SCHEFFERS'S THEORY OF SURFACES.

Einführung in die Theorie der Flächen. Von Dr. GEORG SCHEFFERS. Der "Anwendung der Integral- und Differential-Rechnung auf Geometrie" zweiter Band. Leipzig, Veit & Comp., 1902. Pp. x + 518.

THERE will naturally be a considerable divergence of opinion on the part of mathematicians both as to what such a volume as the one before us should contain, and, from a pedagogical standpoint, as to how the subject matter should be presented. It may be stated at once that in the opinion of the present writer the author has accomplished with marked success, on the whole, the difficult task of placing before the beginner in a clear and readable form an outline of the vast theory of surfaces and surface curves.

Among the advantages of the work which recommend it powerfully to the student taking up this branch of mathematics for the first time, is the almost entire absence of unaccustomed symbols and symbolic operators, the free use of which adds so much to the difficulty of reading Bianchi, or Stahl-Kommerell, for example. Bearing in mind that u, v are the parametric coördinates of a point on the surface, that E, F, G and L, M, N are the fundamental magnitudes, that X, Y, Z are the direction cosines of the surface normal, R_1, R_2 the principal radii of curvature, and finally that K, H represent respectively the Gaussian measure of curvature, and the mean curvature, the student will hardly be at loss to read understandingly any chapter in the book—as far at least as the symbolism is concerned.

The author has, very wisely I think, avoided presenting the theory of surfaces as the theory of invariance of two quadratic differential forms. While there are undoubtedly advantages attached to the latter method of presenting the subject, such as the brevity and elegance of most of the formulas developed, yet this method never appeals to the beginner as being a natural one; and the gain in brevity is apt to be more than compensated for by a loss in clearness. Moreover, to obtain a proper insight into the subject, the student should familiarize himself with both methods of treatment, taking the easier and more natural one first.

Another striking advantage for the beginner is that in this volume, as in the preceding one on space curves, every theorem given, pertaining to the subject, is developed in full from the analytical standpoint, geometrical proofs being appended only in the cases in which they add materially to an understanding of the scope of a theorem, or where they are of especial historical interest. Even when a theorem is given which belongs to an entirely different branch of mathematics, as the one concerning the integration of the so-called unconditionally integrable total differential equations, pages 321–331, the necessary proof is usually given.

Also in this volume, as in the preceding one, the author has taken great care to deduce and formulate all theorems in such manner as to cover the cases in which imaginaries are involved.

If objections are to be made at all to this excellent work, they must, in my opinion, be of a pedagogical nature: and even these are largely matters of individual taste. I will mention, however, a few points in which it seems to me that the author might have added to the usefulness of the book.

Although the volume is already quite bulky, it seems to me unfortunate that it did not open with a separate chapter on the curvature of surfaces from the standpoint of Monge, assuming the surface to be given by an equation of the form $z = f(x, y)$. The author occupies from the beginning the standpoint of Gauss; that is, he assumes the surface to be given by three equations connecting the cartesian coördinates x, y, z with the two parameters u, v. Monge's treatment of the subject is of great historical interest; and the symbols, and many of the formulas occurring, are perfectly familiar to a student who has read only the calculus. It is true that the classic formulas of Monge and Euler are collected in a Table (Tafel XIII.) at the end of the book; but a reader familiar with the treatises of Laurent, Picard, and others, will note with regret the absence from the body of the book of a consistent though brief development of the theory of curvature from Monge's standpoint; and the loss to the beginner is, in my opinion, a serious one.

It also appears regrettable that a chapter was not given on families of surfaces, and their defining partial differential equations. This would seem to be a proper subject to be incorporated in a work on the "Applications of the differential and integral calculus to geometry," and would undoubtedly have added much to the interest of the book. Moreover, the author has shown himself, in several of Lie's works edited by him, to be well fitted to treat this subject in an attractive and instructive manner.

We note with less regret the absence of any chapter on curvilinear coördinates in space.

Another minor objection is the same as that which was made to the volume on space curves: that is, that while many important problems, such as those illustrating the theory of map making, the congruence of surfaces, etc., are worked out more fully than is usual, and with remarkable care, there are no collections of problems for the student at the ends of the chapters—a very desirable feature of most English and of many French works.

I shall now give a general outline of the matter presented; and speak in some detail of a few sections of the work.

