$$
\begin{array}{cccc}
S & T & R & W \\
\begin{pmatrix} 1&0&0&1&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&1&1&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 1&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} &
\begin{pmatrix} 0&1&1&1&1&1\\ 1&0&0&1&0&1\\ 0&1&0&1&1&0\\ 0&0&0&1&0&1\\ 1&0&0&0&0&1\\ 0&0&1&1&1&1 \end{pmatrix} &
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\end{array}
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\begin{array}{cccc}
E_1' & E_2' & E_3' & E_4' \\
\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&1&1&0&1\\ 1&0&0&1&0&1\\ 1&0&1&0&0&1\\ 0&0&0&0&0&1\\ 1&0&1&1&1&1 \end{pmatrix} &
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E_5' & E_6' \\
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CHICAGO,
*August*, 1899.

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# LOBACHEVSKY'S GEOMETRY.

(SECOND PAPER.)

LOBACHEVSKY's earliest published work on geometry, translated by Engel under the title "Ueber die Anfangsgründe der Geometrie," * contains the elements of a system of analytic geometry under the hypothesis of a variable angle of parallelism, together with numerous applications to the determination of the lengths of arcs, areas, and volumes. Some of this matter appears also in Lobachevsky's article on "Géométrie imaginaire" (Crelle, vol. 17) and more in his "Pangéométrie" (Kasan, 1856); but it is probably safe to say that the knowledge of this part of his work is not so general as that of the more elementary side of his theory, partly because of the difficulties involved in reading the last mentioned articles, and partly because of the fact that the widely known "Geometrische Untersuchungen" does not

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* See the BULLETIN for May, 1900, p. 339.

contain this analytic treatment. It is not the least of Professor Engel's services in publishing his volume on Lobachevsky that he accompanies this part of the work with lucid notes, more copious than the text itself, which smooth away many difficulties in the path of the reader. It is the purpose of the present article to present a few of the chief points in this treatment, and in particular to emphasize the analytic connection pointed out by Engel between Lobachevsky's equations and those of the projective measurement of Cayley.

Lobachevsky's angle of parallelism, $\Pi(x)$, where $x$ is the perpendicular distance from a point on one parallel to the other, is analytically defined by the equations

$$\begin{gathered} \tan \tfrac{1}{2}\Pi(x) = e^{-x}, \\ \Pi(0) = \frac{\pi}{2}, \ \ \Pi(+\infty) = 0, \ \ \Pi(-x) = \pi - \Pi(x). \end{gathered} \tag{1}$$

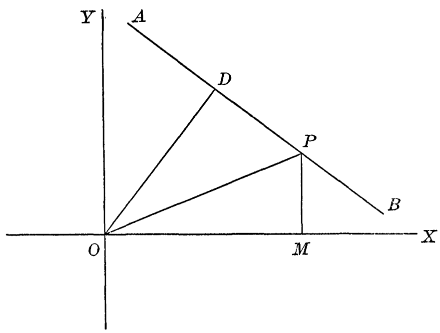
From these the functions $\sin \Pi(x)$, $\cos \Pi(x)$, $\tan \Pi(x)$ are easily written in terms of the exponential or the hyperbolic functions; but we will retain Lobachevsky's notation, the strangeness of which disappears very quickly in practice.* Angles and distances are measured as usual by the repeated application of a unit of measure to the thing measured, but the two units of angle and of length can not be assumed independently of each other. If we adopt the usual unit of angle such that the entire angular magnitude about a point in a plane is equal to $2\pi$, then the unit of length is fully defined by the above equations, if the angle of parallelism for any value of $x$ is assumed or experimentally determined. In the space in which we live, we can only know that the unit of length is extremely great compared with the greatest length which enters into experience.

In a right triangle with sides $a$, $b$, $c$, and with angles $A$, $B$, $\frac{\pi}{2}$, the following relations hold :

$$\begin{array}{ll} \sin \Pi(c) = \sin \Pi(a) \sin \Pi(b), & \sin \Pi(c) = \tan A \tan B, \\ \tan \Pi(c) = \tan \Pi(a) \sin A, & \tan \Pi(c) = \tan \Pi(b) \sin B, \\ \cos \Pi(a) = \cos \Pi(c) \cos B, & \cos \Pi(b) = \cos \Pi(c) \cos A, \\ \sin A = \sin \Pi(b) \cos B, & \sin B = \sin \Pi(a) \cos A, \\ \tan A = \cos \Pi(a) \tan (b), & \tan B = \cos \Pi(b) \tan \Pi(a). \end{array} \tag{2}$$

*The interesting little book by J. Frischauf, "Absolute Geometrie nach Johann Bolyai," Leipzig, 1872, contains some of the equations of this article in the exponential notation.

