THE FIRST MEETING OF THE SAN FRANCISCO SECTION OF THE AMERICAN MATHEMATICAL SOCIETY.

A MEETING of the Pacific Coast Members of the AMERICAN MATHEMATICAL SOCIETY was held in San Francisco in the council room of the Academy of Sciences on Saturday, May 3, 1902. Professor Irving Stringham presided.

A letter from the Secretary of the AMERICAN MATHEMATICAL SOCIETY was read, containing the permission of the Council to organize a Section of the Society to meet near or in San Francisco. Organization of such a Section was effected by the adoption of by-laws and the election of officers, to serve until the December meeting, after which the term of office shall be for one year. Professor Stringham was elected chairman; Professor G. A. Miller, secretary; Professor R. E. Allardice, Professor G. A. Miller and Dr. E. J. Wilczynski, programme committee. The by-laws provide for two meetings a year, one in May and one in December, the election of officers to be held at the December meeting.

The following twenty persons were present at the meeting: Professor R. E. Allardice, Dr. E. M. Blake, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor M. W. Haskell, Dr. D. N. Lehmer, Professor A. O. Leuschner, Dr. J. H. McDonald, Mr. W. A. Manning, Professor G. A. Miller, Dr. H. C. Moreno, Dr. C. A. Noble, Dr. T. M. Putnam, Dr. E. W. Rettger, Professor Irving Stringham, Dr. S. D. Townley, Mr. L. C. Walker, Mr. A. W. Whitney and Dr. E. J. Wilczynski.

The following papers were presented at the meeting:

(1) Professor R. E. ALLARDICE: "On a linear transformation, with some geometrical applications."

(2) Dr. E. M. BLAKE: "A movement whose centrodes are cubics."

(3) Professor H. F. BLICHFELDT: "On the determination of the analytic form of the distance between two points by means of distance relations."

(4) Professor M. W. HASKELL: "A canonical form of the binary sextic."

(5) Dr. D. H. LEHMER: "Constructive theory of the unicursal cubic by synthetic methods."

(6) Dr. SAUL EPSTEEN: "Algebraic relations among the integrals and the reducibility of linear differential equations."

(7) Dr. J. H. McDONALD: "The limits of the minima of definite ternary forms."

(8) Professor A. O. LEUSCHNER: "A short method of deriving osculating elements of the major planets."

(9) Mr. W. A. MANNING: "On the groups of genus one."

(10) Professor G. A. MILLER: "Determination of all the groups of order p^m, p being any prime, which include the abelian group of order p^{m-1} and of type $(1, 1, 1, \cdots)$."

(11) Dr. H. C. MORENO: "On the non-abelian groups in which every subgroup is abelian."

(12) Mr. P. G. NUTTING: "Dynamic effect of stationary waves on immersed bodies."

(13) Dr. T. M. PUTNAM: "Concerning quadruple systems."

(14) Professor IRVING STRINGHAM: "A synthesis of orthogonal substitutions."

(15) Mr. A. W. WHITNEY: "Congruences defined by functions of a complex variable."

(16) Dr. E. J. WILCZYNSKI: "Geometry of the covariants of a binary system of linear homogeneous differential equations."

Mr. Manning was introduced by Professor Miller. Dr. Epsteen's and Mr. Nutting's papers were read by title. Professor Miller's paper appeared in the June number of the BULLETIN. Abstracts of the other papers are given below.

Professor Allardice discussed a problem connected with a system of similar conics through three fixed points. An equation was obtained representing a system of three-cusped hypocycloids inscribed in the triangle of reference. The problem suggested itself of transforming the equation so that the new triangle of reference should be the equilateral triangle with its vertices at the cusps. This transformation was accomplished by the use of the circular points at infinity, and was applied to prove a number of theorems referring to a system of hypocycloids touching three fixed straight lines.

In a paper by Dr. Blake in the first volume of the *Transactions* a list of the then published plane movements by which a carried straight line generates a quartic scroll was given. The present paper proves the existence of another movement of the same kind. It is defined by equations of projective transformation and not by specifying the loci of two points of the moving plane as has usually been done heretofore.

In an interesting paper by M. J. de Tilly, "Essai de géométrie analytique générale," Belgian *Memoires Couronnés*, 1892–93, the existence of a relation connecting the mutual distances of any five points in space is fundamental in defining the analytic functions representing the distances between two points. M. de Tilly does not, however, succeed in defining the functions representing the distances in the euclidean or non-euclidean geometries without introducing some conceptions and properties regarding the straight line and plane.

