were read, and one by Rudel (Nürnberg) on "Die neue bayrische Prüfungsordnung für das Lehramtsexamen der Lehrer für Mathematik und Physik." This was followed by a long discussion on the questions involved in it and in the previous papers of Weber and Hauck.

Before closing, let me add that the mathematical papers mentioned here, together with many others, will appear ere long in the eighth volume of the *Jahresbericht* of the Vereinigung. The few remarks I have made will indicate sufficiently their importance and scope. I have, finally, the pleasure of thanking the amiable secretary of the Vereinigung, Prof. Dr. Gutzmer, for notes of the sessions I could not attend.

JAMES PIERPONT.

YALE UNIVERSITY,
March, 1900.

HILBERT'S FOUNDATIONS OF GEOMETRY.

Grundlagen der Geometrie. Von DR. DAVID HILBERT, o. Professor an der Universität Göttingen. (Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen. Herausgegeben von dem Fest-Comitee.) Leipzig, Teubner, 1899. 8vo, 92 pp.

THE committee in charge of the unveiling of the Gauss-Weber monument at Göttingen has published a memorial volume intended to commemorate the celebration and to serve as a worthy tribute to the genius of the two great men of science. Two professors of the University of Göttingen present in this volume their investigations concerning the foundations of the exact sciences: Professor Hilbert treats of the foundations of geometry; Professor Wiechert discusses the foundations of electrodynamics. The present notice deals only with the former of these memoirs.

It is the object of geometry to analyze and describe our space intuition. The abstraction from spatial intuition leads to three systems of objects: *points*, *straight lines*, and *planes*, which as elements of such intuition, must lie at the basis of any description of space. By means of definitions these elements are brought into certain correlations for which geometry tends to establish general laws. In order to obtain in this way a logically consistent system of propositions certain requirements, called *axioms*, must be satisfied by all imaginable mutual relations between the elements.

Among the axioms of geometry two types can be distinguished: axioms of position and axioms of magnitude. The axioms must have immediate general validity and form a system of propositions independent of each other, not further reducible, and not in contradiction one with any other. Only on the basis of such axioms is any geometrical *definition* possible ; that is, a definition provided it be thinkable, gains its meaning only when it can be shown from the axioms that it has a real content. In addition to these requirements it can be demanded of a system of axioms that it be *simple*, in other words, that the least possible number of propositions be used to establish and sharply circumscribe the relations between the elements, none of the axioms being redundant, *i. e.*, deducible as a corollary from any of the others. The addition of the requirement of *completeness* of the system of axioms will have a meaning only with regard to the particular purpose in view. It is possible to detach (as Professor Hilbert repeatedly does in his researches) certain axioms from the system and build up by means of them alone a geometry forming a logically consistent system and leading to no contradictions. It is, however, perfectly legitimate to inquire what is the complete system of axioms required for laying the foundation of analytic geometry.

Euclid's system of geometry has always been open to objection on two points: the introduction of the axiom of parallels, and the doctrine of proportions and areas. While the latter point, since Euclid's times, has hardly been essentially improved, it is well known how numerous have been the investigations concerning the former. The question whether Euclid's eleventh axiom can be deduced from his other axioms has finally been answered in the negative, a non-euclidean geometry having been constructed by Gauss, Lobachevsky, and Bolyai. The new methods that brought about this final settlement of the old controversy over a much debated problem have led to entirely new views concerning the investigation of the axioms in general. They made it possible for Riemann, Helmholtz, and Lie to found geometry on an analytical basis, a method very different from Euclid's. By conceiving of space as a manifold of numbers these authors dispose at once of a number of geometrical axioms without the necessity of investigating them in detail. In sharp contrast to these analytical attempts we have the purely geometrical researches of Professor Veronese, and also the investigations of Professor Hilbert:

It is our author's aim to lay the proper foundations for euclidean geometry, and beyond this, for analytic geometry. His system thus finds its conclusion with the final recognition that space can be regarded as a manifold of numbers. Among the more important points in which Professor Hilbert's memoir marks a distinct advance I wish to call particular attention to the following:

(1) The introduction of the axioms of congruence, and the definition of motion as based on these; (2) the systematic investigation of the mutual independence of the axioms, this independence being proved by producing examples of new geometries which are in themselves interesting; (3) the principle of not merely proving a proposition in the most simple way but indicating precisely what axioms are necessary and sufficient for the proof; (4) the theory of proportions and areas without use of the axiom of continuity, and more generally, the proof that the whole of ordinary elementary geometry can be treated without reference to the axiom of continuity; (5) the various algebras for segments (Streckenrechnungen), in connection with the fundamental principles of arithmetic.