Let $OX$ and $OY$ be two coördinate axis at right angles to each other. From any point $P$ of the plane drop a perpendicular to $OX$ meeting it at $M$. Then the distances $OM\,(x)$ and $MP(y)$ are the coördinates of $P$, the usual conventions as to signs being maintained. It is essential that the coördinates be always drawn as just stated; a perpendicular




from $P$ to $OY$ will not be equal to $x$, for no rectangle exists on Lobachevsky's plane. If $OP$, the distance of the point from the origin, be denoted by $r$, and the angle $XOP$ be denoted by $\vartheta$, equations (2) give

$$\begin{aligned} \sin \Pi(r) &= \sin \Pi(x) \sin \Pi(y), \\ \sin \vartheta &= \frac{\tan \Pi(r)}{\tan \Pi(y)}, \\ \cos \vartheta &= \frac{\cos \Pi(x)}{\cos \Pi(r)}. \end{aligned} \tag{3}$$

If $r$ is constant, while $x$ and $y$ vary, the first of these equations represents a circle with the center at the origin.

Let $AB$ be any straight line, $p$ the length of the perpendicular $OD$ from the origin, $a$ the angle $XOD$. Then, in the right triangle $ODP$,

$$\cos (a - \theta) \cos \Pi(r) = \cos \Pi(p),$$

whence, by (3),

$$\cos a \cos \Pi(x) + \sin a \sin \Pi(x) \cos \Pi(y) = \cos \Pi(p). \tag{4}$$

This equation, in which $p$ may have any value from 0 to $+\infty$, and $a$ any value from 0 to $2\pi$, is Engel's form of the equation of the straight line. We may obtain the form given by Lobachevsky by assuming that $AB$ cuts $OY$ at a point $(0, l)$ under an angle $A$. Then in the right triangle $ODA$,

$$\cos \Pi(p) = \cos \Pi(l) \cos\left(\frac{\pi}{2} - \alpha\right),$$

$$\tan\left(\frac{\pi}{2} - \alpha\right) = \sin \Pi(l) \cot A,$$

whence by substitution in (4),

$$\cos \Pi(y) = \frac{\cos \Pi(l)}{\sin \Pi(x)} - \sin \Pi(l) \cot A \cot \Pi(x). \qquad (5)$$

This is called by Lobachevsky the general equation of the straight line. It is so in the sense that the equation of any straight line may be given this form by a proper choice of axes. If the axes are fixed, however, the equation fails for lines parallel to $OY$ or divergent from it. The case of parallelism may be included by taking the limiting form of (5) as $l$ increases indefinitely, and the case of divergence may be met by introducing the conceptions of imaginary lengths or angles, which we can only do after we have an analytic expression for each. Engel's equation has no such defect, and we will base our work upon it.

It is worth noticing that $x = c$ represents a straight line, while $y = c$ does not; the former being obtained from (4) by placing $\alpha = 0$, $p = c$, while the latter is not contained in (4). Both of these facts are geometrically evident.

The derivation of expressions for distance and angle is facilitated by first studying the distance $DP$ and the angle $OPD$ of the figure. If we place $DP$ equal to $d$, we have, by (2) and (3),

$$\sin \Pi(d) = \frac{\sin \Pi(r)}{\sin \Pi(p)} = \frac{\sin \Pi(x) \sin \Pi(y)}{\sin \Pi(p)},$$

whence

$$\cos^2 \Pi(d) = \frac{\sin^2 \Pi(p) - \sin^2 \Pi(x) \sin^2 \Pi(y)}{\sin^2 \Pi(p)}.$$

The latter equation gives, by use of (4),

$$\cos \Pi(d) = \pm \frac{\sin \alpha \cos \Pi(x) - \cos \alpha \sin \Pi(x) \cos \Pi(y)}{\sin \Pi(p)},$$

where the sign is determined by assuming a positive direction on $DP$.

