ing from the multiplication of S_1^p by S_2^p. Special example: $T_1^3 \cdot T_2^3$ is absolutely convergent, but T_1^3 and T_2^3 are each divergent when $r < \frac{2}{3}$ and $s < \frac{2}{3}$.

In the divergent series S_1^p the terms increase without limit in numerical value, as v increases without limit. The same is true of S_2^p. Herein lies the difference between this pair of divergent series yielding an absolutely convergent product, and the pair given by Pringsheim.* In the latter the terms of the divergent series remain finite as v increases indefinitely.

From the relation $S_1S_2S_1S_2 \cdots = S_1^p \cdot S_2^p$ we see that there are cases in the multiplication of series in which divergent series may be used with safety—the sum of the final product series being convergent and equal to the product of the sums of the initially given convergent factor series, even when the product of *some* of the given factor series is divergent.

If two or more convergent series, when multiplied together, yield a convergent product series, then the sum of this product series is equal to the product of the sums of the factor series.

This theorem was proved by Abel for the case of two factor series,† and his method of proof is applicable to the general case. The extension is obvious.

COLORADO COLLEGE, COLORADO SPRINGS,
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CONCERNING THE CLASS OF A GROUP OF ORDER p^m THAT CONTAINS AN OPERATOR OF ORDER p^{m-2} OR p^{m-3}, p BEING A PRIME.

BY DR. W. B. FITE.

(Read before the American Mathematical Society, December 28, 1901.)

IF a non-abelian group of order p^m contains an operator of order p^{m-1} it is of the second class.‡ It is the object of

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* *Loc. cit.*, p. 409.

† *Oeuvres complètes* de N. H. Abel, Tome Premier, 1839, "Recherche sur la série $1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \ldots$," Theorem VI.

‡ Burnside, Theory of Groups, p. 76. If we form the group of cogredient isomorphisms G' of G, then the group of cogredient isomorphisms G'' of G', and so on we finally come either to identity or to a group that has no invariant operators except identity, and is therefore simply isomorphic with its group of cogredient isomorphisms. The groups for which this process leads to identity are classified according to the number of these successive groups of cogredient isomorphisms.

the present note to show that a group of order p^m which contains an operator of order p^{m-2} is of class k, where $k \gtreqless 3$, if p is an odd prime; and that $k \gtreqless 4$ if the group is of order p^m and contains an operator of order p^{m-3}, where p is a prime greater than 3.

Let G be a group of order p^m that contains an operator of order $p^{m-\alpha}$, where $\alpha \gtreqless \frac{1}{2} m$ and p is an odd prime. Assume that in all groups of order p^{m_1} that contain an operator of order $p^{m_1-\alpha_1}$, where $\alpha_1 \gtreqless \frac{1}{2}m_1$ and $\alpha_1 < \alpha$, the p^{α_1} power of every operator is invariant. Let A be an operator of G of order $p^{m-\alpha}$. We can assume that G contains no operator of order greater than $p^{m-\alpha}$.

G contains an invariant subgroup G_1 of order p^{m-1} that contains A.* If $G \equiv \{G_1, B\}$, then $A^{p^{\alpha-1}}B^p = B^pA^{p^{\alpha-1}}$, since in accordance with our assumption $A^{p^{\alpha-1}}$ is invariant in G_1. Now

$$B^{-1}AB = AC,$$

where C is an operator of G_1.

$$B^{-1}A^{p^{\alpha-1}}B = (AC)^{p^{\alpha-1}} = A^{p^{\alpha-1}(1+b)},$$

since the $p^{\alpha-1}$ power of every operator of G_1 is in $\{A\}$.

$$B^{-p}A^{p^{\alpha-1}}B^p = A^{p^{\alpha-1}(1+b)^p} = A^{p^{\alpha-1}}.$$

Therefore

$$(1+b)^p \equiv 1 \ (\text{mod. } p^{m-2\alpha+1}) \text{ and } b \equiv 0 \ (\text{mod. } p^{m-2\alpha}).$$

If

$$m - 2\alpha = 0, \qquad b \equiv 0 \ (\text{mod. } p^{m-2\alpha+1}).$$

Therefore $B^{-1}A^{p^\alpha}B = A^{p^\alpha}$, and the order of G' is equal to, or less than, $p^{2\alpha}$.

This result was reached on the assumption that a similar result holds for groups of order p^{m_1} that contain an operator of order $p^{m_1-\alpha_1}$, where $\alpha_1 < \alpha$. Now this result holds when $\alpha_1 = 1$. Therefore it holds for all values of $\alpha \gtreqless \frac{1}{2} m$.

If $\alpha = 2$, the order of G' is equal to, or less than, p^4. It remains to be shown that G' cannot be of the third class. If it were of the third class it would be of order p^4 and contain an operator of order p^2, but none of a higher order.

