tions shown under cartesian form. The surfaces having for curvilinear directors $x - y = 0$ and $xy - \frac{1}{2} = 0$ were studied in detail and models exhibited showing their principal types.

Professor White's paper was a further development of the topic considered in the paper presented by him at the Columbus meeting of the Society. Each mixed concomitant (2, 2) of the cubic defines (as in the paper referred to) two covariant nets of conics. These are polars of two cubics of the syzygetic sheaf; the totality of such is exactly that entire sheaf of cubics. But these concomitants (2, 2) and all the concomitants (3, 3) serve to define also *four* covariant sheaves of cubics, not in the syzygetic sheaf, intimately connected on the one hand with the four inflexional triangles, and on the other hand with the eighteen collineations of the cubic into itself. This paper will be published in the *Transactions*.

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ON CYCLICAL QUARTIC SURFACES IN SPACE OF N DIMENSIONS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1899.)

THE generation of the cyclide as the envelope of spheres which cut a fixed sphere orthogonally and whose centers lie on a quadric can readily be generalized to space of n dimensions.

In ordinary space it appears that the same surface is the envelope of five different systems; that the quadric loci of centers are all confocal and the associated spheres are all orthogonal; that the possibilities of the system are exactly coextensive with the ∞^{13} possible cyclides.

Let

$$(1) \quad (x_1 - ix_{n+2}) \sum_{r=1}^{n} y_r^2 - 2 \sum_{r=1}^{n} x_{r+1} y_i + (x_1 + ix_{n+2}) = 0$$

be the equation of a sphere in R_n; it contains $n + 2$ homogeneous constants x_r, and if the radius be denoted by

$$\frac{ix_{n+3}}{x_1 - ix_{n+2}},$$

the quadratic identity

$$(2) \qquad \varkappa \equiv \sum_{r=1}^{n+3} x_r^2 \equiv 0$$

will exist among these $n + 3$ numbers x_r, which may be called the homogeneous *coördinates of the sphere.*

Two spheres a, b will intersect orthogonally when

$$\sum_{r=1}^{n+2} a_r b_r = 0,$$

the terms defining the radii of the two spheres not occurring ; hence any linear equation of the form

$$(3) \qquad \sum_{r=1}^{n+2} a_r x_r = 0$$

represents the ∞^n spheres which cut a fixed sphere orthogonally.

Now consider a quadratic equation of the form

$$(4) \qquad \varphi_2(x_1, x_2, \cdots, x_{n+2}) = 0$$

which does not contain x_{n+3}, and make it simultaneous with (3); between the two $x_1 + ix_{n+2}$ may be eliminated, leaving a quadratic equation among the point coördinates of the centers of the variable spheres.

Hence, equations (3) and (4) define the ∞^{n-1} spheres which cut a fixed sphere orthogonally, and whose centers lie on a quadric surface M_{n-1}^2. These spheres envelop a new surface whose equation may be found as follows :

Let $x_1 - ix_{n+2}$ be replaced by x_1, as $x_1 + ix_{n+2}$ has been eliminated between (3) and (4). Similarly, let $x_1 + ix_{n+2}$ be eliminated between (3) and (1). Then, with a slight change in the meaning of the coefficients, the problem reduces to that of finding the envelope of the sphere

$$s \equiv x_1 \left[a_{n+2}(y_1^2 + y_2^2 + \cdots + y_n^2) - a_1\right] - 2 \sum_{r=1}^{n} x_{r+1} \left(a_{n+2} y_r - a_{r+1}\right) = 0,$$

subject to the condition, say $f(x_1, x_2, \cdots, x_{n+1}) = 0$.

Since the sphere s is to touch the envelope f, it may be regarded as the equation of the point of contact, and f as the equation of the surface itself, in tangential coördinates, hence

$$\Sigma \frac{\partial f}{\partial x_i} x_i = \Sigma \frac{\partial s}{\partial x_i} x_i = 0,$$

and corresponding coefficients must be proportional,

$$\frac{\partial f}{\partial x_r} = \lambda \frac{\partial s}{\partial x_r}.$$

Between these $n + 1$ linear equations in x_r and the equation $s = 0$ the numbers x_r and λ may be eliminated, giving

$$(\alpha) \qquad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} & a_{n+1}\sum_{r=1}^{n} y_r^2 - a_1 \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} & a_2 - 2y_1 a_{n+2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} & a_3 - 2y_2 a_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+2}\sum_{1}^{n} y_r^2 - a_1 & a_2 - 2y_1 a_{n+2} & \cdots & \cdots & 0 \end{vmatrix} = 0$$

or the Hessian of f bordered by the coefficients of x_r in s.

This proves that the envelope is a cyclical surface, and the number of constants in the most general surface of this kind, viz., $\frac{1}{2}(n^2 + 5n + 2)$, exactly coincides with the number of constants in f, $\frac{1}{2}n(n + 3)$, plus the number in (3), $n + 1$, so that all quartic cyclical surfaces can be generated in this way. This does not show, however, in how many ways the same surface may be generated.