We now proceed to the discussion of particular points.

In the first chapter all the axioms are classed in five main groups. Group I, comprising the *axioms of combination* (Verknüpfung), contains two plane axioms, viz: (I, 1) Any two different points A and B determine a straight line a; (I, 2) any two different points of a straight line determine this line; and five space axioms (the only space axioms of the whole system), viz: (I, 3) any three points not on the same straight line determine a plane; (I, 4) any three points of a plane, not on the same straight line, determine this plane; (I, 5) if two points of a straight line lie in a plane, every point of the line lies in this plane; (I, 6) if two planes have a point in common, they have at least one other point in common; (I, 7) on any straight line there exist at least two points, in any plane at least three points not on a straight line, and in space at least four points not in a plane.

The group II, forming the *axioms of order* (Anordnung), contains four linear axioms, about the order of points on a straight line, such as (II, 3) among any three points on a straight line there always is one, and only one, that lies between the other two; and one plane axiom, viz: (II, 5) Let A, B, C be three points not on a straight line, and a a straight line in the plane ABC, not meeting any one of the points A, B, C; then, if the line a passes through a point within the segment AB, it will always pass through a point of the segment BC, or through a point of the segment AC.

These axioms are sufficient to show that a straight line contains an unlimited number of points, that it divides the plane into two regions, that any one of its points divides the line into two half-rays. It becomes possible to define polygons and it can be proved that a simple polygon divides the plane into two regions. The following very convenient definition of an *angle* might find its place here: an angle is the system of two half-rays issuing in a plane α from the same point O and belonging to different straight lines. The interior of the angle is the region that contains completely the segment joining any two internal points.

Euclid's *axiom of parallels* (group III), "whose introduction *simplifies* the foundations and *facilitates* the building up of geometry" is given in the form: In a plane α it is always possible to draw through a point A, not on the straight line a, one and only one straight line not meeting the line a.

In the fourth group we find, besides the axioms about the congruence of segments and angles, the following: (IV, 6) If for two triangles ABC and $A'B'C'$ the congruences

$$AB \equiv A'B', \quad AC \equiv A'C', \quad \measuredangle BAC \equiv B'A'C'$$

are true, then the congruences

$$\measuredangle ABC \equiv \measuredangle A'B'C' \quad \text{and} \quad \measuredangle ACB \equiv A'C'B'$$

are also satisfied. It is to be noted that, according to Professor Hilbert's definitions, congruence and symmetry are originally not distinguished.

Among the corollaries of the *axioms of congruence* attention may be called to the proof for the congruence of all right angles, a proposition which in Euclid appears as the fourth postulate. It is obvious how the four groups of axioms so far mentioned can serve to define motion in space, even in a non-euclidean space. The circle is defined in the usual way.

The *axiom of Archimedes* (or axiom of continuity), forming group V, completes the system, which contains in all 8 linear, 7 plane, and 5 space axioms. It is stated as follows: (V) Let A_1 be any point on a straight line between two arbitrarily given points A and B; construct the points $A_2, A_3, A_4, \cdots$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, and so on, and that the segments

$$AA_1, \quad A_1A_2, \quad A_2A_3, \quad A_3A_4, \cdots$$

are all equal; then among the points $A_2, A_3, A_4, \cdots$ there will always be a point A_n such that B lies between A and

A_n. While this formulation of the axiom enables us to define the equality of segments in the sense of general projective measurement, it does not involve the continuity of the straight line in the ordinary sense; it only furnishes a condition necessary for an algebra of segments. It would be best to avoid in this connection the use of the term continuity altogether; indeed, the axiom of Archimedes does not relieve us from the necessity of introducing explicitly an axiom of continuity, it merely makes the introduction of such an axiom possible. Thus, for the whole domain of geometry, Professor Hilbert's system of axioms is not sufficient. For instance, while it follows from this system that a circle and a straight line have in common two points, or one point, or no point, it would be impossible to decide geometrically whether a straight line that has some of its points within and some outside a circle will meet the circle; in other words, it remains undecided whether or not the circle is a closed figure. It also follows, for instance, that a right-angled triangle cannot in general be constructed from the hypotenuse and one side.