As presented by the author, the subject matter is arranged in four divisions: I, The Arc Element of the Surface; II, Curvature; III, The Fundamental Equations of the Theory of Surfaces; IV, Surface Curves.

The student who is familiar with the sections of Volume I on parametric coördinates in the plane, will have no difficulty in reading the opening sections of this volume, in which, besides other introductory matter, the quadratic differential form,

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

is deduced for the square of the arc element, and the geometrical meaning of the fundamental magnitudes of the first order E, F, G is explained. The author proceeds, in §§ 5, 6 to the discussion of the problem of the point representation of one surface upon another, the special case considered being that in which areas are invariant. This is a generalization of the corresponding discussion in § 8, Part I, Volume I; and it is in this connection that several inter esting examples of the older methods of map making—those in which areas are invariant—are fully worked out.

The definition and determination of isothermal systems on a surface occupy §§ 7, 8; and here again § 7 is practically the same in form as the corresponding section in Volume I. The author's method for introducing and determining isothermal systems on a surface seems unnecessarily indirect. He first establishes the condition that the parameter curves $u =$ const., $v =$ const., shall divide the surface into a network of infinitesimal rhombi, and then that the diagonals of the rhombi shall meet at right angles. These conditions are satisfied, that is the parameter curves form an isothermal system, when the square of the arc element has the form

$$ds^2 = \Omega\,(u, v)\,(du^2 + dv^2). \qquad (1)$$

He then proceeds to show that all isothermal systems on a given surface can be found when the differential equation

$ds^2 = 0$ of the minimal curves of the surface can be integrated.

It would be more satisfactory, in my opinion, to present the matter, as for example Bianchi does (Differential Geometry, page 70); that is, show first that the square of the arc element can be thrown into the form (1), when the differential equation $ds^2 = 0$ can be integrated. It then follows at once that $u =$ const., $v =$ const. form an isothermal system. Also, if desired, the proof that (1) expresses the necessary as well as the sufficient condition that the parameter curves form an isothermal system can be added in a few lines, as in Darboux, Leçons sur la théorie générale des surfaces, volume I, page 146.

In § 9 we have an excellent exposition of the theory of the conform representation of one surface upon another; and the general theory is illustrated in § 10 by the example of the conform representation of a sphere upon a plane. The important theorem is proved, that the stereographic projections of a given sphere upon a plane are the only conform representations of the sphere upon the plane, by means of which all circles of the sphere appear as circles on the plane. The section closes with a discussion of the theory of Mercator's projection.

The last section of Part I contains a brief outline of the theory of the general point representation of one surface upon another, including Tissot's theorem that in the case of any not conform point representation of one real surface upon another, there is in general one and only one orthogonal system of curves on the one surface which corresponds to an orthogonal system on the other surface. Attention is also drawn to Lie's note (*Mathematische Annalen*, volume 20) in which he shows that this theorem does not always hold for imaginary point representations.

This section is a particularly valuable one, as the problem of the general point representation of one surface upon another is not usually discussed in the text-books.

Part II deals with the theory of curvature of surfaces. The contents of the various sections of this part will be sufficiently indicated by saying that they cover the general topics usually discussed in this connection; namely, conjugate curve systems on a surface; asymptotic curves; lines of curvature; spherical representation of a given surface, after Gauss; minimal surfaces. We note that the author completes in an essential feature the discussion of the curvature of a plane normal section, as it is given in the other current text-books. The curvature, at a point (u, v)

on a given surface, of a plane normal section through (u, v), is represented by

$$\frac{1}{R}=\frac{L+2Mk+Nk^2}{E+2Fk+Gk^2}, \tag{2}$$

where k is written for $dv : du$.

The two cases usually considered here are (i) that the right hand member of (2) is a quadratic fractional function of k, the case which occurs at any ordinary point on a real surface; and (ii) that the right hand member of (2) does not actually contain k, that is, that

$$L : M : N = E : F : G, \tag{3}$$

or that the point (u, v) is an umbilic. As is well known, if these conditions are satisfied at every point on a given surface, the surface is either a plane or sphere.

