Every such factor $x - \nu_i$ *is a simple factor of* $x^{p^{nq}} - x$. The statement being evident for the factor x, we assume $\nu_i \neq 0$, and proceed to prove that $x - \nu_i$ is not a factor in the field of

$$\frac{x^{p^{nq}-1} - 1}{x^{p^n-1} - 1} \equiv (x^{p^n-1})^{r-1} + \cdots + (x^{p^n-1}) + 1,$$

where $$r \equiv (p^{nq} - 1)/(p^n - 1).$$

But for $x = \nu_i \neq 0$, this function reduces in the $GF[p^n]$ to the value r, which is evidently not divisible by p.

It follows that the quotient $(x^{p^{nq}} - x)/(x^{p^n} - x)$ breaks up into factors of degree q belonging to and irreducible in the $GF[p^n]$. Any such factor may be used to define the $GF[p^{nq}]$.

MÉRAY'S INFINITESIMAL ANALYSIS.

Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques. Par M. CH. MÉRAY, professeur à la Faculté des Sciences de Dijon. Paris, Gauthier-Villars. Première partie: *Principes généraux*, 8vo., 1894, xxxiii + 405 pp.; Deuxième partie: *Étude monographique des fonctions principales d'une seule variable*, 1895, xi + 495 pp.; Troisième partie : *Questions analytiques classiques*, 1897, vi + 206 pp.; Quatrième partie : *Applications géométriques classiques*, 1898, xi + 248 pp.

As a work of art—and mathematics is preëminently a fine art—this monumental work of Méray's is a masterpiece; as a treatise to induct students of mathematics into the mysteries of the infinitesimal analysis under the direction of any other than the author* its success is hardly so unqualified. Constructed so as to require technically as preliminary training nothing but a knowledge of elementary algebra and the theory of systems of linear algebraic equations, the first volume would indicate either or both of two conditions: that the author is a most marvelous teacher, that his pupils are of a race superior to that brilliant body of mathematicians now at Paris. It is one thing to be possessed of suffi-

* The author states in the preface to the first volume (1894) that for twenty-four years he has found his method uniformly satisfactory in teaching.

cient elementary knowledge to undertake the serious study of Méray's treatise and quite another thing for that preliminary training to have developed the power of abstraction necessary to work through it.

Méray seems never to have had a particular idea, except at second thought or second hand. He lisped in generalities, for the general ideas came. Near the beginning of his career he wrote an elementary geometry in which the usual distinctions between the geometry of space and that of the plane were abandoned; the perusal of his present work leaves one with the impression that it must have been a life long regret to the author that his Euclid did not spring forth in n dimensions.

The first volume, which is an elaboration of the author's Nouveau précis d'analyse infinitésimale published in 1872, comprises in four hundred pages all the general notions of analysis from the integral number to the integrals of partial differential equations, without mention of a particular function, a straightforward, iron-heeled, uninterrupted march of general principles. The power of abstraction demanded is something terrific; the reader experiences all of the symptoms of that peculiar malady encountered by travellers at high altitudes, and if modern theories be true, some of the author's ethereal oxygen ought to be replaced by the carbon dioxide of regions nearer us common mortals. It is upon this ground that the volume cannot be judged as cordially on pedagogic as upon æsthetic principles; this criticism would be more unjust had not the author addressed his work not only to savants but primarily to students. It may be remarked here however, parenthetically, that the first volume is not half so disheartening when accompanied by the second.

But on the other hand when the work is contemplated as the consistent and logical development of an idea, as the persistent adherence to a method until it had mastered the difficulties interposed by its critics, as the epic of a human ideal, Méray's work is a triumphant success. No student who has had sufficient discipline to maintain him in its perusal can read the work without being profoundly grateful; no student who is endeavoring, for his own peace of mind and safety of soul, to construct his own theory of the number system and to frame his own theory of functions, can afford not to have availed himself of Méray's admirable work; many who cannot accept all of his dogmas—and dogmas are as deplorable in science as in religion—will recall with regret that they never sat under the author's tutelage.

A detailed analysis of this extensive treatise would ten times exceed the space that the BULLETIN can place at the disposal of the reviewer ; in order to respect this restriction of space and at the same time acquaint the reader with the gist of Méray's method, the writer undertakes a succinct account of the author's memoir* on the theory of functions to which reference has already been made, noting what modifications or additions have been made in the treatise.