Let the two forms

$$\varkappa, \quad \varphi_2 + a_n x_{n+3}^2$$

be subjected to any linear transformation, such that $\varkappa = 0$ may go into itself, and φ_2 may become a sum of squares of the form

$$(5) \qquad F \equiv \sum_{r=1}^{n+3} b_r x_r^2 = 0;$$

further, suppose the restriction be also imposed that x_{n+3} shall go into itself. Equation (5) represents a complex of spheres; in the vicinity of any of its spheres z it may be replaced by the tangent linear complex

$$\sum_{r=1}^{n+3} \frac{\partial F}{\partial z_r} x_r = 0,$$

so that all of the spheres which belong to F and touch z must touch it in points of its $M_{n-2}^{2,2}$ of intersection with the fundamental sphere of the tangent complex. When

$$\sum_{r=1}^{n+3} \left(\frac{\partial F}{\partial z_r}\right)^2 = 0,$$

these two spheres touch each other. The $M_{n-2}^{2,2}$ reduces to a point, and z is a singular sphere.

Let $\partial F/\partial z = t_r$; now t is also a sphere which touches z, and the whole tangent pencil can be represented in the form

$$m_r = t_r + \lambda z_r = b z_r + \lambda z_r = z_1 (b_r + \lambda).$$

The sphere m touches z, which is also a sphere, hence

$$\sum_{r=1}^{n+3} m_r x_r = 0, \qquad \sum_{r=1}^{n+3} z_r^2 = 0,$$

or, replacing z_r by its values,

$$A \equiv \sum \frac{m_r^2}{b_r + \lambda} = 0, \qquad \sum \frac{m_r^2}{(b_r + \lambda)^2} = 0.$$

Those values of m which satisfy these two equations define the singular sphere of a quadratic complex for every value of λ, hence the whole pencil of complexes have the same surface of singularities. The original complex is contained in the series, corresponding to $\lambda = \infty$.

The λ eliminant of these two equations will give the equation in tangential coördinates of the surface of singularities; the surface is seen to be of class $4n$.

Among the quadratic complexes of the pencil are $n + 3$ simple ones counted twice, corresponding to $\lambda = -b_r$. When $\lambda = -b_{n+3}$ this becomes the complex of points in R_n which, combined with $x = 0$ and with the other terms of the complex

$$\sum_{r=1}^{n+2} \frac{x_r^2}{b_r - b_{n+3}} = 0,$$

defines a general cyclical surface. But the envelopes of the other $n+2$ systems corresponding to $\lambda = -b_r$ $(r = 1, \cdots, n+2)$ define the same surface. Now $x_r = 0$ represents the totality of spheres which cut a fixed sphere orthogonally, and when associated with $A = 0$ defines all those spheres whose centers also lie on a quadric, hence by (α) they envelop a cyclical surface. The fundamental spheres of these $n+2$ systems are by (3) mutually orthogonal, and the quadrics are confocal since $A = 0$. Hence

An M_{n-1}^4 *in (euclidean)* R_n *which contains the absolute as a double* $M_{n-2}^{1,2}$ *can be generated in* $n+2$ *ways as the envelope of those hyperspheres which cut a fixed hypersphere orthogonally and whose centers lie on a* $M_{n-2}^{2,2}$. *The fixed spheres are mutually orthogonal and the quadrics are confocal.*

The $M_{n-2}^{2,2}$, intersections of the M_{n-1}^2 and the hyperspheres, are all focal spreads of the cyclical surface. Through the center of each sphere passes a bitangent cone M_{n-1}^2 whose elements are perpendicular to the elements of the asymptotic cones of the M_{n-1}^2.

By giving λ different values in the system

$$\sum_{r=1}^{n+2} \frac{x_r^2}{a_r - \lambda} = 0, \qquad x_{n+3} = 0,$$

a series of confocal cyclical surfaces is obtained. By substituting the coördinates of a point sphere in the equation, n different values for λ can be found; hence, *n cyclical surfaces of a confocal system pass through every point in space;* by applying the tangent complex to each and using (3) it appears that *these cyclical surfaces intersect orthogonally.*

For $n = 2$ these surfaces (bicircular quartic curves) have been systematically studied, from a different point of view, by Casey, Darboux, Cox, Loria, and others; for $n = 3$ (cyclides) by Casey, Maxwell, Cayley, Darboux, Reye, Loria, Bôcher, Domsch, Loewy, Moutard; and many special points have been noticed in numerous other papers.

The method here given is a generalization of that first employed by Darboux, using Lie's more general coördinates. The latter were first systematically employed in an article by the author, read at the Toronto meeting of the AMERICAN MATHEMATICAL SOCIETY, and published in the BULLETIN, volume 4 (new series), pp. 144–154.

For $n = 4$, the number of distinct types is 58, and for larger values of n the number of types has not been determined.

CORNELL UNIVERSITY,
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