But there is a third possibility: that is, the numerator and denominator of the right hand member of (2) may have a common factor which is linear in terms of k, in which case (2) can be written

$$\frac{1}{R}=\frac{\alpha k+\beta}{\gamma k+\delta}. \tag{4}$$

This case, which is of interest and importance, seems to have been first discussed by Stäckel, *Leipziger Berichte*, 1896. At a point (u, v) on a given real surface, where (4) is satisfied, there cannot exist principal radii of curvature, since, by (4), $\frac{1}{R}$ has no finite maximum or minimum at that point. An example of a real surface with such a point is the following: We construct in each plane through the z-axis the circle which is tangent to the xy-plane at the origin, and of which the ordinate, $z = R$, of the center, is a linear fractional function of the tangent of the angle which the plane of the circle makes with the zx-plane. The locus of the ∞^1 circles constructed in this manner is a real surface; and the origin is clearly a point of the required nature on that surface.

If the condition (4) is satisfied at every point on a surface, Euler's theory of curvature does not hold for the surface. A surface of this nature can be shown to be imaginary, and to contain one system of minimal straight lines, and one system of minimal curves. The surfaces for which the conditions (3) hold at every point, contain, it will be remembered, two distinct families of minimal straight lines.

In §13 the author inserts a portion of the theory of the ruled surface, and develops the theory of curvature of the ruled surface as an example of the application of the formulas deduced for the general theory of curvature. It would have been better, in my opinion, to devote a division of the book to a consecutive development of the theory of the straight line system—as is done by Laurent and Jordan, for example—deducing thence in outline both the theory of the line complex, which is so intimately connected with that of the curvature of surfaces, and the theory of the ruled surface. At present the reader has to look through several sections of both Volume I and Volume II to find what is said of the ruled surface, or of the line complex.

In the example, page 183, of the asymptotic curves of a ruled surface, the author shows in the usual manner that the curvilinear asymptotic curves of a ruled surface are defined by a Riccati equation

$$\frac{dv}{du} = A(u) + B(u)v + C(u)v^2;$$

but, contrary to his usual custom, he does not complete the discussion by giving Bonnet's theorem, that the general solution of this equation can be found, when one particular solution is known.

Although the author's discussion of the theory of the lines of curvature of a surface is quite comprehensive, he omits to state that the number of lines of curvature through an umbilic is generally finite, and to show how to obtain the differential equation defining the directions of the lines of curvature through such a point.

The above objections are, however, trivial in comparison with the general excellence of the treatment of the subjects covered by Part II.

Part III, which is considerably more difficult for the beginner than the other divisions of the book, deals primarily with the derivation of the three fundamental partial differential equations of a surface: that is, the three equations connecting the fundamental magnitudes E, F, G, L, M, N with their derivatives. The principal object of this Part is to show that when any six functions E, F, G, L, M, N of u and v satisfy the three fundamental equations, then E, F, G, L, M, N can be considered the six fundamental magnitudes of a certain surface, and that they do, in fact, define the surface except as regards its position in space. The proof of this great theorem, which is given in several consecutive steps, and which occupies some thirty pages of the book, is, in my opinion, decidedly to be preferred for a beginner to the comparatively compressed proof given by Stahl-Kommerell, or Bianchi. The author recommends, however, that these pages be omitted on a first reading of the volume.

In § 6 of this Part we have the derivation of all differential invariants of a surface under the movements in space

$$x_1 = \alpha_1 x + \alpha_2 y + \alpha_3 z + a,$$
$$y_1 = \beta_1 x + \beta_2 y + \beta_3 z + b,$$
$$z_1 = \gamma_1 x + \gamma_2 y + \gamma_3 z + c.$$

These differential invariants, which are, of course, of prime importance in the theory of congruence of surfaces, are easily seen to be functions of the fundamental magnitudes E, F, G, L, M, N, and their derivatives with respect to u and v. The fundamental magnitudes are not invariant, however, when a new system of parameters u, v is introduced on the surface. Therefore, to decide finally whether two surfaces are congruent or not, it is advantageous, as Lie pointed out (*Leipziger Berichte*, 1896), to make use of magnitudes which are invariant both under all movements in space, and under all changes of the parameter system on a surface. Such a magnitude is called a differential invariant "in the wider sense": and represents, of course, some geometrical property of the surface at a point (u, v) which is unchanged either by a movement of the surface in space, or by the introduction of a new parameter system on the surface. The principal radii of curvature are seen to be differential invariants of this nature. From this standpoint the author develops in §10 a complete theory of congruence for those surfaces for which no relation of the form

$$W(R_1, R_2) = 0$$

exists between the principal radii of curvature. The latter, the so-called Weingarten-, or W-surfaces, are specially investigated in §11.