In the first chapter of the memoir, and the first three chapters of the treatise, Méray constructs the ensemble of fictitious numbers employed by analysis, without the aid of any other notions than those of the integral number and of the addition of such numbers. He banishes all consideration of arguments from his study of imaginaries. His notion of variant (almost identical with that of infinite rational sequence), employed to give precision to the most difficult concept of analysis, makes his exposition of the irrational both clear and luminous. In the preface to the first volume of the treatise, Méray substantiates the claim made by his friends that the latter solution was first given by him. The invention of this theory of incommensurables has been credited to Heine and the first use made of it to Lipschitz, Du Bois-Reymond, and G. Cantor. Heine's memoir entitled "Die Elemente der Functionenlehre" appeared in the seventy-fourth volume of *Crelle's Journal* with the date of 1872 ; Méray's Nouveau Précis is of the same date, 1872, but three years previously he had presented the theory at a congress of learned societies in Paris, in a note entitled "Remarques sur la nature des quantités définiies par la condition de servir de limite à des variables données" pub-

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* Nouveau Précis d'Analyse infinitésimale, Paris, F. Savy, 1872 ; 8vo, xxiii + 310. This work is now included in the list of publications of Gauthier-Villars. The memoir was dedicated to M. Faurie, General Inspector of secondary instruction, and was seriously designed to bring about a revolution against the traditional methods of teaching the infinitesimal calculus. For convenience of comparison the order of ideas exhibited in the subjects of the chapters is here appended :—1° preliminaries; 2° generalities on series ; 3° series arranged in the ascending powers of the independent variables ; 4° olotropic functions, classification of their derivatives, general properties ; 5° limits of convergence of the developments of olotropic functions ; 6° composite functions ; 7° principles of the discussion of functions ; 8° various propositions on olotropic or quasi-olotropic functions of a single variable ; 9° fundamental principle of the integral calculus ; 10° theory of implicit functions ; study of some implicit functions of a single variable defined by a unique equation ; 12° analytic theory of the exponential function and of its related functions ; 13° inverse functions ; 14° ordinary differential equations, principal case ; 15° general case of the preceding ; 16° total differential equations ; 17° simple integrals.

lished in the *Revue des Sociétés savantes* (*Sciences mathématiques*, vol. 4, p. 284, 1869).

The number system thus established in a manner that leaves nothing to be desired in the way of cold logic, the author proceeds to present the properties of series, and in particular those of integral power series. He loses no opportunity to impress his conviction that all analytic functions are identical with integral series. To Méray's mind no function exists which cannot be expanded by Taylor's theorem. All other functions he rules out as "phantasies of the mathematical imagination;" but does the reader clearly understand the meaning of the last phrase? To the same exile he consigns the non-euclidean geometry. The demands of analysis required that the idea of the number system be enlarged to include certain fictitious elements that could equally be dubbed fantastic creatures of a mathematical imagination; Méray fell in with the call of the hour and led the procession. For this, among other reasons, it is difficult to understand why the author refuses to respect the discoveries of the last twenty years and accord to the notion function the possibility of a similar enlargement. Méray is a pure mathematician, an analyst of power, and yet he would prescribe a definite boundary to the theory of functions and to all physical phenomena to come within the ken of men because "Tout phénomène naturel est représentable exactement par des séries entières, approximativement par leurs premiers termes, dont des observations de plus en plus précises fournissent empiriquement les coefficients dans l'ordre même où l'Analyse les range."

Lie averted the current objections to his method from being hurled against his volume on differential equations by limiting it to differential equations that admit of infinitesimal transformations; this was thoroughly scientific and did not carry with it the absurd denial of the existence of differential equations not admitting of infinitesimal point transformations, the number of which equations is legion. Similarly no complaint could be lodged against Méray had he contented himself with the study of functions developable by Taylor's theorem, without defining analysis to be the systematic study of the general properties of functions. He got hold of a large idea, and exhausted it ten, perhaps twenty, years ago; since then it seems to have been master of the man. Would that the same vigorous analytical power had been turned in the meantime on those spurned subjects.

The exposition of the general properties of integral series is considerably simpler in the treatise than in the memoir because of the introduction of the author's new demonstration * of a lemma of Cauchy; the demonstration gives the lemma the position of a new theorem. The discussion of the continuity and convergence of series is admirable.

Passing to the notion of function, of which more or less has been remarked already, the author defines a function to be olotropic in the areas S_x, S_y, $\cdots$, with the olometers δ_x, δ_y, $\cdots$, when, for every system of numbers, x_0, y_0, $\cdots$, taken within the areas in question, we can develop the function in a convergent integral power series in $x - x_0$, $y - y_0$, $\cdots$, provided that the moduli of these differences are respectively less than δ_x, δ_y, $\cdots$. The areas S_x, S_y, $\cdots$, are any whatsoever having simple or multiple contours.

