group of isomorphisms of C is cyclic only when $a_0 = 0$ or 1 and just one of the other exponents differs from 0, or when $a_0 = 1$ or 2 and all the other exponents are 0.*

CORNELL UNIVERSITY,
February, 1901.

BESSEL FUNCTIONS.

Einleitung in die Theorie der Bessel'schen Funktionen. By PROFESSOR J. H. GRAF und DR. E. GUBLER. Zweites Heft: *Funktionen zweiter Art.* Bern, Wyss and Co., 1900.

THE first part of this work appeared in 1898 and was reviewed in the BULLETIN, February, 1899, pp. 253–8. The general arrangement of the second part is similar to that of the first, the authors again emphasizing the fact that the work is done in the spirit of Schläfli's lectures, the manuscripts of which were in their hands, though many problems are extended and modernized. This fact explains the absence of many important phases of the theory of the Bessel functions which one might expect in a symmetric treatise. Moreover, the authors have been rather overgenerous in their references to papers originating at Bern, omitting others which contained proofs of fundamental theorems prior to their discovery by the Bern school, although probably no plagiarism could be charged. Several fundamental theorems by American authors have received no recognition in the book.

Here, as in Volume I, the loop integral is the principal factor in the investigation, and next in importance is the expansion in series. The differential equation is less frequently used. The procedure is rather original, and frequently markedly different proofs for well-known theorems are given, which in some instances have led to detection of error in papers already published.

The only attempt at a concrete illustration or application is the expansion of a few functions in terms of Bessel functions, though the relations which exist between these functions and others are quite fully brought out.

The second part begins with the expansion of $\dfrac{1}{x-y}$ in terms of Bessel functions, the result being

* Gauss, Disquisitiones Arithmeticæ, 1801, Art. 92 .

$$\frac{1}{x-y} = J_0(y) \cdot \frac{1}{x} + \sum_{n=1}^{\infty} J_n(y) \int_0^{\frac{N}{x}} e^{-xs}\left(t^n - \frac{(-1)^n}{t^n}\right) ds.*$$

$$[s = \tfrac{1}{2}(t - t^{-1})], \quad (|x| > |y|), \quad (\lim N = \infty).$$

The part under the integral sign is denoted by $2\,O_n(x)$, and $O_n(x)$ is called the Bessel function of the second kind. This terminology is unusual, since the differential equation for $J_a(x)$ is not satisfied by $O_a(x)$, but in other respects $O_a(x)$ is quite analogous to $J_a(x)$. The authors suggest the analogy between the Bessel functions of the first and second kind on the one hand and spherical harmonics of the first and second kind, as defined by Neumann, on the other.

The symbol n is used to denote an integral parameter; the form of the infinite series for $O_n(x)$ is then derived, and the numerical coefficients calculated for $n = 1$ to $n = 11$. This method is then compared with that of Neumann for obtaining equation (1). $O_n(x)$, $J_n(x)$ are both shown to exist in a Laurent ring. Any continuous and differentiable function can be expanded in but one way in terms of Bessel functions.† The discussion of integrals of products of J, O closes the chapter.

In the following chapter the related function $S_n(x)$ is introduced:

$$S_n(x) = \int_1^{\frac{N}{x}} e^{-xs}\left(t^n - \frac{(-1)^n}{t^n}\right)\frac{dt}{t},$$

$$\left(O_n(x) = \frac{\cos^2 \frac{1}{2}n\pi}{x} + \frac{n}{2x} S_n(x)\right).$$

The expression $\cos^2 \frac{1}{2}n\pi$, $= 0$ or 1, $n = 1 \bmod 2$, $0 \bmod 2$ causes some confusion, both in this and later chapters, but the difficulty is easily removed by changing a limit in a summation. The numerical calculation of $S_n(x)$ is given for $n = 1$ to $n = 12$. $S_n(x)$ is always a polynomial in x^{-1}.

$$\frac{\cos^2 \frac{1}{2}\pi n}{x}$$

is eliminated and $S_n(x)$ expressed as a (finite) series in terms of $O_\lambda(x)$. The differential equation is found to be

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* The notation $\overset{a}{J}$ of the work reviewed is here replaced by J_a, etc.

† In my review of Part 1, I carelessly attributed this theorem to Schlömilch. This was simply an error of my own; the statement was not made in the book, but Schlömilch's name was used in a different connection.

$$x^2 \frac{d^2}{dx^2} S_n + x \frac{dS_n}{dx} + (x^2 - n^2) S_n = 2x \sin^2 \tfrac{1}{2} n\pi$$

$$+ 2n \cos^2 \tfrac{1}{2} n\pi = 2x \text{ or } 2n \text{ as } n \equiv 0,\ 1 \bmod 2\ ;$$

that for O is

$$\square\, O_n + O_n = x \cos^2 \tfrac{1}{2} n\pi + n \sin^2 \tfrac{1}{2} n\pi,$$

$\square\, J_n(x) = 0$ being the differential equation for $J_n(x)$.

The chapter closes with the integration of this differential equation, which results in expressing $S_n(x)$ and $O_n(x)$ in terms of $J_n(x)$, $K_n(x)$.

