compelled to notice, however, that the edition is not wholly satisfactory on account of the errors mentioned above. In its present form it can scarcely be recommended to students unless they can read neither German nor French; and then it should be used only under competent guidance. But most of the serious errors could be corrected by changing a few words, and if a page of such corrections were inserted in the volume it could be used by students with greater safety.

To other than university graduate students *no* edition of the book appeals. For the statement* that it has pedagogical value, or that it is to be of influence on elementary instruction, only means that students of very advanced mathematics, by their knowledge of it, may reflect into elementary books or teaching which they may influence, something of its general spirit, almost nothing of its actual contents. To insert the system of axioms proposed by Hilbert in an elementary (high school) book or, for instance, to try to introduce into such a text-book the theory of motion as based upon congruence, or even the proof that all right angles are equal, would be as ill considered as to expect the average high school teacher to grasp the meaning of the original in its entirety.

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*Sommer, l. c., p. 299; Townsend, preface to English edition.

We must rejoice, however, in proof of the wide circulation of Hilbert's ideas, that both a French and an English translation have actually been published. A widely diffused knowledge of the principles involved will do much for the logical treatment of all science and for clear thinking and clear writing in general.

E. R. HEDRICK.

YALE UNIVERSITY,
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DICKSON'S LINEAR GROUPS.

Linear Groups with an Exposition of the Galois Field Theory. By L. E. DICKSON, Assistant Professor of Mathematics in the University of Chicago. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Volume VI. Leipzig, B. G. Teubner, 1901. 8vo, x + 312 pp.

SHORTLY after the appearance of the first few numbers of the Encyclopädie der Mathematischen Wissenschaften the publishers announced a series of text-books on advanced mathematics to be issued in connection with the Encyclopädie. While the authors of articles in the Encyclopädie were especially requested to take advantage of this series to develop their subjects more fully and thus make them more accessible to the student, other writers were asked to assist to make the series as complete as possible. More than fifty different volumes of this series have already been announced, by almost as many different writers of various countries.

Never before has there been such extensive collaboration to bring the developments in the various parts of our subject within the reach of the student. It is hoped that this series will do much towards increasing the number of well-equipped investigators and thus exert a strong influence towards more substantial progress in various directions. The fact that the authors belong to so many different countries emphasizes the cosmopolitan element in mathematical work and the absence of national prejudices among its devotees.

The present work is the sixth volume of the series and is devoted to a subject which has been developed principally on French and American soil. The fundamental ideas are due to Galois and were published by him at the early age of eighteen