Is the system of axioms outlined above consistent in itself? Does it not contain any statement, or statements, whose application may finally lead to something unthinkable or self-contradictory? As geometry is built up by the indefinitely repeated application of the axioms, the possibility is not excluded that a contradiction might appear only after an unlimited repetition of such application. J. H. Lambert compares the axioms to as many equations that can be combined in innumerable ways. Professor Hilbert, to decide the question of consistency, imagines the domain of an enumerable ensemble of numbers and represents a point by two numbers of the domain, a straight line by the ratios of three numbers. With the aid of certain conventions about the order of the points on a line, etc., about translation and rotation, a geometry is thus defined for which all five groups of axioms hold. The question is thus transferred from the domain of geometry to that of arithmetic; any contradiction in the geometry must appear in the arithmetic of the imagined domain of numbers. But just because the question is merely transferred, the same problem remains open for arithmetic. It would seem desirable to find a decision in the geometrical domain itself and not to leave it to a lucky chance of future times. The importance of a final decision as to the absence of contradictions among the axioms is apparent; it is higher even than the question as to their mutual independence.

In the present memoir the author discusses the axiom of parallels, the congruence axiom for triangles, and the axiom of Archimedes, each as to its independence of the axioms of the other groups. An exhaustive investigation of the whole subject of the mutual independence of axioms was given by Professor Hilbert in a course of lectures on euclidean geometry in the University of Göttingen, 1898–99, which thus supplements the printed memoir. The method of proof in every case consists in showing that there exists a consistent system of postulates by means of which it is possible to construct a geometry in which the axiom under discussion does not hold.

Thus, to prove the independence of axiom (I, 5) of all other axioms of the first group, Professor Hilbert proceeds as follows. Let the points of the new geometry be those of euclidean space with the exception of a single one, O; let the planes be planes; and let us take as straight lines the circles through the point O, which does not exist in the new geometry. For this geometry all the axioms of the first group hold excepting the fifth. It should be observed that in this method of reasoning "the properties of euclidean space are used merely as abbreviating notations for certain *arithmetical* relations."

The proofs for the independence of the axioms become the more involved the greater the number of axioms regarded as valid. In his lectures Professor Hilbert enters upon a broader discussion of these questions, especially in regard to non-euclidean geometry, the proof of the congruence theorems in this geometry, etc.

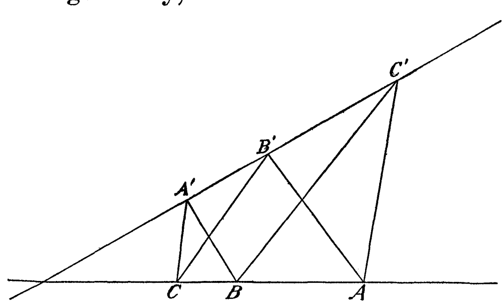

FIG. 1.

For the further development of *plane* geometry two well-known propositions appear to be of fundamental importance; the recognition of this fact must be regarded as an important advance. The author denotes these propositions briefly as Pascal's and Desargues's, formulating them as follows:

Pascal's proposition: Let A, B, C and A', B', C', respectively, be any three points on each of two intersecting straight lines, all different from the point of intersection; then, if CB' be parallel to BC' and CA' parallel to AC', BA' will be parallel to AB'.

Desargues's proposition: If two triangles be situated in a plane in such a way that any two corresponding sides are parallel, then the joins of the corresponding vertices pass through one and the same point or are parallel.

The proof of Pascal's proposition, as a theorem of plane geometry, is readily obtained by means of the axioms (I, 1), (I, 2), II, III, and IV (axioms of congruence), without the aid of the principle of Archimedes. The essential difference between the two proofs given by Professor Hilbert for this proposition lies in the fact that in the second proof not all the axioms of congruence are used, the axiom of congruence for triangles being replaced by one for isosceles triangles.

