# THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 22, 1902, Vice-President Professor Maxime Bôcher in the chair. The attendance at the morning and afternoon sessions was about thirty-five, including the following thirty members of the Society:

Dr. Grace Andrews, Professor Maxime Bôcher, Professor F. N. Cole, Dr. W. S. Dennett, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Dr. G. B. Germann, Dr. Carl Gundersen, Dr. E. R. Hedrick, Dr. G. W. Hill, Dr. E. V. Huntington, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Professor James Maclay, Mr. H. B. Mitchell, Professor W. F. Osgood, Dr. M. B. Porter, Professor J. K. Rees, Miss I. M. Schottenfels, Professor D. E. Smith, Professor P. F. Smith, Dr. Virgil Snyder, Professor E. B. Van Vleck, Professor A. G. Webster, Dr. Ruth G. Wood, Professor R. S. Woodward.

The Council announced the election of the following persons to membership in the Society: Professor Edward Brand, Howard College, East Lake, Ala.; Mr. David Raymond Curtiss, Harvard University; Miss Alice Bache Gould, Boston, Mass.; Dr. Carl Gundersen, New York City; Mr. A. F. van der Heyden, Middlesbrough High School, England; Dr. Jean de Seguier, S. J., Paris, France; Mr. J. W. Young, Cornell University. Fourteen applications for membership were received.

Professor T. S. Fiske, whose term as member of the Editorial Board of the *Transactions* had expired, was reelected to this position. Dr. Edward Kasner was reëlected Assistant Secretary of the Society. A petition for the authorization of a Pacific Section of the Society, signed by seventeen members representing the Pacific slope, was received and laid over for consideration at the next meeting of the Council. It was decided to hold the next summer meeting of the Society at Evanston, Ill., in the latter part of August or the first part of September.

The following papers were presented at the meeting:

(1) Dr. E. V. HUNTINGTON: "A complete set of postulates for the theory of absolute continuous magnitude."

(2) Dr. E. V. HUNTINGTON: "Complete sets of postulates for the theories of positive integral and positive rational numbers."

(3) Dr. E. V. HUNTINGTON: "Simplified definition of a group."

(4) Dr. M. B. PORTER: "On the arithmetic nature of the zeros of Bessel functions."

(5) Dr. W. B. FITE: "On a property of groups of order $p^m$, $p$ being a prime."

(6) Professor L. E. DICKSON: "The groups of Steiner in problems of contact (second paper)."

(7) Dr. VIRGIL SNYDER: "On the forms of quintic scrolls."

(8) Mr. PERCIVAL LOWELL: "On the capture of comets by Jupiter."

(9) Mr. H. L. RIETZ: "On primitive groups of odd order."

(10) Professor MAXIME BÔCHER: "On systems of linear differential equations of the first order."

(11) Dr. E. J. WILCZYNSKI: "Covariants of systems of linar differential equations."

(12) Professor JAMES MACLAY: "On some associated surfaces of negative curvature."

(13) Professor E. W. BROWN: "On the small divisors in the lunar theory."

(14) Mr. OTTO DUNKEL: "Some applications of Green's theorem in one dimension."

(15) Mr. J. W. YOUNG: "On a certain group of isomorphisms."

(16) Dr. A. S. GALE: "On the rank, order, and class of algebraic minimum curves."

(17) Mr. W. H. ROEVER: "Brilliant points and loci of brilliant points."

Mr. Rietz was introduced by Dr. W. B. Fite, and Mr. Roever by Professor Osgood. Mr. Lowell's paper was presented to the Society through Dr. Hill, Mr. Dunkel's through Professor Bôcher, and Mr. Young's through Dr. J. I. Hutchinson. In the absence of the authors, Dr. Fite's paper was read by Dr. Snyder, Mr. Lowell's by Dr. Hill, Mr. Dunkel's by Professor Bôcher, and Mr. Young's by Mr. Rietz. The papers of Professor Dickson, Dr. Wilczynski, and Professor Brown were read by title. At the close of the regular programme Professor A. G. Webster gave a résumé of a paper, also read before the American Physical Society, on the motion of a spherical pendulum, illustrating it with photographs and stereopticon views.

Abstracts of the papers are given below.

Dr. Huntington's first paper presents a complete set of postulates or primitive propositions from which the mathematical theory of absolute continuous magnitude (positive real number) can be deduced.* The fundamental concept involved is that of a "rule of combination" in an assemblage. According to such a rule, any two elements $a$ and $b$ of the assemblage, in a given order, determine the third element $a \circ b$. (The same idea is the fundamental concept in the definition of a "group.") The rule of combination here considered is one upon which six fundamental restrictions are imposed, these requirements being expressed in the following postulates:

1. Whatever elements $a$ and $b$ may be ($a = b$ or $a \neq b$), $a \circ b$ is also an element of the assemblage.

2. $a \circ b \neq a$.