By means of some general axioms regarding space and distances, chief among which is the existence of a relation connecting the mutual distances between any five points, Professor Blichfeldt has defined the analytic forms of the distance between two points for the euclidean or non-euclidean spaces. These axioms are essentially as follows:

(1) A point in space is determined by means of three real coördinates, x, y, z.

(2) The distance between two points is a real function of the coördinates of the two points.

(3) This function is unaltered by interchanging the corresponding coördinates of the two points.

(4) The ten mutual distances of any five points in space are connected by one relation independent of the coördinates of these five points.

(5) It is impossible to pass through two points of general position two real curves, one through each point, possessing the property that the distance between any point of one and any point of the other is a constant.

(6) Through no point in general position does there pass a real curve, the distance between every pair of points of which is indeterminate, infinite or a constant. It may be remarked that the first five conditions are satisfied by eleven distinct functions.

The form $x^6 + y^6 + z^6 + kx^2y^2z^2$ would, by analogy with the cases of the binary quartic and octavic, seem to be a natural and desirable canonical form of the binary sextic, but the effort to perform the reduction has hitherto been unsuccessful. Professor Haskell shows that the method attempted by Sylvester in his paper "On a remarkable discovery in the theory of canonical forms and of hyperdeterminants" (*Philosophical Magazine*, fourth series, volume 2, pages 391–410), may be completed so as to furnish the reduction desired, which is accomplished by the aid of three simultaneous quadratics in three unknown quantities, requiring only a quartic equation for their solution.

Defining the unicursal cubic as the locus of the intersections of corresponding rays of two projective pencils, one of the first and one of the second order, Dr. Lehmer starts with this problem: "Given three rays a, b, c of a pencil of the first order s and the three corresponding rays α, β, γ of a pencil of the second order k, to construct the cubic." Two lines are drawn through (a, α), one of them u arbitrary in direction perspective to s, the other v tangent to and perspective to k. The point rows u, v are in perspective position and serve to determine linearly any point on the cubic. The whole theory of the cubic is then built up in a very simple manner by the use of the properties of the point Σ, the center of perspectivity of u and v.

In Dr. Epsteen's paper "reducibility" is taken in Koenigsberger's sense, *i. e.*, a linear differential equation is said to be reducible if it has integrals in common with a differential equation (linear or not), of lower order, and with coefficients belonging to the same domains of rationality as those of the original equation. Dr. Epsteen shows that if the group of a linear homogeneous differential equation of the nth order is intransitive of the $(n - m)$th degree (Lie, Transformationsgruppen, volume 1, page 216), the linear differential equation is reducible to an equation of the mth order.

A limit of the minima of definite binary quadratic forms is deducible from the conditions for a reduced form. In the Disquisitiones Arithmeticæ Gauss gave the limit $\frac{4}{3}\sqrt[3]{D}$, and later Hermite gave the exact limit $\sqrt[3]{2D}$ for definite ternary forms, D being the determinant of the form. In the *Mathematische Annalen*, volume 6, Korkine and Zolotareff investigate the problem of finding the relative limit minima of forms, or of finding forms whose minima diminish when their coefficients undergo infinitesimal changes subject to the restriction that the determinant shall remain unaltered. They find in addition to $\sqrt[3]{2D}$ the relative limit $\sqrt[3]{\frac{27}{16}D}$. It is not apparent that these two are all. By the consideration of the deformations which leave a reduced parallelism of lines or planes still reduced, a proof is obtained that the limits of Hermite and Korkine and Zolotareff are all that exist, and this is the object of Dr. McDonald's paper.

In the calculations of the perturbations of the minor planets it is not necessary that the elements of the disturbed and the disturbing body should osculate for the same epoch, but the epochs of osculation may differ by a series of ten or more years (cf. P. A. Hansen, "Methode zur Berechnung der absoluten Störungen der kleinen Planeten," *Abhandlungen d. k. S. Gesellschaft der Wissenschaften*, V, page 185). Hansen states that in those cases in which the difference in epoch affects the perturbations, allowance may be made for this when the perturbations of the second order are determined. In another place he admits that considerable labor may be saved by introducing, at the beginning, elements which osculate for the same epoch.