Chapter VIII introduces two new functions, $T_n(x)$, $U_n(x)$, defined by series, somewhat analogous to the partial summations of $J(x)$ and of $K(x)$. They are next expressed as integrals,

$$T_n(x) = \frac{2}{\pi i} \int_0^\pi \left(\varphi - \frac{\pi}{2} \right) e^{-i(x \sin \phi - n\phi)}\, d\varphi.$$

In deriving the differential equation for $T_n(x)$, $\cos^2 \frac{1}{2} n\pi$ intrudes again and the authors have chosen an infelicitous expression for removing it. In fact, taking the statement pp. 50–52 literally, an actual error would be made. This is about the only difficult part to follow in the text.

The function T is expressed in terms of J_λ and also as a definite integral—a similar discussion follows for U. K, J, T, S are shown to satisfy the relation

$$(-1)^n y_{-n} = y_n.$$

At the end of the chapter the relation between these functions and the y of Neumann, of Hankel, and of Weber are given, the first being expressed by the equation

$$y_n(x) = \log x . J_n(x) - \tfrac{1}{2} S_n(x) + \tfrac{1}{2} T_n(x) - U_n(x),$$

and the others are also given in detail. The functions S_n, T_n, U_n, are called Schläfli functions.

Chapter X deals with the addition theorem ; it is prefaced by a historical introduction which concisely gives the development of the problem. The method of proof is quite consequent : it consists in expressing $J_n(x + y)$ as a definite (loop) integral

$$J_n(x + y) = \frac{1}{2\pi i} \int e^{(x+y)s} t^{-n-1} dt, \qquad [s = \tfrac{1}{2}(t - t^{-1})],$$

$$e^{ys} = \sum_{\lambda = -\infty}^{\infty} J_\lambda(y) t^\lambda,$$

$$J_n(x+y) = \sum_{\lambda=-\infty} J\,(y) J_{n-\lambda}(x), \qquad (|y|<|x|)$$

By simple and natural transformations, the functions

$$J_n(x), \quad K_n(x), \quad O_n(x), \quad S_n(x), \quad T_n(x),$$

are shown to satisfy the same addition theorem, namely,

$$Z_n(x+y) = \sum_{\lambda=-\infty}^{\infty} Z_{n-\lambda}(x) J_\lambda(y)\,;$$

similar expressions are derived for the argument $x - y$, for parameter n and $-n$. The chapter concludes with a similar discussion of the product of two Bessel functions. To me this chapter appears very successful in its consistent method of procedure.

The next chapter deals with the expansion into a continued fraction of the ratio of two Bessel functions whose parameters differ by unity; it is preceded by a sketch of the historical development of the problem, a feature that is repeated in every subsequent chapter. Several pages are devoted to the expansion of particular functions.

The Schläfli function $P_m^{(a)}(x)$ is shown to be a polynomial of order m in x^{-1}. It is developed for values of m from -2 to 8 in terms of an arbitrary parameter a; then a recurring process gives the value of the function for other negative values of the integer m. The chapter closes with the application to some particular cases, $a = \frac{1}{2}$, $a = -\frac{1}{2}$, $\lim_{m=\infty} P_m^{(a)}(x)$.

Chapter XII treats of the classic problem of the relation of the Bessel function to the hypergeometric series. The treatment is quite original and direct, the method being somewhat independent of the older memoirs. The problem is to determine the value of the integral

$$S = \int_0^\infty J_a(x)\, e^{-bx} x^{c-1} dx\,;$$

the result is

$$S = \frac{\Gamma(a+c)}{2^a \Gamma(a+1) b^{a+c}} F\left(\frac{a+c}{2},\ \frac{a+c+1}{2},\ a+1,\ -\frac{1}{b^2}\right);$$

the conditions for convergence are carefully discussed.

An interesting case is discussed wherein $b = \pm i$; the region in which the function exists and the transformed path of the loop integral are well treated. A second proof is given, depending on a curvilinear integral; some new relations between F functions are incidentally found, among them being

$$F\left(\frac{\alpha+\gamma-1}{2},\ \frac{\alpha+\gamma}{2},\ \gamma,\ -\frac{4x(1-x)}{(1-2x)^2}\right)$$
$$=(1-2x)^{\alpha+\gamma-1}(1-x)^{1-\gamma}F(\alpha,\ 1-\alpha,\ \gamma,\ x).$$

This last equation is then developed directly by means of the definite integral; the path of integration is varied and the moduli of periodicity obtained by crossing the section are then discussed. Here again the special power of the curvilinear integral is exhibited, in the use of which the authors have shown considerable skill. The function considered is

$$T=\frac{t^2}{4(t-1)},$$

and the curve of section (Grenzscheide) is a limaçon. The chapter closes with a discussion of a few particular cases, including besides some well known results a few new ones regarding integrals of Bessel functions.

In the final chapter Weber's discontinuous integral

$$\int_0^\infty J_1(x)J_0(ax)dx$$

is discussed and generalized for any parameters; this is done by Dr. Gubler. The method employed is an application of the principles established in the preceding chapter, with the selection of an appropriate path of integration for each case.

A short appendix is added, in which still another form of the integral is obtained, and another section reduces the differential equation of the Bessel function of the first kind to a Riccati equation; and gives the form of the solution as a function of two independent $J(x)$ functions.

The bibliography which introduces each chapter, and the list of sources quoted which is appended to each part, while not always complete and not always giving the original source, still form a valuable part of the book. This work and the treatise of Gray and Mathews supplement each other, each being somewhat one sided when studied alone.

A number of typographical errors already found are corrected in a list inserted in Part II; but a good many more are still in the text, though few, if any, would prove a source of annoyance to the reader.

Any worker in Bessel functions will find the work a helpful text.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
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