proof but had not yet anything to take its place, until in 1826 he had found his way successfully through all his difficulties. He adds his own testimony as to the origin of his great theories in the opening sentences of the "Neue Anfangsgründe" in which he declares that the futility of the efforts made during two thousand years since Euclid to complete the theory of parallel lines aroused in him the suspicion that the ideas sought to be proved were not necessarily true. While it is remarkable that the solution of a two-thousand year old problem should be given almost simultaneously by three men, it should be remembered that these three were not the only mathematicians who had worked upon the problem. More than one had missed the solution by a hairsbreadth only; Lobachevsky, Bolyai, and Gauss succeeded in finding it.

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VOGT'S ALGEBRAIC SOLUTION OF EQUATIONS.

Leçons sur la Résolution algébrique des Équations. Par H. VOGT, professeur adjoint à la Faculté des Sciences de Nancy. Paris, Nony et Cie., 1895. 8vo., viii+201 pp.

THE present work is, we suppose, intended to be an introduction to the *modern* theory of the algebraic solution of equations. It is true that the word *modern* does not appear in the title, but however elementary the character of a *new* book of this kind may be, it is natural to suppose that the author will present his material in accordance with modern points of view, as far as these are elementary and simple.

This, however, is not the case with the volume in hand, as we proceed to show. First and foremost we have the following serious criticism to make. The rockbed of the modern theory of the algebraic solution of equations is the principles of Galois. A text book on this subject which does not explain these with all detail and use them systematically from start to finish cannot be called modern.

That the present volume sins grievously in this respect can be shown at once. Galois' theory proposes a perfectly simple and uniform scheme for the solution of any given equation. In a work of this kind this scheme should be developed at the start and then undeviatingly employed throughout the work whenever the solution of a particular class of equations is being effected. Almost the first step in this scheme relates to the adjunction of a rational function of the roots of the given equation and the determination of the ensuing Galoisian group. Is it possible to fancy a modern treatment of the subject this book deals with, which takes up this problem, as fundamental as it is simple, just ten pages before its close! Such a fact means simply that the author throws Galois' theory to the winds.

The reason for this is nowhere explicitly given. It cannot be that Galois' principles are not fruitful; for they occupy a central position in the great field of modern mathematical speculation. It cannot be because they are too difficult to be treated in an elementary text book; for as we shall see the author treats questions much more abstruse than the elementary Galoisian theory offers. It cannot be that the problems it enables us to solve are uninteresting; for every one familiar with Galois' theory knows the contrary. It may be that M. Vogt thinks Galois' methods are not all rigorous. Chapter IX. at least lends color to this supposition. This chapter treats of the algebraic solution of equations and culminates in the theorem of Ruffini and Abel that the general equation of degree greater than four cannot be solved algebraically. The author frankly states that it is taken *en grande partie* from Chapter XIII. of Netto's book on Substitutionentheorie und ihre Anwendungen auf die Algebra. Now Netto in this chapter takes a stand which is either trivial or which is an impugnment of the correctness of those parts of Galois' theory which treat of problems of this character. As the question is altogether fundamental and as put by Netto may easily lead one astray, it is worth spending a few words on it. The question at issue is this: Does Galois' theory enable us to draw into the circle of our reasoning irrational resolvents, *i. e.*, resolvents whose roots are not rational functions of the roots of the given equation, or does it not? The statement of Netto's in this connection is this:* since the theory of substitutions treats only of *rational* functions of the roots it is impossible to employ this theory when dealing with irrational functions. To use the theory of substitutions, then, in a problem which requires the consideration of irrational resolvents would be a *petitio principii.* Such problems he declares can be settled only by algebraic reasoning. Strictly speaking this is doubtless quite correct, but taken in the strict sense these remarks are trivial. If taken in the sense that the

* L. c., p. 235.

demonstration, for example, of the Ruffini-Abel theorem by the Galoisian theory is unsound, and that it must be replaced by the laborious algebraical reasoning which Netto employs* and in which he is followed by Vogt, Netto's position is incorrect and pernicious. That irrational resolvents can be employed directly in Galois' theory follows from the simple fact that, whenever a reduction of the Galoisian group takes place on adjoining an accessory irrationality, the same effect can be produced by a rational function of the roots. The theorem which is fundamental in these questions, and which cannot be insisted on too much, is due to Jordan and given in his Traité, p. 269. The demonstration of this theorem is entirely substitution-theoretical. How does this agree with Netto's declaration: "treten daher * * * irrationale Funktionen der Wurzeln auf, so befinden wir uns auf einem Gebiet, in dem von Substitutionen überhaupt keine Rede mehr sein kann." It would certainly enlighten many if Professor Netto would explain how these remarks are to be put in harmony with §§ 230, 231 of his book, particularly with the statement made at the close of § 230.