By devising an algebra of segments (Streckenrechnung) based on Pascal's proposition the true import of this theorem for the construction of the system of geometry is brought out very clearly. The sum of two segments on the same line being defined in the usual way, let the product be defined as follows: On one side of a right angle lay off from the vertex O the segment a, on the other the segments 1 and b; then draw $1a$ and through b the parallel to $1a$; this parallel will cut off on the other side a segment c (counted from O) which is defined as the product

$$c = a \cdot b$$

of the segment a into the segment b.

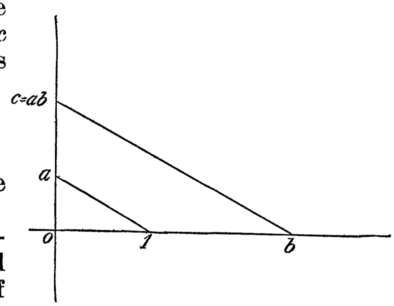

FIG. 2.

In this algebra of segments the commutative and associative laws hold of course for the sum; but they hold also, and this is the meaning of Pascal's proposition, for multiplication. Finally, it can be shown that the distributive law,

$$a(b + c) = ab + ac,$$

holds likewise.

It is obvious how closely this algebra is connected with the theory of proportions. Let the proportion

$$a : b = a' : b',$$

where a, b, a', b' are any segments, be defined as equivalent to the equation of segments

$$a \cdot b' = b \cdot a'.$$

If, moreover, similar triangles be defined in the usual way, it is easy to prove, with Professor Hilbert, on the basis of the algebra of segments, the general validity of the theorem of proportions, and it can further be shown that a straight line is represented by a linear equation.

While this use of Pascal's proposition for the theory of proportions is certainly an important advance it is still more surprising to see the great importance assumed by this proposition as basis for the theory of the areas of plane figures.

Two polygons are said to have *equal area* (flächengleich) if they can be resolved into a finite number of triangles that are congruent in pairs. They are said to have *equal content* (inhaltsgleich) when it is possible to add to them polygons of equal area so that the new polygons so arising have equal area. These two definitions are quite distinct because the investigation is to be carried on independently of the validity of the axiom of Archimedes. It can then be proved that rectangles, parallelograms, and triangles of equal bases and equal altitudes have *equal content.* The fundamental theorem formed by the converse of the last proposition, viz., *if two triangles of equal content have equal bases, they must have equal altitudes,* requires the introduction of the idea of *measure of area* (Flächenmass), which in the case of the triangle is simply half the product of base and altitude ; the theorem is then readily proved by very clear, though somewhat lengthy, considerations. As an elegant application of these results appears finally the proposition (previously discussed by other authors):

If, after cutting up a rectangle by means of straight lines into a number of triangles, any one of these triangles be omitted it will be impossible to make up the rectangle from the remaining triangles.

As regards Desargues's proposition it is known that it can easily be proved with the aid of all the axioms of group I (including the space axioms) as well as those of groups II and III. This fact can be expressed by saying that the existence of Desargues's proposition in the plane is a necessary condition if the plane geometry is to be part of a solid geometry, or the plane a part of space. Leaving out the space axioms it is impossible to prove Desargues's proposition with the aid of the remaining among the above-named axioms. Indeed, Professor Hilbert shows that this proposition cannot even be true in a plane geometry in which *all* axioms hold excepting the axiom of congruence for triangles. This proof will be read with great interest inasmuch as it leads to the construction of such a geometry which invites further investigation. It must, however, not be regarded as proved that the axioms of Hilbert's system are all necessary for the truth of Desargues's proposition; it is not impossible that its existence is independent of the axiom of parallels.

The importance of Desargues's proposition in the system of geometry and its relation to Pascal's proposition appear clearly from an algebra of segments based upon it. In this algebra, in which the constructions do not differ from those of the algebra referred to above except in so far as an arbitrary angle takes the place of the right angle, the associative and commutative laws hold for addition; for multiplication the associative and distributive laws are true, but not the *commutative* law.