The Astronomical Ephemerides are arranged to fully meet the needs of computers of special perturbations, but they do not give the osculating elements of the major planets. The Ephemerides contain, however, the heliocentric coördinates of the major planets with sufficient accuracy to permit of the calculation of the osculating elements for any desired epoch.

The method proposed by Professor Leuschner is first to reduce the heliocentric places of the disturbing planet to the same equinox for a number of tabulated dates on both sides of the epoch of osculation of the minor planet, and then to determine the coördinates and velocities of the disturbing planet for the epoch by interpolation and numerical differentiation. In this respect the method is similar to the "Short method of deriving orbits from three observations," recently published by the same author (*Publications of the Lick Observatory*, volume 3, part 1). From the coördinates and velocities thus obtained the elements may be determined by Encke's formulæ, *Berliner Jahrbuch*, 1858.

For the sake of a more accurate determination of the osculating elements of the major planet, the polar coördinates given in the Ephemerides should be transformed into rectangular equatorial coördinates before the application of numerical differentiation.

The method has been successfully applied by Drs. R. T. Crawford and F. E. Ross, who are at present engaged in determining the perturbations of the Watson asteroids for the Watson Trustees. They have made some valuable suggestions, which have been incorporated in Professor Leuschner's method.

One of the four infinite systems of discontinuous groups of genus one is that generated by two operators of order 4, whose product is of order 2. Mr. Manning studied this system by analytic methods and arrived at the following two theorems: The commutator subgroup H of such a group G is abelian and has at most two invariants. The quotient group G/H is either the cyclic group of order 4 or the abelian group of order 8 and of type (2, 1). By means of these theorems Mr. Manning determined all the possible groups in case H is cyclic, and found simple formulas exhibiting the results.

In his paper on groups all of whose subgroups are abelian, Dr. Moreno shows that all such groups are solvable. If the order is p^m, where p is any prime, the commutator subgroup is invariant and of order p. Let P_{m-1} represent any subgroup of order p^{m-1} which contains an operator of the highest order involved in the entire group. Then P_{m-1} has at most three invariants. If it has only two invariants, they are of different orders, and if it has three, one of them is the commutator. Dr. Moreno determines all of the possible groups for these two cases. The groups which are possible when P_{m-1} is cyclic have already been determined by Burnside.

It has been observed that, if a balloon filled with hydrogen be immersed in an organ pipe, it moves toward the node of a stationary sound wave, while the same balloon filled with carbon dioxide moves toward the loop of the wave. It is the purpose of Mr. Nutting's paper to discuss the dynamics of such a motion, and to test or illustrate Lord Kelvin's general theorem that the kinetic energy of such vibrating systems tends towards a maximum. Neglecting pulsations in the balloon and considering only oscillations, Mr. Nutting shows that Lord Kelvin's condition is satisfied.

A quadruple system of n elements is an arrangement of the n elements in quadruples, in such a way that any triple of elements enters into one, and only one, quadruple. There are $\frac{1}{24}n(n-1)(n-2)$ quadruples in the system, where n must be of the form $6h \pm 2$. The existence of quadruple systems of order 2^k, and $2^{k+1}.5$ $(k = 1, 2, 3, \cdots)$ is demonstrated in Dr. Putnam's paper, and a concrete method of constructing them is given. Given a quadruple system on n elements the existence of one on $2n$ elements is proven, and a method of construction exhibited. The quadruples of a system on 8 or on 16 elements may be written down by considering the vertices of a cube in three or in four dimensions, four vertices which lie in a plane or are the vertices of certain tetrahedra forming the quadruples of the system.

Professor Stringham's paper sets up a series of orthogonal transformations of special type in four variables and shows how, by properly combining them, the general transformation involving six independent parameters may be constructed. The fundamental, or generating substitution has the form

$$\theta w + \lambda x + \mu y + \nu z, \quad -\lambda w + \theta x - \nu y + \mu z,$$

$$-\mu w + \nu x + \theta y - \lambda z, \quad -\nu w - \mu x + \lambda y + \theta z,$$

$$[\theta^2 + \lambda^2 + \mu^2 + \nu^2 = 1].$$

If θ, λ, μ, ν be replaced by $\cos\varphi$, $a_1 \sin\varphi$, $a_2 \sin\varphi$, $a_3 \sin\varphi$, the condition for orthogonality becomes

$$a_1^2 + a_2^2 + a_3^2 = 1,$$

independently of the value of φ, and the substitution is completely characterized by the angle φ and a *directrix* a_i whose direction cosines are a_1, a_2, a_3. The symbol of the substitution being (φ, a_i), its inverse $(\varphi, a_i)^{-1}$ is obtained by changing the signs of all the terms of (φ, a_i) except those of the principal diagonal.