Similarly, if the angle $OPD$ is called $\beta$,

$$\sin \beta = \frac{\sin \Pi(x) \sin \Pi(y) \cos \Pi(p)}{\cos \Pi(r) \sin \Pi(p)},$$

$$\cos \beta = \pm \frac{\sin \alpha \cos \Pi(x) - \cos \alpha \sin \Pi(x) \cos \Pi(y)}{\cos \Pi(r) \sin \Pi(p)},$$

the sign of $\cos \beta$ corresponding to that of $\cos \Pi(d)$.

Passing to the general case, suppose two lines $(\alpha_1, p_1)$ and $(\alpha_2, p_2)$ intersect at $P$. The angle between them is the algebraic difference of the angles $\beta_1$ and $\beta_2$ made by each with $OP$. Placing $\varphi = \beta_1 - \beta_2$, we find

$$\cos \varphi = \frac{\cos (\alpha_1 - \alpha_2) - \cos \Pi(p_1) \cos \Pi(p_2)}{\sin \Pi(p_1) \sin \Pi(p_2)}, \tag{6}$$

the simplification being effected by means of (3) and (4). This equation affords a simple analytic criterion for the relative position of two lines, they being intersecting, parallel, or divergent according as $\cos \varphi$ is numerically less than, equal to, or greater than unity.

In a similar manner, the distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is the algebraic difference of the two distances $d_1$ and $d_2$ measured as above along the line joining the two points. If we call $d_1 - d_2 = D$, and make use of Lobachevsky's formula,

$$\sin \Pi (d_1 - d_2) = \frac{\sin \Pi(d_1) \sin \Pi(d_2)}{1 - \cos \Pi(d_1) \cos \Pi(d_2)},$$

we find

$$\sin \Pi(D) = \frac{\sin \Pi(x_1) \sin \Pi(x_2) \sin \Pi(y_1) \sin \Pi(y_2)}{1 - \cos \Pi(x_1) \cos \Pi(x_2) - \sin \Pi(x_1) \sin \Pi(x_2) \cos \Pi(y_1) \cos \Pi(y_2)}. \tag{7}$$

Neither (6) nor (7) is given by Lobachevsky, but his equation for the length of a line measured from its intercept on $OY$ and expressed in the parameters of equation (5) may be derived from (7).

Engel calls attention to the fact that equation (4) is linear in two functions of $x$ and $y$, and that a simple substitution leads to a system of projective measurement as developed by Cayley. Following this line of thought, let us place, with Engel,

$$\begin{aligned}\xi &= \cos \Pi(x), \\ \eta &= \sin \Pi(x) \cos \Pi(y),\end{aligned} \tag{8}$$

thus reducing (4) to

$$\xi \cos \alpha + \eta \sin \alpha = \cos \Pi\,(p).$$

Now from equations (3), together with (1), if a point is at a finite distance $r$ from the origin,

$$0 < \sin\ \Pi(x)\ \sin\ \Pi(y) < 1,$$

while as $r$ increases indefinitely, $\sin \Pi(x)\ \sin \Pi(y)$ approaches zero as a limit. Therefore, since

$$\sin\ \Pi(x)\ \sin\ \Pi(y) = \sqrt{1 - \xi^2 - \eta^2},$$

if $(\xi, \eta)$ be interpreted as Cartesian coördinates on a plane, the entire plane of Lobachevsky corresponds to the interior of the conic

$$\xi^2 + \eta^2 - 1 = 0$$

and the infinitely distant portion of the plane corresponds to the conic itself. We will take this conic as the fundamental conic of a system of projective measurement, so choosing the arbitrary constants involved that, if $M$ is the distance between two points $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$, and if $\psi$ is the angle between two lines $(u_1, v_1, w_1)$ and $(u_2, v_2, w_2)$,*

$$\cos iM = \frac{1 - \xi_1 \xi_2 - \eta_1 \eta_2}{\sqrt{(1 - \xi_1^2 - \eta_1^2)\,(1 - \xi_2^2 - \eta_2^2)}}\,, \tag{9}$$

$$\cos \psi = \frac{u_1 u_2 + v_1 v_2 - w_1 w_2}{\sqrt{(u_1^2 + v_1^2 - w_1^2)\,(u_2^2 + v_2^2 - w_2^2)}}\,. \tag{10}$$

