ON A NEW EDITION OF STOLZ'S ALLGEMEINE ARITHMETIK, WITH AN ACCOUNT OF PEANO'S DEFINITION OF NUMBER.

Theoretische Arithmetik. Von O. STOLZ (Innsbruck) und J. A. GMEINER (Wien). I. Abtheilung: *Allgemeines. Die Lehre von den rationalen Zahlen.* Leipzig, B. G. Teubner, 1901. iv + 98 pp.

THIS publication of about 100 pages is the first instalment of a new and revised edition of Stolz's Allgemeine Arithmetik (1885–86), — a work which has long since proved indispensable to all who desire a systematic and rigorous development of the fundamental elements of modern arithmetic.

The revision thus far completed (probably about one seventh of the entire work) covers the first four chapters:

I. On quantities (Grössen) and operations (Verknüpfungen) in general.

II. On the natural numbers and the four fundamental operations.

III. On the general properties of any direct operation (Thesis, $a \circ b$) and its inverse (Lysis, $a \smile b$), as deduced from certain fundamental formal laws; in particular, the analytic theory of (absolute) rational numbers.

IV. On the synthetic theory of (absolute) rational numbers, with a treatment of systematic fractions.

The remaining chapters in the first part will contain (if the order in the old book is preserved) the theory of negative and irrational numbers, with an account of euclidean "ratios," followed by an elaborate treatment of the theory of limits as applied to functions of a real variable and to infinite series of real terms. The second part will then contain the theory of operations on complex numbers, including chapters on infinite series, infinite products and continued fractions. The complete work will belong to the Teubner series of mathematical textbooks.

As one turns the pages of the new edition one is struck first of all by the great improvement in the general appearance of the book. The title itself, "theoretical arithmetic," is much more lucid than the old title of "general arithmetic," which to the uninitiated might mean common algebra; the notes, historic and other, which used to be difficult of reference, are now placed conveniently as footnotes on the proper pages; and best of all the paragraph numbers in the several sections are now placed at the top of each page. All these changes will be appreciated by those who have had to refer often to the older book. The addition of some thirty problems or exercises is also very welcome.

In the body of the text there are numerous changes which greatly enhance the clearness of the presentation. The introduction of the rational numbers (III, 7) and the opening paragraph of IV should be especially mentioned. We notice that in III no attempt is made to prove the mutual independence of the postulates $A) - E)$, and as a matter of fact if E, 2) is made to read: "when $b > b'$ then $a \circ b > a \circ b'$" the postulate D_1 can be at once deduced as a theorem.

The only radical change from the plan of the first edition, however, is found in the second chapter, where the treatment of the natural numbers is now based on Peano's definition. The details of Peano's theory are here for the first time made accessible to readers unfamiliar with his mathematical language. Since, however, no systematic treatment of Peano's method is obtainable in English, it may be not without interest to devote the remainder of our space to a connected account of its principles.

PEANO'S THEORY OF THE NATURAL NUMBERS.*

The fundamental concepts which form the starting-point of Peano's theory are three: (1) a class or assemblage, denoted by N; (2) a special object, denoted by E; (3) what we may call a "rule of succession," (denoted by $\circ$), according to which every object, a, of the given class N determines uniquely another object, $a \circ$, called the successor of a.

The nature of the assemblage N, of the special object E, and of the rule of succession $\circ$ is left wholly undetermined, except for the following five restrictions, or postulates:

* G. Peano: Arithmetices principia nova methodo exposita, Turin, 1889; "Sul concetto di numero," *Rivista di Matematica*, vol. 1 (1891), pp. 87–102, 256–267; *Formulaire de Mathématiques*, vol. 2 (1898), no. 2, pp. 1–59; vol. 3 (1901), pp. 39–44. Where I have written N and E, Peano uses N_1 and 1 or later N_0 and 0; and for $a \circ$, Peano uses $a +$ or later $a + 1$. I have also changed the wording of postulate 5. Stolz and Gmeiner use "Zahl," 1 and $a + 1$, for N, E and $a \circ$, and add another postulate, to the effect that every number shall be called equal to itself.

1. *The special object E shall belong to the class N.*

2. *If an object a belongs to N then its successor $a\circ$ shall also belong to N.*

3. *If an object a belongs to N then its successor $a\circ$ shall be different from E.*

4. *If $a\circ = b\circ$ then $a = b$.*

5. *If a is any given element of N, then a can be found in the sequence E, $E\circ$, $(E\circ)\circ$, . . .*

The meaning of these postulates will be clearer, if we consider at once the following systems, in which N, E and $\circ$ are so chosen that all the postulates, or all but one, are satisfied.