It is easily shown that the product of any two substitutions of the type (φ, a_i) is a third substitution of the same type. It then follows that all the substitutions of this type satisfy the conditions defining a group. The product of two substitutions having a common directrix a_i, but different arguments φ, φ', is

$$(\varphi, a_i) \cdot (\varphi', a_i) = (\varphi + \varphi', a_i).$$

It follows by induction that

$$(\varphi, a_i)^n = (n\varphi, a_i),$$

Thus, all the substitutions (φ, a_i), having a given directrix, form a group. The path curves of the transformations of this group are Clifford parallels.

With every substitution (φ, a_i) is coördinated another $(\varphi, -a_i)$, obtained from (φ, a_i) by interchanging the signs of its first column with those of its first row. All that has been said of (φ, a_i) is obviously true of $(\varphi, -a_i)$. The substitutions $(\varphi, -a_i)$ form a group simply isomorphic with the group of the (φ, a_i). All the substitutions of the type (φ, a_i) are permutable with those of the opposite type $(\varphi, -a_i)$, but not in general with one another.

The general orthogonal substitution with six independent parameters may be compounded of two substitutions of opposite types (φ, α_i), $(\varphi', -\alpha_i')$, whose arguments and directrices are distinct. Interpreted geometrically such a resultant transformation represents a general displacement in elliptic or hyperbolic space. It has no invariant points.

But two substitutions of opposite types having a common argument compound into a new special form representing a rotation about a fixed axis. If the components of this substitution be $(\varphi, -\alpha_i)$ and $(\varphi, \alpha_i')^{-1}$, then there is a complementary substitution of the same form and having the same geometrical significance, whose components are (φ', α_i') and $(\varphi', -\alpha_i)$, and the product of all of these

$$(\varphi, -\alpha_i) \cdot (\varphi, \alpha_i')^{-1} \cdot (\varphi', \alpha_i') \cdot (\varphi', -\alpha_i)$$

is, again, a general substitution with six independent parameters. [Cf. the author's paper in the Publications of the Congress of Mathematicians held at Paris in 1900.]

Mr. Whitney's paper called attention to the fact that relations between two complex variables define congruences of lines, the lines for instance obtained by joining corresponding points in the w-plane and z-plane. In the case of a linear relation between w and z with a certain simple choice of coördinates two linear complexes of a special character are obtained related to each other involutorically; the axes of the congruence are imaginary. In general there is the problem of determining the special character of the complexes and the relations between them due to the differential equations that express the fact that w is a monogenic function of z.

Dr. Wilczynski shows that the covariants determined in a previous paper prove a number of fundamental theorems in the projective theory of ruled surfaces. There are only three such covariants. The first leads to a congruence of lines made up of the generators of the first kind of all of the hyperboloids osculating a given ruled surface. One particular ruled surface of this congruence is especially studied, and is closely associated with the given surface. The conditions are discussed under which the asymptotic lines of the two surfaces correspond to each other. The second covariant determines an involution on each generator of the ruled surface, whose double points are the flecnodes (using Cayley's term) on that generator. The flecnode curve is fundamental in this theory. A number of Cremona's theorems on ruled surfaces with straight line directrices are generalized to apply to all ruled surfaces. Dr. Wilczynski gives both analytic and synthetic proofs of these theorems. The third covariant furnishes another congruence associated with a given surface, and in particular a third ruled surface associated with the original one and the one already mentioned. A few brief remarks are made, showing how these covariant surfaces may serve to simplify the integration of the original system of differential equations. This paper will be combined with the previous paper on covariants for publication in the *Transactions*.

E. J. WILCZYNSKI.

MATHEMATICAL PROBLEMS.*

LECTURE DELIVERED BEFORE THE INTERNATIONAL CONGRESS OF MATHEMATICIANS AT PARIS IN 1900.

BY PROFESSOR DAVID HILBERT.

WHO of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones. If we would obtain an idea of the probable development of mathematical knowledge in the immediate future, we must let the unsettled questions pass before our minds and look over the problems which the science of to-day sets and whose solution we expect from the future. To such a review of problems the present day, lying at the meeting of the centuries, seems to me well adapted. For the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future.

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