3. $(a \circ b) \circ c = a \circ (b \circ c)$ whenever $a \circ b, b \circ c, (a \circ b) \circ c$ and $a \circ (b \circ c)$ belong to the assemblage.

4. When $a \neq b$ at least one of the following conditions is satisfied: either 1° there is an element $x$ such that $a = b \circ x$ or 2° there is an element $y$ such that $a \circ y = b$.

5. (Let the notation $a < b$ indicate that an element $y$ exists such that $a \circ y = b$; and let $a \leqq b$ indicate: $a < b$ or $a = b$.) If $S$ is any infinite sequence of elements, $a_1$, $a_2$, $a_3, \cdots, a_\kappa, a_{\kappa+1}, \cdots$, such that

$$a_1 < a_2 \,;\; a_2 < a_3 \,;\; \cdots \,;\; a_\kappa < a_{\kappa+1} \,;\; \cdots,$$

and if there is an element $c$ such that $a_\kappa < c$ whenever $a_\kappa$ belongs to $S$, then there is an element $A$ having the following properties: 1° for every element $a_\kappa$ of $S$ $a_\kappa \leqq A$; and 2° if $y$ and $A'$ are such that $y \circ A' = A$ then there is at least one element of $S$, say $a_r$, for which $A' < a_r$.

6. Whatever element $a$ may be, there are two elements $x$ and $y$ such that $x \circ y = a$.

The object of the work which follows is to show that these six postulates form a "complete set"; that is, to show that they are 1° consistent, 2° sufficient, and 3° independent. By these three terms is meant: 1° there is at least one assemblage in which the chosen rule of combination satisfies all the six requirements; 2° there is essentially only one such assemblage possible; and 3° none of the six postulates is a consequence of the other five. This

* Cf. O. Hölder, "Die Axiome der Quantität," *Leipziger Berichte*, 1901.

being shown, the postulates 1–6 may be said to define essentially a single assemblage, which is called the "system of absolute continuous magnitude." The element $a \circ b$ is then the "sum" of $a$ and $b$. From another point of view, the propositions 1–6 may be accepted as expressing in precise mathematical form the essential characteristics of "magnitude" in the popular sense of the word; from this point of view they may properly be called the "axioms" of (absolute continuous) magnitude. From either point of view the propositions 1–6 form a complete logical basis for a deductive mathematical theory. The paper closes with a statement of some unsolved problems concerning possible modifications of the postulates.

By properly modifying the above set of postulates Dr. Huntington constructs in his second paper two complete sets of postulates such that every assemblage which satisfies either of these new sets will be equivalent to the system of positive integers. The first of these sets contains six postulates and the second four. By a further modification, a complete set of five postulates for the theory of positive rational numbers is obtained.

Dr. Huntington's third paper appears in the present number of the BULLETIN.

Except in the case of Bessel functions whose indices are halves of odd integers (in which case the non-zero roots are evidently transcendentally irrational), there seem to be no known theorems concerning the arithmetic nature of the roots of Bessel functions. Dr. Porter applies a theorem of Hurwitz (*Mathematische Annalen*, volume 32) to show that the non-zero roots of all Bessel functions of real rational index are irrationalities of higher order than the second.

If a metabelian group $G$ of order $p^m$ ($p$ being a prime) contains an abelian subgroup of order $p^{m-\alpha}$, the $p^\alpha$ power of every one of its operators is invariant. Dr. Fite considered the more general case of a group $G$ of order $p^m$ and class $k$ that contains an abelian subgroup of order $p^{m-\alpha}$. It was shown that in such a group the $p^\alpha$ power of every operator is invariant, if $k \equiv p$. It follows as a corollary that a group of order $p^m$ that contains an abelian subgroup of order $p^{m-1}$ and type $(m_1, m_2, \cdots, m_n)$, where $m_i > 1$ $(i = 1, 2, \cdots, n)$, is either metabelian or of class $k$, where $k > p$.

Professor Dickson's paper is the continuation of the article with the same title which appeared in the *Transactions* for

January, 1902. Both papers investigate the group $G$ of the equation for the determination of the curves of order $n-3$ having simple contact at $\frac{1}{2}n(n-3)$ points with a given curve $C_n$ of order $n$ having no double points. For $n=4$ these are the 28 bitangents to a quartic curve. The first paper discussed the case $n$ odd; the present paper treats the case $n$ even. The group $G$ is a subgroup of the group of Steiner, the latter being holoedrically isomorphic with the first hypoabelian or the abelian linear group on $2p$ variables with coefficients taken modulo 2, according as $n$ is odd or even. The proofs are simpler, and more direct, than those of Jordan, Traité des Substitutions, pages 229–249.

