other questions if he has once become interested in them, and a great merit of this book is that he will almost inevitably become interested in many of the subjects treated.

We have noticed the absence of only one important subject which would seem to belong in a treatment of this sort. We refer to the theory of the conformal transformation effected by the ratio of two solutions of a homogeneous linear differential equation of the second order (Schwarz's s-function). The volume before us does not, of course, touch those sides of the subject with which Lie's name is connected, nor is any special attention devoted to the theory of the real solutions of differential equations, but in those parts of the theory which he professes to treat the author has achieved more than ordinary success.

MAXIME BÔCHER.

Ιωάννου Ν. Χατζιδάκι — Εἰσαγωγὴ εἰς τὴν Ἀνωτέραν Ἀλγέβραν. Second edition. Athens, 1898.

THE work of Professor Hatzidakis is a scientific development by Weierstrass's methods of the general principles of arithmetic which form the basis of algebra, of calculus, and of the theory of functions. The natural numbers are defined by the process of counting and in the first chapter is placed the *formal* reasoning by which the associative and commutative laws are deduced from the special cases

$$\beta + \alpha = \alpha + \beta, \tag{1}$$

$$\alpha + (\beta + \gamma) = \alpha + \beta + \gamma. \tag{2}$$

The second chapter deals with fractional numbers, which are regarded as collections of fractional units, and the fraction $\frac{\alpha\beta}{\beta}$ is equal to the natural number α. Then any two numbers of the enlarged system are said to be equal if integral equimultiples of them are equal. Multiplication of fractional numbers is defined according to the distributive law, *i. e.*, 1.41×1.73 is the sum of the products obtained by multiplying every one of the units of the one factor by every unit of the other. The product of two units is defined in like manner so that equals multiplied by equals shall give equals. Thus, since we wish the two products $\frac{10}{10} \times \frac{100}{100}$ and 1×1 to be equal and since

$$\tfrac{10}{10} \times \tfrac{100}{100} = 1000\,(\tfrac{1}{10} \times \tfrac{1}{100}),$$

the product $\frac{1}{10} \times \frac{1}{100}$ is defined to be $\frac{1}{1000}$. From this definition it is easy to deduce the three fundamental identities

$$\beta\alpha = \alpha\beta, \qquad (4) \qquad \alpha(\beta\gamma) = \alpha\beta\gamma, \qquad (5)$$

$$(\alpha + \beta)\,(\gamma + \delta) = \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta. \qquad (6)$$

A feature worthy of imitation is a short chapter on "Zero as a number," where it is shown that when the fractional system is enlarged by the introduction of zero the laws of multiplication still hold with the exception of the law of equals multiplied by unequals and therefore that division by zero is impossible. After this the system of rational numbers is completed by defining the negative numbers (as assemblages of negative units) and establishing the five fundamental identities for the enlarged system.

In the system of rational numbers many problems admit no solution, *e. g.*, the equation $x^2 = 2$. The attempt to solve this by the algorithm for square root determines an unlimited succession of digits 1, c_1, c_2,... such that

$$\left(1 + \frac{c_1}{10} + \cdots + \frac{c_\nu}{10^\nu}\right)^2 < 2 < \left(1 + \frac{c_1}{10} + \cdots + \frac{c_\nu + 1}{10^\nu}\right)^2$$
$$(\nu = 1, 2, \cdots).$$

But such sequences also arise in dealing with certain problems which admit rational solutions, *e. g.*, the equation $3x = 4$. Here again the algorithm for division by the decimal scale gives an unlimited succession of digits. In each case we have an assemblage in which the number of units of each kind is definite, although the number of different units is unlimited, and in each case, however many units of the assemblage are taken, the aggregate is always less than 2. The repeating decimal is naturally regarded as defining a number. Further there exists a rational number $\frac{4}{3}$ such that whatever part M of the infinite decimal is taken, $\frac{4}{3}$ can be separated into two parts one of which is greater than M and, whatever part N is taken out of $\frac{4}{3}$, a part of the decimal can be found greater than N. Accordingly the number represented by 1.333... is said to be *equal* to $\frac{4}{3}$.

On the above considerations is based the following extension of the idea of number. Given any determinate assemblage (πλῆθος) of units such that, however many of them we take, the aggregate is always less than some integer, the *totality* (σύνολον) of the units of the assemblage shall be called a number. This is not of itself a satisfactory definition of an irrational number, but what follows can be easily used to supply the deficiency; for we find next the rules for comparison and combination of these new objects of thought. Two numbers are equal when and only when every *part* of each is contained in the other. The sum of two numbers is the totality of all their units. Their product is the totality obtained by multiplying every unit of the one by every unit of the other. From these definitions the associative, commutative, and distributive laws follow at once.

Thus the system of real numbers is completed, and the next chapter deals with the common complex numbers along similar lines. In fact the same words (τὸ σύνολον πολλῶν μονάδων) are used for the definition of every kind of number. A short chapter at the end of the book is devoted to proving that the system of common complex numbers is the only one in which all the ordinary laws of the four species hold, and more general systems are considered in two appendices.

The intervening chapters, however, contain matter of more vital interest in ordinary analysis. On the definition of irrational numbers in Chapter V can be based the treatment of limits and the convergence of infinite processes. In Chapter VII, "On limits," it is proved that if a positive variable constantly increases but always remains less than some natural number it approaches a limit and therefore that $\lim_{\nu=\infty}\left(1+\frac{1}{\nu}\right)^{\nu}$ exists. In Chapter VIII it is shown that e^x is a continuous function of x, and the exponential a^x when x is irrational is defined as $e^{x \log a}$. Chapter IX is an introduction to the theory of equations comprehending a proof of the fundamental proposition of algebra. Chapter X is devoted to determinants and the solution of simultaneous linear equations.

The above topics are treated in an elementary manner, but it is elementary work which breathes the spirit of the higher analysis, which appeals to the æsthetic faculties by the simplicity and uniformity of its methods no less than to the logical faculty by rigor and conciseness. The work exhibits throughout the clearness which is characteristic of Greek writing and proves the vitality and power of the language of Aristotle and Euclid.

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