Comparing these with $D$ and $\varphi$, computed above for the corresponding points and lines, we have $D = M$, $\psi = \varphi$. The latter assertion follows at once from (6), by placing $\cos \alpha = u$, $\sin \alpha = v$, $- \cos \Pi(p) = w$. To establish the former, we substitute from (8) in (7), obtaining

$$\sin \Pi(D) = \frac{\sqrt{(1 - \xi_1^2 - \eta_1^2)(1 - \xi_2^2 - \eta_2^2)}}{1 - \xi_1\xi_2 - \eta_1\eta_2} = \frac{1}{\cos iM}\,,$$

* For these and other references to projective geomety, see Klein's Autographirte Vorlesungen über Nicht-Euklidische Geometrie, vol. 1. (Göttingen, 1892).

or
$$\frac{2}{e^D + e^{-D}} = \frac{2}{e^M + e^{-M}},$$

of which the only solution in positive real qualities is $D = M$.

It was first shown by Klein and is now well known that the projective geometry in which a real conic serves as the basis of measurement has all the characteristic features of Lobachevsky's geometry. The equations (8) establish a direct analytic relation between the two geometries, since they image the entire Lobachevsky plane upon the interior of the $(\xi, \eta)$-plane with preservation both of angle and of distance, with the understanding that these magnitudes are to be measured in the proper way on each plane. To any real point $(\xi, \eta)$ lying within the fundamental conic, corresponds one and only one real point $(x, y)$ and conversely. In order to extend this correspondence to imaginaries without destroying the one to one relation, we may agree to consider on the $(x, y)$-plane only complex quantities of the type $a + bi$, where $0 \leqq b < 2\pi$. With this convention any two straight lines intersect in one and only one point, since two equations of type (4) have always an analytic solution. Lobachevsky's distinction between intersecting, parallel, and divergent lines appears accordingly as one in real finite variables only. Divergent lines meet in an imaginary point, corresponding to a real point $(\xi, \eta)$ outside of the conic.

The use of the $(\xi, \eta)$-plane offers a two-fold advantage: first, it often makes the peculiar properties of Lobachevsky's geometry directly visible; and secondly, it simplifies the analytic treatment. As an example of the first use, consider Lobachevsky's theorem that two straight lines either intersect, are parallel, or have a common perpendicular. On the $(\xi, \eta)$-plane two lines are always perpendicular to the polar of their point of intersection, and this polar appears in the interior of the conic only when the two lines intersect on the exterior; so that, Lobachevsky, working in reals, could only recognize the common perpendicular for divergent lines. In the same connection, the impossibility of the existence of a rectangle is evident from the properties of poles and polars.

The nature of the circle is also easily studied on the $(\xi,\eta)$-plane. If, in equation (7), we hold $D$ constant and take $(x_1, y_1)$ as a fixed point, while replacing $(x_2, y_2)$ by the variable point $(x, y)$, we have the equation of the circle. The corresponding equation (9) is

$$k(1-\xi_1\xi_2-\eta_1\eta_2)^2 = 1-\xi^2-\eta^2,$$

which is the equation of a conic tangent to the fundamental conic at the points where it is cut by the polar of $(\xi_1, \eta_1)$.

There are then three cases, according as $(\xi_1, \eta_1)$ is within, on, or outside the conic. These correspond on Lobachevsky's plane respectively to an ordinary circle with real center and finite radius, to a limit circle approached by any circle as the radius is indefinitely increased and the center becomes indefinitely remote, and to a curve all points of which are equidistant from a real straight line. The first two kinds are given by Lobachevsky, but it is doubtful if he knew that the last was also a circle with imaginary center and radius.

An expression for the element of arc may be obtained readily from (9) by placing $\xi_1 = \xi$, $\eta_1 = \eta$, $\xi_2 = \xi + d\xi$, and $\eta_2 = \eta + d\eta$ and neglecting infinitesimals of higher order. There results

$$ds = \frac{\sqrt{d\xi^2 + d\eta^2 - (\eta d\xi - \xi d\eta)^2}}{1-\xi^2-\eta^2}.$$