(a) $N =$ class of positive integers ; $E = 1$; $a\circ = a + 1$.
(b) $N =$ class of positive integers with 0 ; $E = 0$; $a\circ = a + 1$.
(c) $N =$ class of positive integral powers of 2 ;

$$E = 2^1 ; \quad a\circ = 2a.$$

Each of these systems (a), (b), (c) satisfies all the five postulates.

(S_1) $N =$ class of positive integers ; $E = 0$; $a\circ = a + 1$.
(S_2) $N =$ class of positive integers from 1 to 9 ;

$$E = 1 ; \quad a\circ = a + 1.$$

(S_3) $N =$ class of positive integers from 1 to 9 ;

$$E = 1 ; \quad a\circ = a + 1, \text{ except that } 9\circ = 1.$$

(S_4) $N =$ class consisting of two elements 0 and 1 ;

$$E = 0 ; \quad 0\circ = 1, \; 1\circ = 1.$$

(S_5) $N =$ class of positive integers ; $E = 1$; $a\circ = a + 2$.
Here the system S_k ($k = 1, 2, 3, 4, 5$) satisfies all the other postulates, but not the kth.

The possibility of constructing such systems as (a), (b), (c) proves that *the postulates 1–5 are consistent ;* and the consideration of the systems S_1–S_5 shows us that *the postulates 1–5 are independent of one another.* This proof of independence, by the way, which forms a most interesting feature of Peano's work, is not reproduced by Stolz and Gmeiner.

Further, it can be easily shown that any two systems, S and S', which satisfy all the postulates 1–5 can be brought into one-to-one correspondence with each other in such a way that

E will correspond with E', and whenever a in S corresponds with a' in S' then $a\circ$ will correspond with $a'\circ$. This fact may be stated in the theorem: *All systems* $(N, E, \circ)$ *which satisfy postulates 1–5 are equivalent* to one another (cf. the systems a–c above). On the basis of this theorem Peano then states his definition:

A "system of natural numbers" shall mean any system $(N, E, \circ)$ *which satisfies the postulates 1–5.*

It is clear that such a definition is a very different thing from a psychological analysis of the process of counting. It is rather the construction of a symbolic "calculus of operations," which contains indeed the arithmetic calculus as a special case when a special interpretation, viz., (a), is given to its symbols, but in which the validity of the deductions is quite independent of any such interpretation.

The first step in the development of the theory is the definition of the symbol $a + b$ as follows: Let a and b be any two natural numbers, that is, any two elements of a system $(N, E, \circ)$ which satisfies Peano's five postulates; then $a + b$ denotes the number determined by the following recursion formulæ:

$$a + E = a\circ\,; \quad a + (E\circ) = (a + E)\circ\,; \quad \cdots; \qquad a + (k\circ) = (a + k)\circ. \tag{6}$$

Thus, in order to compute the value of $a + b$ we must compute in succession the values of

$$a + E, \quad a + (E\circ), \quad a + [(E\circ)\circ], \quad \cdots, \tag{6$'$}$$

until we strike the desired element $a + b$.

The statement of Stolz and Gmeiner, defining $a + b$ as that number which is obtained by starting from a and proceeding b steps forward in the series of numbers, would seem to imply that the *ability to count* is after all involved in this process. Such is, however, not the case. In order to find the number $a + b$ by Peano's definition we require simply the ability to determine from any given element a the element $a\circ$; we can thus compute one after the other the elements of the sequence (6′), and we know by postulate 5 that this sequence will contain the element $a + b$. How far we shall have to go to find this element we do not know; we do not count the steps already taken at any stage, nor do we ask how many more there are to take; it is sufficient to know that if we persevere, one step at a time, the desired number will surely be reached.*

The element $a + b$ thus determined is called the "sum" of the given elements a and b; when the system bears the interpretation (a) then $a + b$ will clearly be the sum of a and b in the ordinary sense.

Similarly, the "product," ab, of two elements a and b is defined by the recursion formulæ

$$aE = a; \quad a(E\circ) = (aE) + a; \quad \cdots; \quad a(k\circ) = (ak) + a. \quad (7)$$

From the definition of $+$ follows next the theorem

$$a + (b + c) = (a + b) + c$$

the proof of which we give in full as a typical illustration of the use of postulate 5 (method of mathematical induction). Thus: the theorem is obviously true when $c = E$, by (6). And if it is true when $c\circ$ is any particular element k then it will be true also when $c = k\circ$.† Therefore, being true for $e = E$, it is true for $c = E\circ$, and hence for $c = (E\circ)\circ$, etc. But by 5 any given element can be reached in this way; hence the theorem is true for all values of c.