Several years ago Schwarz found 15 types of quintic scrolls. By means of the dual of the configurations employed by Schwarz, Dr. Snyder extends the subdivision of the forms and finds 28, of which 21 are unicursal, 5 of genus 1, and 2 of genus 2. The most general equation of any type can be written at once when the configuration of nodal lines is given.

Mr. Lowell's paper expresses the action of Jupiter on a comet passing within the sphere of activity of the planet by the equation of energy, and thence by the principle of moving axes deduces Professor H. A. Newton's two important formulæ, extending them to the general case. Tisserand's criterion and Jacobi's integral are shown to be corollaries from the general deduction. The conditions of the relative orbit are then considered and a critical angle found within which, if the comet approaches the planet, the latter's action upon it can never be such as to render its absolute orbit retrograde. All the members of Jupiter's present family are shown to satisfy this criterion, so that the directness of their orbits is, within the limits of the approximation, not only a present but a permanent phenomenon. Lastly the angle of stability is deduced within which Jupiter's control on the capture is complete.

Denoting by $G$ a primitive group of degree $n$, and by $G_1$ its maximal subgroup leaving a given letter fixed, the results obtained by Mr. Rietz may be stated as follows: 1° If $p^\alpha$ is the highest power of a prime $p$ which divides the order of $G$, and if a subgroup $P$ of order $p^\alpha$ is of degree $n-1$, then, unless $P$ is a regular group, or is formed by establishing a simple isomorphism between regular groups, $G$ has an intransitive subgroup of degree $n$ which has a transitive constituent of degree $1 + kp$, and of order $lp$. 2° If $G_1$ has an invariant subgroup $H$ of degree $n - a$ $(a > 1)$, then $G_1$ has a transitive constituent whose degree exceeds the degree of any transitive constituent of $H$. 3° If all the transitive constituents of $G_1$ are primitive groups of the same degree, $G_1$ is formed by establishing a simple isomorphism between them. By means of these theorems the determination of the primitive groups of odd order (see BULLETIN, volume 8, page 18) is extended from degree 150 to 243. It results that aside from invariant subgroups of the metacyclic group there are only two such groups within the given limits. They are of degree 169 and of orders 169.7 and 169.21. Each of these groups is solvable. There is then no simple group of odd composite order of degree less than 243.

Professor Bôcher considered the system of differential equations

$$y_i' = a_{i1}y_1 + \cdots + a_{in}y_n + \beta_i \qquad (i = 1, 2, \cdots, n),$$

where the coefficients $a_{ij}$ and $\beta_i$ are assumed to be functions of the real variable $x$, which throughout a certain finite interval have at most a finite number of discontinuities, these discontinuities being of such a nature that the integrals

$$\int |a_{ij}| dx, \qquad \int \beta_i dx$$

always converge. The method of successive approximations is used, and it is shown first that for the above system of equations the fundamental existence theorem holds in precisely the same form as it does in the case ordinarily considered where the $a$'s and $\beta$'s are continuous ; and secondly that a small variation in the $a$'s, $\beta$'s, and the initial values produces a small variation in the solutions.

The covariants considered in Dr. Wilczynski's paper are those of a system of differential equations of the form

$$(1) \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

under the transformation

$$(2) \quad \xi = \xi(x), \quad \eta = a(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z,$$

where $\xi$, $\alpha$, $\beta$, $\gamma$, $\delta$ are arbitrary functions of $x$. The results obtained are as follows:

An integral rational function of $y$, $z$, $p_{ik}$, $q_{ik}$ and the derivatives of these quantities which is a covariant must be homogeneous in $y$, $z$, $y'$, $z'$, etc., and isobaric in all of the quantities entering into it. The assignment of weight is as follows: $y$ and $z$ are of weight zero, $p_{ik}$ of weight one, $q_{ik}$ of weight two; every differentiation increases the weight by unity, and the weight of a product is the sum of the weights of its factors. If $C_{\lambda, w}$ is an integral rational covariant of weight $w$ and degree $\lambda$, and if the transformation (2) is made, $C_{\lambda, w}$ will be transformed in accordance with the equation

$$I_{\lambda, w} = \frac{(\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}}}{(\xi')^{w}} C_{\lambda, w}.$$

Its degree $\lambda$ is always even.