The successive derivatives of a function of one or more variables are naturally defined after Lagrange's theory of analytic functions and Arbogast's method of derivatives, the classical objections to these being avoided by considering only olotropic functions. The derivatives are obtained algebraically without reference to infinitesimals by developing the series $f(x + h, y + k, \cdots)$ and throwing the development into the form $f(x, y, \cdots) + hf_x' + kf_y' + \cdots$, the quantities f_x', f_y', $\cdots$ being convergent series.

A capital element in the generation of new functions is the operation which the author calls *cheminement;* this operation is the ensemble of the following: To obtain in X, Y, $\cdots$ the value of a function $f(x, y, \cdots)$ it is necessary to start from an initial system x_0, y_0, $\cdots$, and interpolate, if possible, between it and the proposed system X, Y, $\cdots$ intermediate systems x_1, y_1, $\cdots$; x_2, y_2, $\cdots$; $\cdots$; x_k, y_k $\cdots$, such that the differences between the corresponding values of the variables in two consecutive systems shall have moduli less than the olometers of the function for the values of the smaller index; then to calculate the function $f(x_1, y_1, \cdots)$ by Taylor's formula, by putting

$$f(x_1, y_1, \cdots) = f[x_0 + (x_1 - x_0), \quad y_0 + (y_1 - y_0), \cdots];$$

then $f(x_2, y_2, \cdots)$ in a similar manner by means of $f(x_1, y_1, \cdots)$; and finally $f(X, Y, \cdots)$ from $f(x_k\, y_k, \cdots)$.

The author studies the theory of indefinite integrals in the eighth chapter of the treatise under the title inverse calculus of derivatives. Up to this point inclusive it has been a question only of isolated functions, that is to say

**Bulletin des Sciences mathématiques*, vol. 15 (1891).

functions considered independently, as if no other functions existed. The remainder of the volume is occupied with general operations which imply the composition of functions.

In the tenth chapter of the treatise's first volume the author demonstrates the convergence of the elementary developments of the integrals of a system of total differential equations, by a direct examination of these developments themselves and without the intervention of auxiliary differential equations. It is at this juncture that Méray introduces an auxiliary notion, attributed to Weierstrass but employed by Méray since 1868 and given at § 139 of the Nouveau Précis, and § 300 of the first volume. This auxiliary artifice he calls the *majorante* and defines it as follows: Given an olotropic function $f(x, y, \cdots)$ of any variables, we call majorante of this function at $x_0, y_0, \cdots$ every function $\varphi(x, y, \cdots)$ of the same variables, possessed of the property that, for $x = x_0$, $y = y_0, \cdots$, its value and that of all its derivatives shall be real and positive, and superior to the moduli of the corresponding values of $f(x, y, \cdots)$ and its derivatives.

The memoir contains practically nothing of the chapter XII. of the first volume on partial differential equations; the material is drawn largely from two papers published by the author with the collaboration of M. Riquier.*

In the last chapter of the volume on systems of total differential equations all reference to unsolved equations and to those of order higher than the first is omitted. In his Nouveau Précis the author demonstrated for the first time the possibility of resolving normally the integral equations with regard to the arbitrary constants; a demonstration neither without use nor without value.

As remarked before, the first volume contains no reference to any special form of function; the chapters of the memoir relating to particular functions were reserved for the second volume.

Among the elements of the first chapter we find the author's more recent demonstration† of the existence of roots of every integral algebraic equation; this demonstration is as simple and elegant as could be desired.

The limits of space prescribed forbid calling attention to the details of this second volume. Certain striking ones

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**Annales scientifiques de l'École Normale supérieure*, vol. 6 (1889), vol. 7 (1890).

† "Méthode directe fondée sur l'emploi des séries pour prouver l'existence des racines des équations entières à une inconnue par la simple exécution de leur calcul numérique," *Bulletin des Sciences mathématiques*, vol. 15 (1891).

however demand a word. In the third chapter the author develops the theory of radicals as a simple corollary of his theory of implicit functions elaborated in the first volume. He incorporates his memoir* which treats of this question in harmony with the fundamental point of view of Abel, whom Méray places with Lagrange as the first to found the infinitesimal analysis on a solid basis. The author's method is long, but he avoids all trigonometric considerations and thus adds a further illustration that the intrusion of geometric facts is not a necessity in analysis.