In a similar way the theorems

$$a + b = b + a, \quad (ab)c = a(bc), \quad ab = ba,$$

and

$$(a + b)c = ac + bc$$

are established.

The definition of the symbols $>$ and $<$ depends on the following theorem, which is proved also by the use of (6): If $a \neq b$ then there is either a number λ such that $a = b + \lambda$ or else a number μ such that $b = a + \mu$ (and not both). In the first case a is called "greater than" b $(a > b)$, in the second case "less" $(a < b)$. (8)

When the system bears the interpretation (a) the use of the symbols $>$ and $<$ as thus defined will clearly agree with the ordinary use of them.

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* Of course this implies unlimited time at our disposal—an assumption not explicitly stated by Peano or by Stolz and Gmeiner.

† For on the given assumption we have, by 6,

$$(a + b) + (k\circ) = [(a + b) + k]\circ = [a + (b + k)]\circ$$
$$= a + [(b + k)\circ] = a + [b + (k\circ)].$$

From the definition of $>$ and $<$ we have at once that $a + b > a$, and also the theorems: If $a > a'$ then $a + b > a' + b$ and $ab > a'b$. For example, from $a > a'$ follows $a = a' + x$, whence $ab = (a' + x)b = a'b + xb$, or $ab > a'b$. This proof of Peano's is simpler than the induction proof used by Stolz and Gmeiner.

After these fundamental definitions and theorems have been established the further development of the theory — including the introduction of the rational, irrational, negative and complex numbers — proceeds along lines already familiar and need not be enlarged upon here.

In a sense, the five postulates of Peano may be said to contain, implicitly, all the results of algebra and analysis, including the theory of functions of a complex variable. It should be clearly understood, however, that the business of the mathematical theory is solely to draw logical deductions from these postulates, not to discuss the question whether the results possess any real significance in the objective world. On this deeper question, Peano's work merely reduces the problem to its lowest terms; if we can show that his fundamental postulates are capable of real interpretation in the objective world, then all their consequences will likewise be capable of such interpretation. Whether, however, a real interpretation of the postulates—(for example, the system (a) on p. 42)—is psychologically possible or not, is a problem of epistemology with which the mathematical theory has nothing whatever to do.

A precisely similar remark applies to any deductive theory such as that contained in Hilbert's Grundlagen der Geometrie (1899). At the basis of Hilbert's theory stand certain symbols such as "point," "line," "between," etc., whose meaning is left wholly undetermined except for the imposition of certain postulates. The development of the theory consists in deducing the theorems which follow from the fundamental postulates by the laws of formal logic, quite independently of any special interpretation of the undefined symbols. To be sure, when the undefined symbols are interpreted as meaning point, line between, etc., in the ordinary sense then the whole theory becomes identical with our ordinary geometry of space. But the question whether such an interpretation is possible or not is a question which the deductive theory leaves wholly untouched. Hence Hilbert's theory cannot be said to give us any new knowledge of the real nature of space, any more than

Peano's theory gives us any new knowledge of the real nature of the natural numbers. Both these theories, however, enable us to state the philosophical problem with a definiteness which has not heretofore been possible.

Further comment on Stolz and Gmeiner's book is impossible in the space at our disposal. The remaining instalments of this great work will be awaited with keen interest.

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HARVARD UNIVERSITY,
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LAZARUS FUCHS.*

FUCHS is dead. This announcement must have caused deep sorrow in the heart of many American mathematicians. For many of us have been his pupils, and to some of us his example has been the greatest inspiration of our lives. The writer of this little sketch is one of these. He remembers how he looked forward to the time when he would be fitted to attend Fuchs's lectures. He remembers the small and crowded lecture-room in the University of Berlin, poorly ventilated, stuffy and hot in the summer days, but so full of meaning and inspiration to the earnest and thoughtful student. Fuchs was not a brilliant lecturer. He spoke in a quiet, undemonstrative manner, but what he said was full of substance. To the student there was the inspiration of seeing a mathematical mind of the highest order full at work. For Fuchs worked when he lectured. He was rarely well prepared, but produced on the spot what he wished to say. Occasionally he would get lost in a complicated computation. Then he would look around at the audience over his glasses with a most winning and child-like smile. He was always certain of the essential points of his argument, but numerical examples gave him a great deal of trouble. He was fully conscious of this failing, and I remember well one occasion when, after a lengthy discussion, he laid considerable emphasis upon the fact that "*in this case*, two times two is four."

The mathematical papers of Fuchs are very numerous, but excepting a few of his earliest attempts, they are all connected directly or indirectly with the theory of linear differential equations. This was the province which, to quote the words of Auwers when he introduced Fuchs to the Berlin Academy of

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