By means of (8) and the formula of differentiation

$$d\Pi(x) = -\sin \Pi(x) dx,$$

this becomes
$$ds = \sqrt{dy^2 + \frac{dx^2}{\sin^2 \Pi(x)}},$$

a result which may be obtained with more difficulty from (7) and is derived by Lobachevsky directly from an infinitesimal triangle. Either of these expressions applied to the circle

$$\sin \Pi(x) \sin \Pi(y) = \sin \Pi(r)$$

or
$$1-\xi^2-\eta^2 = \sin^2 \Pi(r)$$

gives as the entire circumference

$$C = 2\pi \cot \Pi(r) = \pi(e^r - e^{-r}).$$

Similarly the expression given by Klein for an area on the $(\xi, \eta)$-plane

$$S = \int\int \frac{d\xi d\eta}{(1-\xi^2-\eta^2)^{3/2}}$$

becomes by change of variables

$$S = \int\int \frac{dxdy}{\sin \Pi(y)},$$

and either of these applied to the circle gives as the entire area

$$A = \pi(e^{\frac{r}{2}} - e^{-\frac{r}{2}})^2.$$

The integration of the last integral with respect to $y$ gives as the area $dS$ of a narrow strip on the plane

$$dS = \cot \Pi(y). \ dx.$$

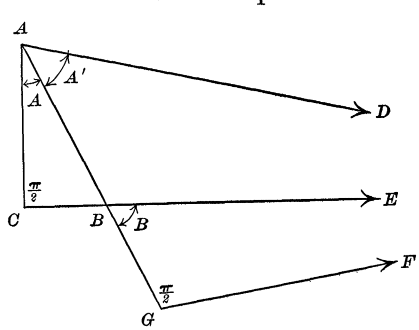
This expression is found by Lobachevsky as a result of the theorem that the area of a triangle is equal to $\pi$ minus the sum of its angles. Conversely the latter theorem may be derived by integration as shown in the "Pangéométrie." To see this, consider first an infinite strip of the plane bounded by the positive part of $OX$, a portion of $OY$, and a line parallel to $OX$ cutting $OY$ at a distance $a$ from $O$. The equation of this parallel, found by placing $l = a$, and $A = \Pi(a)$ in (5), is

$$\cos \Pi(y) = \cos \Pi(a). \ e^{-x}.$$

Hence

$$S = \int_0^\infty \cot \Pi(y) \ dx = -\int_a^0 \sin \Pi(y) \ dy = \frac{\pi}{2} - \Pi(a) = \frac{\pi}{2} - A.$$

Consider now a triangle right-angled at $C$, and draw through $A$ a line parallel to $CB$. Prolong $AB$ to $G$, so that $GF$ perpendicular to $AB$ shall be parallel to $CB$.




The triangle $ABC$ equals the sum of the infinite strips $DACE$ and $EBFG$ diminished by $DAGF$. Substituting the areas of the strips, as found above,

$$ABC = \frac{\pi}{2} - (A + A') + \frac{\pi}{2} - B - \left(\frac{\pi}{2} - A'\right)$$
$$= \frac{\pi}{2} - A - B$$
$$= \pi - (A + B + C).$$

The theorem is thus proved for a right triangle, and is readily extended to an oblique triangle by dividing it into two right triangles by a perpendicular from any vertex.

In the foregoing pages no attempt is made to give an exhaustive statement of Lobachevsky's methods and results on the plane nor to indicate his extension of his methods to space.

FREDERICK S. WOODS.

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY, *May*, 1900.

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## BURKHARDT'S ELLIPTIC FUNCTIONS.

*Functionentheoretische Vorlesungen.* Von HEINRICH BURKHARDT. Zweiter Theil: *Elliptische Functionen.* Leipzig, Veit and Company, 1899. 8vo., x + 373 pp.

THE theory of elliptic functions has developed so rapidly and in so many different directions in recent years that an elementary treatise of moderate compass which would afford a rapid survey of its many and heterogeneous parts has been a long felt want. The admirable little treatise by Appell and Lacour is perfect in its way, but it addresses itself only to students who do not care to go very far into the theory of functions. It makes no pretentions to satisfy the needs of another large class of students, namely those who regard the theory of elliptic functions as merely one division of a greater theory and who thus study the elliptic functions not only on account of the interesting properties they offer *per se*, but also as a means of becoming more familiar with the principles and methods of the theory of functions, or as a stepping stone to the more abstruse theories of the abelian transcendents and automorphic functions.

The present volume meets the wants of this latter class most successfully. We are so impressed with its many merits that we do not hesitate to predict for it a rapid and widespread popularity.