In the fourth chapter he reproduces and improves the method given in the Nouveau Précis for extending the capital discovery of Puiseux on algebraic irrationals to the roots of any olotropic equation. A principle is introduced under the name principle of the conservation of the number of roots which later brings into evidence certain fundamental properties of biperiodic functions; the principle is one of those which exist only by virtue of extending the ordinary algebraic operations to imaginaries. The author regrets, as does the reader, that the inflexible rigor which he has made his law would not permit of abridging these earlier chapters.

The Napierian logarithm, being the first transcendent having a derivative, naturally suggests itself as the first transcendent to be studied. The author then presents the exponential as the simple result of the analytic inversion of the logarithm. The inversion of the indefinite integral of a rational differential gives other functions, no longer indefinitely olotropic as the exponential, but indefinitely meromorphic; in this way the circular functions are introduced. Méray returns to the differential equation as the starting point for his elliptic functions, regarding his demonstration† of the non-nullity of the moment of the periods as sufficiently simple, and considering the dominant rôle assumed by the theta functions in modern expositions as artificial. As usual, it is the most general function that he studies.

As to the functions of Euler studied in the last chapter, they also have their origin in integration, not directly or indirectly from indefinite integrals considered as functions of their principal variables, but as functions of parametric variables which accompany the principal variable in certain

* "Théorie des radicaux fondée exclusivement sur les propriétés générales des séries entières," *Revue bourguignonne de l'Enseignement supérieur*, vol. 1 (1891).

† "Sur l'existence effective des deux périodes des fonctions elliptiques," *Ann. scient. de l'École Normale sup.*, 3d series, vol. 1 (1894).

artificial definite integrals; by the latter Méray means the integral of $f(x)$ from x_0 to X when the path of integration contains any value of x_1 singular for $f(x)$, or when the path is unlimited in any direction.

The course of the development of the whole treatise is admirably exhibited by the subjects and order of the chapters of the several volumes, which are appended below in conclusion.* The elegant applications presented in the last two parts are more than worthy of a direct reference, for they demonstrate again the success of the method in the hands of the author.

I. 1° Preliminary generalities comprising a review of the fictitious quantities upon which the speculations of modern analysis are founded; fractions, positive and negative quantities; 2° continuation of the preceding, variants in general; incommensurable quantities; 3° continuation of the two preceding chapters, imaginary quantities; 4° series in general; 5° integral series; 6° derivatives of olotropic† functions, usual genesis of these functions; 7° fundamental properties of functions which are olotropic within given areas; 8° inverse calculus of derivatives; 9° composite functions; 10° essential principle of the theory of total differential equations; 11° implicit functions in general; 12° essential principles of the theory of partial differential equations; 13° further study of immediate systems of total differential equations; addition, 1° on an

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* The first volume contains a list of Méray's principal publications. The works have already been referred to; the memoirs are to be found in the following journals: *Nouv. Ann. de Math.*, 1st ser., vol. 13 (1854), 3d ser., vol. 8 (1889), vol. 9 (1890), vol. 11 (1892); *Comptes rendus*, vol. 40 (1855), vol. 106 (1888); *Ann. di Matem.*, 1st ser., vol. 3 (1860); *Ann. scient. de l'Ec. Norm sup.*, 1st. ser., vol. 4 (1867), 2d ser., vol. 6 (1877), vol. 8 (1879), vol. 12 (1883), 3d ser., vol. 1 (1884), vol. 2 (1885), vol. 6 (1889), vol. 7 (1890); *Revue des Sociétés savantes* (*Sc. math., phys., et nat.*), 2d ser., vol. 3 (1868), vol. 4 (1869); *Liouville's Journal*, 3d ser., vol. 10 (1884); *Bull. des Sc. math.*, 2d ser., vol. 12 (1888), vol. 5 (1891); *Revue bourguignonne de l'Enseign. sup.*, vol. 1 (1891), vol. 2 (1892); *Ann. de la Fac. des Sc. de Toulouse*, vol. 4 (1891).

† Mathematics of necessity mutilates language occasionally in the construction of its technical terms, but it is to be regretted that Méray did not give heed to a correction in the form of this term suggested by the editors of the *Bulletin des Sciences mathématiques* in 1874; orthography should respect etymology and the Greek derivative is ὅλος. It is interesting to observe that Méray used this term in 1872, but that not until 1875 did its successful rival *holomorphic* appear; see Briot and Bouquet's Theory of elliptic functions, 2nd edition, p. 14. In the preface to the first volume Méray anticipates a day when both will disappear, namely at that epoch when mathematicians shall have opened their eyes to the accuracy of the conceptions of Lagrange and have ceased to contemplate functions non-integrable, continuous but without derivatives, et cætera.