The equation of the straight line is found to be of the form

$$ax + by + c = 0,$$

a, b, c being segments of the given system, x and y coördinates; in the products ax and by the order of the letters is essential. It can now be shown analytically that a solid geometry is possible in which all the axioms I to III hold. It follows that, the plane axioms of group I and the axioms of groups II and III being assumed, the existence of Desargues's proposition is the necessary and sufficient condition under which the plane geometry is part of the solid geometry based on those axioms.

To examine further the relations of Desargues's to Pascal's proposition we may proceed as follows, with the aid of the algebra of segments based on Desargues's theorem.

Let us take on one side of an arbitrary angle the segments

$$OA = a, \ OB = b, \ OC = c,$$

and on each side of the angle a unit segment

$$OE = 1, \ OE' = 1.$$

If through A, B, or C a parallel be drawn to the "unit line" EE', meeting the other side at A', B', C', then we put

$$OA = OA' = a, \ OB = OB' = b, \text{ etc.}$$

The sum of two segments b and c is formed by the usual rules: In the figure, $B'L$ is drawn parallel to OC, CL parallel to OB', and LD parallel to BB', which is parallel to EE'; hence

$$OB = CD$$

and

$$OD = b + c.$$

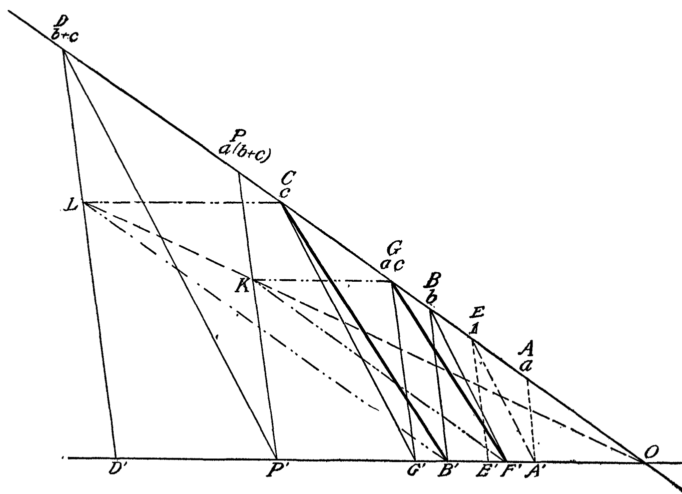

FIG. 3.

The product of the segment a into the segment b is defined simply by drawing through B a parallel BF' to EA' and putting

$$OF' = a \cdot b.$$

The commutative law does not hold for this product because we have not introduced the axioms of congruence, nor the axiom of Archimedes, so that these axioms cannot be used for the proof. Professor Hilbert has shown, however, that the distributive law holds for our multiplication, *i. e.*, we have for instance

$$a(b + c) = ab + ac.$$

The figure shows the construction of the two expressions $a(b + c)$ and $ab + ac$: we have

$$OF' = ab, \qquad OG' = OG = ac,$$
$$OD = b + c, \qquad OP' = OP = a(b + c) = ab + ac.$$

The distributive law requires that the line PP' which is parallel to AA' should pass through K.

It is now seen without difficulty that there exists one and only one segment x corresponding to the equation

$$x \cdot a(b + c) = ab.$$

At this stage we add explicitly to the axioms so far used, viz. (I, 1), (I, 2), II, III, the assumption that in our algebra the commutative law shall hold for multiplication on a straight line. Then the equation

$$ax(b + c) = ab$$

is true, and hence the segment x also satisfies the equation

$$x(b + c) = b,$$

which means for the figure the parallelism of the lines $F'P$ and $B'D$.

Thus the sides of the triangles $F'PK$ and $B'DL$ are parallel in pairs so that, according to the converse of Desargues's proposition, the points O, K, L lie on a straight line. From the triangles $F'GK$ and $B'CL$ it follows further that

$$F'G \text{ is parallel to } B'C.$$

In the hexagon $BF'GG'CB'$ the points B and G' can be selected arbitrarily, also the directions

$$BF' \text{ parallel to } CG', \text{ and } BB' \text{ parallel to } GG'.$$

We are thus led to Pascal's proposition for two straight lines; and it can now easily be shown that the commutative law of multiplication and Pascal's proposition hold for the whole plane.

