A DEMONSTRATION OF THE IMPOSSIBILITY OF A TRIPLY ASYMPTOTIC SYSTEM OF SURFACES.

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Consider in space any system whatever of curvilinear coordinates, ρ_1, ρ_2, ρ_3 and let the Cartesian coördinates x, y, z of a point with respect to fixed rectangular axes be given in terms of the preceding by the equations

$$(1) \quad x = f(\rho_1, \rho_2, \rho_3), \quad y = \varphi(\rho_1, \rho_2, \rho_3), \quad z = \psi(\rho_1, \rho_2, \rho_3).$$

It is evident that the coefficients of the system of equations

$$(2) \quad \begin{aligned} \frac{\partial^2\theta}{\partial\rho_1^2} &= a_{11}\frac{\partial\theta}{\partial\rho_1} + a_{12}\frac{\partial\theta}{\partial\rho_2} + a_{13}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_2^2} &= a_{21}\frac{\partial\theta}{\partial\rho_1} + a_{22}\frac{\partial\theta}{\partial\rho_2} + a_{23}\frac{\partial\theta}{\partial\rho_3}, \\ \frac{\partial^2\theta}{\partial\rho_3^2} &= a_{31}\frac{\partial\theta}{\partial\rho_1} + a_{32}\frac{\partial\theta}{\partial\rho_2} + a_{33}\frac{\partial\theta}{\partial\rho_3} \end{aligned}$$

can be so determined that it will admit x, y, z, as particular simultaneous solutions.

Any point (x, y, z) may be looked upon as the intersection of three surfaces, one of each of the families for which ρ_1, ρ_2, ρ_3 are the respective parameters.

Consider a surface of each family and the point defined by them. From the form of equations (2)* it is evident that the necessary and sufficient condition that the surfaces $\rho_1 =$ const. and $\rho_2 =$ const. cut out upon the surface $\rho_3 =$ const. the asymptotic lines through the given point, is expressed by

$$a_{13} = a_{23} = 0.$$

Since similar results are obtained for the other two surfaces through the point, and in turn for every point and the surfaces determining it, it follows that the necessary and sufficient condition that the surfaces of a triple system cut one another along asymptotic lines is that the coefficients of the equations (1) satisfy the condition

* Bianchi, Lezioni, . 109.

$$(3) \qquad a_{13} = a_{23} = a_{12} = a_{32} = a_{21} = a_{31} = 0.$$

The square of the linear element on the surface $\rho_3 =$ const. through a given point may be written in the form

$$(4) \qquad ds_3^2 = E_3 d\rho_1^2 + 2F_3 d\rho_1 d\rho_2 + G_3 d\rho_2^2,$$

where

$$E_3 = \Sigma\left(\frac{\partial x}{\partial \rho_1}\right)^2, \quad \Sigma F_3 = \frac{\partial x}{\partial \rho_1}\frac{\partial x}{\partial \rho_2}, \quad G_3 = \Sigma\left(\frac{\partial x}{\partial \rho_3}\right)^2,$$

and where, by hypothesis, the parametric lines are the asymptotic lines.

Denoting by $\left\{ {rs \atop t} \right\}_3$ the Christoffel symbols formed with respect to (4) we have

$$a_{12} = \left\{ {11 \atop 2} \right\}_3, \quad a_{21} = \left\{ {22 \atop 1} \right\}_3,$$

Hence, by (3),

$$\left\{ {11 \atop 2} \right\}_3 = \left\{ {22 \atop 1} \right\}_3 = 0,$$

that is,* the ρ_1 and ρ_2 lines are geodesics; but this is impossible, since they are asymptotic lines. Since similar results follow for surfaces $\rho_1 =$ const. and $\rho_2 =$ const., we have the theorem

There is no triple system of surfaces cutting mutually in the asymptotic lines of these surfaces.

This theorem leads to several interesting negations.

Consider a triple system of surfaces cutting one another in a system of conjugate lines on each. Suppose now that we have another triply conjugate system of surfaces, which are such that to each point determined by three surfaces of the first system there corresponds a point determined by three surfaces of the second system and in such a way that the respective surfaces defining the first point correspond by orthogonality of linear elements to those defining the second point. From the theory of infinitesimal deformation of surfaces we know that if (x, y, z) and (x_1, y_1, z_1) denote corresponding points in these two systems, then the triple system of associate † surfaces of the first system are defined by the equations ‡

$$d\xi = z_1 dy - y_1 dz, \quad d\eta = x_1 dz - z_1 dx, \quad d\zeta = y_1 dx - x_1 dy,$$

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* Bianchi, Lezione, p. 146.

† Ibid., p. 279.

‡ Genty, *Ann. de la Fac. des Sci. de Toulouse*, vol. 9, no. 1.

where (ξ, η, ζ) are the coördinates of the point corresponding to (x, y, z). Consider a set of these corresponding surfaces, one from each system; the conjugate system on the first, whose correspondent on the second is a conjugate system, corresponds to the double system of asymptotic lines on its associate.* Hence the triple system of associate surfaces is composed of surfaces cutting mutually in their asymptotic lines. But such a system is impossible; hence

No two triply conjugate systems of surfaces can correspond by orthogonality of linear elements.

Since, in the infinitesimal deformation of a surface that conjugate system of lines is preserved which has a conjugate system for its correspondent on a surface corresponding by orthogonality of linear elements, it follows from the above result that

A triply conjugate system of surfaces cannot be deformed infinitesimally into another triply conjugate system.

Suppose there is given a triple system of minimal surfaces cutting one another along the conjugate lines of length zero for each surface. Corresponding to the three surfaces determining a point we have three other surfaces—the *adjoints* of the former—determining the corresponding point in the system of adjoint surfaces. Moreover, it is evident that the three surfaces of this latter system cut one another in the conjugate lines of length zero. Since a minimal surface and its adjoint correspond by orthogonality of linear elements, this second triply conjugate system of mimimal surfaces would have this kind of correspondence with the first system. Since this is impossible, we have

There cannot be a triply conjugate system of minimal surfaces cutting one another in the conjugate lines of length zero.

Again suppose that there can be obtained a triply orthogonal system of minimal surfaces. By the Dupin-Darboux theorem we have that these surfaces intersect one another in lines of curvature. In connection with this system consider the triple system of adjoint minimal surfaces of the given system. Since the lines of curvature on a minimal surface correspond to the asymptotic lines on its adjoint, the surfaces of the second system will cut one another in asymptotic lines. Hence the hypothesis was incorrect, and therefore

There cannot be a triply orthogonal system of minimal surfaces.

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* Bianchi, Lezioni, p. 279.