In the other sections of this Part, the author discusses the deformation of surfaces. If S' is a surface which is a deformation of a given surface S, that is, if S' is applicable to S, the application of S' to S is equivalent to a point representation of S' upon S, which is conform and which leaves areas invariant. This point representation will not, however, in general be equivalent to a continuous transformation of S' into S. The skew helicoid, a minimal surface, is an interesting example of a surface which can be transformed into a surface to which it is applicable, the catenoid, by means of a continuous deformation. The original surface is a minimal surface in each of the ∞^1 forms which it assumes under this operation.

In first discussing the problem of deformation, § 3, the minimal curves are used as parameter curves on the surface. It is clear that if S' is applicable to S, the minimal curves of S' will coincide with those of S when the two surfaces are brought to superposition. In order to use the minimal curves as parameter curves, it is necessary to integrate the differential equation $ds^2 = 0$ on both surfaces. If that has been done, and if the expression for ds^2, the square of the arc element, when written in the new variables, can be thrown into the same form on both surfaces, S' is applicable to S.

In § 12, the last section of this Part, the author introduces for the first time the first, second, and mixed differential parameters of a surface; and, following Darboux, as he himself states, completes the discussion of the problem of deformation by showing that the question as to whether S' is a deformation of S can always be determined by mere algebraic eliminations, whenever two independent differential parameters "in the wider sense" of S, or S', are known. It is to be regretted that no example of the application of this theory is given.

The fourth, and last, Part of the volume treats of the theory of the surface curve, by far the greater portion of this Part being devoted to the theory of geodesics. After defining a geodesic, and deducing the differential equation of all geodesics on a given surface, it would have been well to bring out clearly the fact that a geodesic through a point P is definitely determined when the direction of its tangent at P is given : but that the geodesic is not always definitely determined when a second point Q on the curve is given, no matter how near Q may be to P.

The fact that the miminal curves are geodesics is sufficiently emphasized.

In § 2, under the caption of the geodesic representation of one surface upon another, we have the proof of Liouville's important theorem that the geodesics on any surface, for which ds^2 can be thrown into the form

$$ds^2 = (U(u) + V(v))(du^2 + dv^2),$$

can be found by quadratures. All surfaces of constant curvature, and surfaces of revolution, belong, of course, to this class.

After showing that a geodesic point representation of one surface upon another is not always equivalent to a deformation, the author closes this section with a pretty example of the general theory, in which the equations defining the most general geodesic representation of a plane upon itself are derived. They are, of course, the equations of the general projective point transformations in the plane.

The next two sections serve to introduce the Gaussian systems of geodesic coördinates. After proving that the total curvature of a geodesic triangle is equal to the sum of the angles of the triangle, diminished by two right angles, the author proceeds to remark that a geometry of the geodesics on a surface of constant curvature can be developed, which is analogous to that portion of the geometry of the straight line in the plane which does not depend upon the fact that the sum of the three angles of a plane triangle is equal to two right angles. This follows from the fact that the finite equation, in u and v, of the geodesics on a surface of constant curvature, is linear.

The interesting fact, however, is not mentioned that since a surface of constant positive curvature, K, is applicable to a sphere of radius $\frac{1}{\sqrt{K}}$, the formulas of spherical trigonometry, which give the relations between the sides and angles of a spherical triangle, hold also for the sides and angles of any geodesic triangle on the given surface of constant curvature.

A discussion of the geometry of the surface of centers, and the development of the conditions under which the ∞^2 generators of a line complex can be the normals of a surface, occupy §§ 5, 6.

The volume closes with § 7, the curvature and torsion of a general surface curve.

After defining the geodesic, or tangential, curvature, and the normal curvature of a surface curve, the author is careful to point out that the term "geodesic circle" is used by some writers to mean a surface curve of constant geodesic curvature, and by others to mean a surface curve the points of which are at a constant geodesic distance from a fixed point on the surface. The proof is given, in a line or two, that the two definitions are not generally coincident.

The expression for the absolute torsion of a surface curve is also derived; but no mention is made of geodesic torsion,