Without introducing the new assumption it would not have been possible to show that

$$F'P \text{ is parallel to } B'D;$$

i. e., Pascal's proposition would not hold.

Let us sum up these results in the two propositions:

(1) Pascal's proposition can be proved by means of the axioms I, II, III, and the commutative law of multiplication, *i. e.*, without any use of the axioms of congruence.

(2) Pascal's proposition can *not* be proved by means of the axioms I, II, III alone, *i. e.*, without using the axioms of congruence and the axiom of Archimedes.

The algebras of segments based on Desargues's and Pascal's propositions, in which the totality of all segments is regarded as a number system, leads the author repeatedly to discussions concerning the principles of arithmetic.

The real numbers form a system of objects subject to certain laws of combination and order and to the proposition of Archimedes. The twelve laws of combination enumerated by Professor Hilbert refer to addition, multiplication, division, and involve the laws of association, distribution, and commutation. There are four propositions of order. The proposition of Archimedes is as follows :

If $a > 0$ and $b > 0$ be any two numbers, it is always possible to add a a sufficient number of times to itself so that the resulting sum shall have the property :

$$a + a + \cdots + a > b.$$

Any system of objects that satisfies at least some of these 17 propositions is called by Professor Hilbert a *complex number system.*

The two algebras of segments, which do not satisfy the 17th postulate, are called *non-archimedean* number systems ; the segments of the algebra based on Desargues's proposition, in particular, form an *arguesian* number system.

Of particular importance is the following result of the investigation as to the mutual interdependence of the 17 propositions :

In any number system in which the proposition of Archimedes holds, the commutative law of multiplication can be deduced from the other laws of operation ; and :

There exist non-archimedean number systems in which the commutative law of multiplication can not be deduced from the other laws.

In connection with the investigation of the principles of geometry Professor Hilbert discusses geometrical constructions. The old problem of constructions performable with the ruler or with the compasses alone is generalized to the question of indicating all problems solvable only by drawing straight lines and laying off segments. It appears that this condition is satisfied by the geometrical problems that can be solved by means of the axioms I to V. The closely related question as to whether any given geometrical problem can be solved by drawing straight lines and laying off segments alone, is answered by Professor Hilbert on the basis of profound speculations in the theory of numbers published by him elsewhere.

I conclude this review by expressing the hope that the important new views on the foundations of geometry opened up in this memoir may soon become generally known and be introduced into the teaching of elementary geometry.

J. SOMMER.

GÖTTINGEN, OCTOBER, 1899.

KOENIGS' LECTURES ON KINEMATICS.

Leçons de Cinématique professées à la Sorbonne par GABRIEL KOENIGS, avec des notes par M. G. DARBOUX et par MM. E. et F. COSSERAT. Paris, Hermann, 1897. 8vo., x + 499 pp.

WITH this book Professor Koenigs begins the publication of a treatise consisting of two or three volumes, which is to present the development of a course of lectures on kinematics delivered annually either at the École Normale or the Sorbonne for the last eight years. The first volume, the first ten chapters of which were printed in 1894, is devoted to theoretical kinematics ; the rest of the work will be occupied with applied kinematics.

Kinematics as a distinct science is of comparatively recent origin. The formulæ which give the variations of the coördinates of the points of a movable solid in space were published by Euler in 1750. D'Alembert suggested the importance of studying the laws of movements separately. Ampère drew a definite demarcation between mechanics and the geometry of movement, but his object was to develop kinematical science solely for its use in the theory of mechanisms ; the term kinematics is due to Ampère. Previously, in his geometry of position, Carnot predicted a much wider career for this science then to be, by calling attention to the fact that mechanics and hydromechanics would be infinitely simplified if the theory of geometrical motions were thoroughly investigated, since then the analytic difficulties encountered in the study of equilibrium and motion would be reduced to the general principle of the communication of motions, which is only another form of the principle of action and reaction. In 1838 Poncelet included the geometric properties of moving bodies in his course at the Faculty of sciences of Paris ; with the exception of the notions of Chasles, we owe to Poncelet the theory of the continuous motion of a solid in space. Willis, of