Let us look now at the selection of material M. Vogt has to offer the reader. In a work that treats Galois' theory so shabbily we are not surprised to find only some twenty pages devoted to the theory of substitution groups. This is certainly unfortunate. The theory of groups is every day winning in importance. One has only to think of the rôle they play in geometry and in the theory of differential equations. The Galoisian theory of equations offers a splendid opportunity to introduce the student to an important part of this great theory, namely, the theory of *finite* groups. At the very start of the Galoisian theory, substitution groups demand our attention. The notions of transitivity, primitivity, invariant subgroups, series of composition, and isomorphism present themselves simply and naturally at once. A little later when we begin to consider more carefully the groups of the resolvents we find it necessary to pass from the narrow notion of a substitution group to the broad and fertile notion of a group in the abstract. A little later still we come to the groups of the regular bodies, *i. e.*, to particular cases of the all important linear group. It is with regret that we see such an opportunity entirely ignored.

Instead then of a treatment of such questions, we find a wearisome reproduction of some of Kronecker's algebraical

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* It is Kronecker's modification of Abel's proof. Cf. *Monatsber. d. Berl. Akad.*, 1879, p. 205.

theories. A chapter is devoted to cyclic and metacylic functions of n independent variables; another chapter presents Kronecker's celebrated method of decomposing a form in n variables for an arbitrary domain of rationality. A third chapter gives Kronecker's treatment of abelian equations; and a fourth, Kronecker's researches on metacylic equations of prime degree. We do not wish to be understood as underestimating Kronecker's methods; on the contrary, we are an ardent admirer of them. But the methods of Kronecker here given form but a small part of the equipment of this mathematical Hercules. To try to give what is necessary would utterly crush the reader; to give no more than M. Vogt has given seems to us wholly inadequate. With half the space, all the results of importance the author has given could be demonstrated by Galois' theory and with far greater ease to the reader.

I pass now to a few criticisms of detail. The systematic employment of indeterminates should certainly be preceded by a few words of explanation. The author does not bring out with sufficient emphasis the meaning of the very fundamental terms: valeur numérique, valeurs numériquement distinctes, algébriquement distinctes. The author seems to be influenced by Kronecker's dictum which forbids the use of purely logical definitions. At least in two important instances he has followed it, viz., in the decomposition of a form into irreducible factors, and in the actual determination of the Galoisian group for a given equation. A third equally important case he has not treated, viz., the problem of determining whether a given rational function of the roots belongs to a given group or not.

On p. 62 we are given a definition of a general equation, and the remark is made that equations whose coefficients are integers are special. On p. 80 special equations are said to be those whose Galoisian group is not the symmetric group. There is a confusion of terms here. As Hilbert showed for the first time there are an infinity of equations whose group is the symmetric group and whose coefficients are integers.*

On p. 146 the rule for forming cyclic functions is not universally applicable, as simple examples show.

Consider the Abelian equation for $R(1)$, $x^4 + 1 = 0$. Let $\varepsilon = e^{2\pi i/8}$; set $x_0 = \varepsilon$, $x_1 = \varepsilon^3$, $x_2 = \varepsilon^5$, $x_3 = \varepsilon^7$; also let $x_1 = \vartheta_1 x_0$, $x_2 = \vartheta_2 x$. Then Kronecker's scheme for the roots becomes

$$\vartheta_1^{h_1} \vartheta^{h_2} x_0 \qquad (h_1, h_2 = 0, 1).$$

* Crelle, vol. 110.

The functions y_α become here

$$y_0 = \sum_{h_2} \vartheta_2^{h_2} x_0 = x_0 + x_2; \quad y_1 = \sum_{h_1} \vartheta_1^{h_1} x_0 = x_1 + x_3.$$

Now both y_0 and y_1 are zero; they are thus not cyclic, and the rule breaks down.

Finally we observe that the treatment in Chapters X. and XII. of Kronecker's problem, of finding the necessary and sufficient form of the roots of all algebraically solvable equations of prime degree n, is far too condensed for so abtruse a matter. It is also lacking in rigor in two essential points. The question whether the functions

$$\psi_\mu = \sum_{r=0}^{n-1} \omega^{\mu r} x_r \qquad (\omega^n = 1)$$

vanish or whether the functions

$$y_q = \psi_{g^q} \psi_{g^{q-1}}^{-q} \qquad (q = 1, 2, \cdots, n-1)$$

are distinct is not discussed.

Before closing we beg to have it clearly understood that our criticisms have been made on the supposition that the volume in hand is to serve as an introduction to the modern theory of the algebraic solution of equations. To one who is already familiar with the elements of this theory, the present work will give much interesting and valuable information, particularly in regard to the methods peculiar to Kronecker. It may then serve in some measure as a preparation toward studying the papers of this great master.

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YALE UNIVERSITY,
March, 1900.

ELEMENTS OF THE CALCULUS.

The Elements of the Differential and Integral Calculus, based on the Kurzgefasstes Lehrbuch der Differential- und Integralrechnung, von W. Nernst und A. Schönflies. By J. W. A. YOUNG and C. E. LINEBARGER. New York, D. Appleton and Co., 1900. 8vo., xvii + 410 pp.

OF the various new text-books on the calculus, this recent joint publication by a teacher of mathematics and a teacher of physics and chemistry will doubtless attract much interest, based as it is upon the German work, intended primarily for chemists, which appeared in 1896