* Frobenius, *Berliner Sitzungsberichte*, 1895, p. 173. Burnside, *Proc. London Math. Soc.*, vol. 26 (1895), p. 209.

In this case if B^p were not in $\{A\}$, G' would be generated by two independent operators of order p^2. But there is no such group.*

If $B^p = A^{lp}$ we must have $l \equiv 0 \pmod{p}$, since otherwise A^p would be invariant in G, and the order of G' would be less than p^4. But if we had $l \equiv 0 \pmod{p}$, G' would be generated by an operator of order p^2 and two of order p. This is impossible.† Therefore:

A group of order p^m, where p is an odd prime, that contains an operator of order p^{m-2} is of class k, where $k \gtreqless 3$.

Suppose now that $\alpha = 3$ and $p > 3$. The order of G' is equal to or less than p^6. If this order equals p^6, G' contains an operator of order p^3, but none of higher order. If G_2 is a subgroup of G_1 of order p^{m-2} that contains A, there must be an operator D in G_2 of order p that is not contained in $\{A\}$, since otherwise G_2 would contain only one subgroup of order p and therefore would be cyclic.‡ But we have supposed that G contains no operator of order higher than p^{m-3}. Therefore if C were an operator of G_1 that is not in $\{G_2\}$ and that corresponds to an operator of G' of order p^3,§ we should have $C^p = A^{lp}$, where $l \not\equiv 0 \pmod{p}$, or $C^p = A^{lp}D$. In the former case, A^{p^2} would be invariant in G and the order of G' would be less than p^6. In the latter, $k \gtreqless 4$. If $B^{p^2} = A^{lp^2}$, where $l \not\equiv 0 \pmod{p}$, A^{p^2} would be invariant in G, and the order of G' would be less than p^6. Therefore if the order of G' is p^6, we need to consider only the cases in which G' is generated either by two independent operators of order p^3 or by an operator of order p^3 and one or more operators of orders less than p^3. The latter is impossible, and the former is possible only in case G' is of class k, where $k \gtreqless 3$, since if $k > 3$ one of the successive groups of cogredient isomorphisms would have one generator of order greater than the order of any other generator. This is impossible.

If the order of G' is less than p^5, then $k \gtreqless 4$. We consider therefore the case in which G' is of order p^5. If A^{p^2} is not invariant in G, G' has an operator of order p^3 and therefore $k \leqq 4$. Accordingly, we suppose that A^{p^2} is invariant in G. Then G' has an operator of order p^2. Now

*Young, *Amer. Jour. of Math.*, vol. 15 (1893), pp. 124–178; Hölder, *Math. Annalen*, vol. 43 (1893), pp. 371–412.

† This is proved in an article offered for publication to the *Transactions of the American Mathematical Society*.

‡ Burnside, Theory of Groups, 1897, p. 73.

§ We assume that G and G' are arranged in an $(\alpha, 1)$ isomorphism, the α invariant operators of G corresponding to identity of G'.

every group of the fourth class, of order p^5, where $p > 3$, that has an operator of order p^2 is generated by an operator of order p^2 and three of order p.* No such group can be a group of cogredient isomorphisms. Therefore:

A group of order p^m, where p is a prime > 3, that contains an operator of order p^{m-3} is of class k, where $k \gtreqless 4$.

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PROOF THAT THE GROUP OF AN IRREDUCIBLE LINEAR DIFFERENTIAL EQUATION IS TRANSITIVE.

BY DR. SAUL EPSTEEN.

(Read before the American Mathematical Society, December 28, 1901.)

WE will define with Frobenius † the linear differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 \tag{1}$$

to be irreducible when it has no integral in common with a differential equation of the same character, but of a lower order.

Picard has shown ‡ that for the equation (1) a linear homogeneous group

$$Y_i = \sum_k a_{ik} y_k \tag{2}$$

plays the same rôle as the group of substitutions plays in the Galois theory of algebraic equations.

It is the object of this brief paper to show that the group (2) of the equation (1) will be transitive § when the equation is irreducible and intransitive when the equation is reducible, and we shall carry the proof out on lines analogous to the corresponding theorem in algebra. ||

The basis of our proof is the Lagrange-Vessiot theorem, ¶ that if the rational differential function $S(y)$ of the in-

* Bagnera, *Ann. di Matematica*, ser. 3, vol. 1 (1898), p. 218.

† Frobenius, *Crelle*, vol. 76.

‡ Picard, *Comptes rendus*, 1883.

§ Lie-Engel, Transformationsgruppen, I., ch. 13.

|| Netto, Substitutionstheorie, § 154.

¶ E. Vessiot, *Annales de l'Ec. Norm. Sup.*, 1892.