The system (1) has but three essentially distinct covariants. They are all of the second degree. Putting

$$\rho = 2y' + p_{11}y + p_{12}z, \quad \sigma = 2z' + p_{21}y + p_{22}z,$$

then

$$C_1 = P = z\rho - y\sigma$$

is one of them, of weight 1. There is one of weight 2

$$C_2 = C = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz,$$

where the quantities $u_{ik}$ are the same as those so denoted in the author's paper on invariants. Finally there is one of weight 3

$$C_3 = E + 2N,$$

where

$$E = v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz,$$

$$N = 2(u_{12}z\sigma - u_{21}y\rho) + (u_{11} - u_{22})(z\rho + y\sigma),$$

the quantities $v_{ik}$ also being the same as those so denoted in the author's previous paper on invariants.

In Professor Maclay's paper two related representations of a function of a complex variable by means of surfaces are given, each surface completely representing the function. Some relations between the two surfaces are then exhibited, following which a discussion is made of a class of surfaces embracing the two above named. All the surfaces under consideration are of negative curvature and have the same spherical representation.

In Professor Brown's paper the difficulties arising in the integration of the equations of motion of the moon from the presence of small divisors are pointed out. The degree of accuracy in terms of higher order is seriously affected thereby. Various methods are developed to avoid, as far as possible, this loss of accuracy, when the author's treatment of the theory is in question. The question as to what degree of accuracy is necessary for the intermediate orbit is fully considered and it is shown that if this orbit be found accurately to $m^{10}$, all coefficients up to and inclusive of the fourth order can be found to $m^8$, and within the practical limits set by observation to $m^{12-q}$ for terms of the $q$th order. If a numerical development be used, it is shown that 10 places of decimals are sufficient for the intermediate orbit.

Mr. Dunkel's paper appears in the present number of the BULLETIN.

The main object of Mr. Young's paper is to determine, by a purely analytic method, the defining relations of the group of isomorphisms $I$ of the non-abelian group of order $p^m$ which contains operators of order $p^{m-1}$. It is found that $I$ is defined by

$$I^{p^{m-2}(p-1)} = 1,\ J^p = 1,\ K^p = 1,\ I^{-1}JI = J^x,\ I^{p-1}K = KI^{p-1},$$

$$I^{-1}KI = K^s I^{tp^{m-3}(p-1)},\ K^{-1}JK = JI^{p^{m-3}(p-1)},$$

where $x$ is any number belonging to the exponent $p-1$ mod $p^{m-1}$, and where $s$, $t$ are defined by the relations

$$sx - 1 \equiv 0 \bmod p, \quad 2tx + x - 1 \equiv 0 \bmod p.$$

Dr. Gale considered the equations of a minimum curve in the form

$$x = (1 - s^2)F''(s) + 2sF'(s) - 2F(s),$$

$$y = i(1 + s^2)F''(s) - 2isF'(s) + 2iF(s),$$

$$z = 2sF''(s) - 2F'(s),$$

where $F(s)$ is an algebraic function such that $F'''(s) \neq 0$. Rules were given for the determination of the rank, order, and class of the curve which can be applied if the developments of $F(s)$ in series be known in the vicinity of its poles, branch points, and the point at infinity. Detailed application was made to the case in which $F(s)$ is a two valued function, the case in which $F(s)$ is a rational function having been treated by Lie (*Mathematische Annalen*, volume 14 (1879), pp. 368, 371, 372). Finally attention was called to the bearing of the paper on the theory of algebraic minimum surfaces.

In a polished surface an observer sees images of a source of light. The points in which the surface is pierced by the right lines connecting the eye of the observer with the images are the brilliant points considered in Mr. Roever's paper. When the surface moves, the brilliant points move, and when the motion is very rapid it is possible for the observer to see the locus of the brilliant points. This locus is called a brilliant curve. Such a curve is seen in a carriage wheel with polished spokes when rays of sunlight or rays from a street lamp fall upon it. Closely packed assemblages of brilliant points look like continuous regions of light when the consecutive brilliant points are so close that the eye cannot separate them. A striking example of this is presented by a circular saw which has been polished with emery in a lathe. The consecutive scratches are so close that the corresponding brilliant points cannot be separated by the eye. Each scratch may be considered as being the special position of a variable circular scratch, and hence the isolated brilliant points are points of a brilliant curve. In the present paper are given the analytical conditions which determine the brilliant points of a space curve. The equation of the locus of brilliant points of a one or a two parameter family of curves is obtained. It is shown that both the "carriage wheel curve" and the "saw curve" are represented by equations of the fourth degree, and geometrical constructions are obtained for them. Photographs of the latter curve, in which an arc light is the source and the optical center of the camera lens the recipient, accompany the article. The curves themselves were exhibited by means of very simple apparatus.

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