

## ON ELLIPTIC FUNCTIONS.

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How simply and naturally to present the theory of elliptic functions is an embarrassing question. Since the functions of Jacobi and Weierstrass are both used by eminent mathematicians it is necessary to treat them both. This is usually done by giving one of the two classes the preference and deducing the properties of the other as corollaries. Thus in the works of Thomae and Weber the functions of Jacobi are given the preference; in the works of Halphen, Jordan, and Tannery and Molk it is the functions of Weierstrass.

I do not believe this course tends to show to best advantage the peculiar nature of each class of functions nor to show up their similarities and differences.

In my opinion the true foundation of Jacobi's theory is the *thetas*, of Weierstrass's the *sigmas*. To deduce the properties of the functions of Jacobi as Jordan has done in the new edition of his excellent Cours seems to me to rob the reader of half the beauties of this theory. The same is true of the functions of Weierstrass when deduced from those of Jacobi.

Another feature we should look for in a satisfactory presentation of the theory is one too often overlooked. We should demand that, since the sigmas and thetas are the fundamental elements of our theory, these should not be brought on to the scene like a kind of *deus ex machina* but that they appear as necessary elements of our reasoning. The introduction of the sigmas is made easy and natural by virtue of Weierstrass's factor theorem. To introduce the thetas naturally is less easy. Weber has shown with great success how the properties of the functions of Jacobi may be studied by means of the *T* functions, that is one valued transcendental functions which satisfy the relations

$$\begin{aligned} T(u + 2\omega_1) &= e^{-\pi i[2a_1(u + \omega_1) + b_1]} T(u) \\ T(u + 2\omega_2) &= e^{-\pi i[2a_2(u + \omega_2) + b_2]} T(u) \end{aligned} \quad (1)$$

where 
$$R\left(\frac{\omega_2}{\omega_1}\right) > 0 \quad (2)$$

and 
$$2a_2\omega_1 - 2a_1\omega_2 = m, \quad \text{an integer.} \quad (3)$$

The motives Weber gives for introducing such a function are so meager that the reader is left in wonder how they were ever thought of. He is also held in painful suspense for twenty-two pages whether such functions, whose existence is postulated, really exist.

Now the  $T$  functions, containing the thetas and sigmas as special cases, afford an excellent standpoint from which the functions of Jacobi and Weierstrass may be viewed, if we can only arrive at them in a natural manner, by the aid of a few à priori considerations. This I suggest doing in the following manner by starting from the notion of periodicity.

I would begin by remarking that having examples of one valued functions which are *simply* periodic and without essentially singular points in the finite part of the plane, it is natural to inquire after such functions that have more than one period. We show easily that such functions can be at most doubly periodic and that their periods, say  $2\omega_1, 2\omega_2$ , cannot have a real ratio. For brevity call such functions elliptic. That they exist is shown at once by simple examples, as

$$f(u) = \frac{\sum}{\tilde{\omega}} \frac{1}{(u - \tilde{\omega})^m} \quad (\tilde{\omega} = 2\lambda_1\omega_1 + 2\lambda_2\omega_2, \quad m > 2.)$$

Next we would show that an elliptic function takes on every value in a parallelogram of periods  $P = (2\omega_1, 2\omega_2)$  as often, say  $m$  times, as it becomes infinite, and that  $m > 1$ . This we call the order of the function. Take now the most general elliptic function  $\varphi_m(u)$  of order  $m$ . Let its zeros and poles in  $P$  be respectively

$$a_1, a_2 \dots a_m,$$

$$b_1, b_2 \dots b_m.$$

Let the totality of numbers  $\equiv a_\kappa \pmod{2\omega_1, 2\omega_2}$  be represented by  $\alpha_\kappa$  while those congruent to  $b_\kappa$  may be  $\beta_\kappa$ . Call  $\alpha$  the totality of  $\alpha_\kappa$ , and  $\beta$  the totality of  $\beta_\kappa$ .

Then since

$$\sum \frac{1}{|\alpha|^3}, \quad \sum \frac{1}{|\beta|^3}$$

converge, Weierstrass's factor theorem gives at once

$$\varphi_m(u) = e^{g(u)} \frac{\prod_{\alpha} \left(1 - \frac{u}{\alpha}\right) e^{\frac{u}{\alpha} + \frac{1}{2}\left(\frac{u}{\alpha}\right)^2}}{\prod_{\beta} \left(1 - \frac{u}{\beta}\right) e^{\frac{u}{\beta} + \frac{1}{2}\left(\frac{u}{\beta}\right)^2}}.$$

The products  $\prod_{\alpha}$  and  $\prod_{\beta}$ , being absolutely convergent, can be broken up into partial products so that

$$\varphi_m(u) = e^{g(u)} \frac{\prod_{\kappa=1}^m \prod_{\alpha_{\kappa}} \left(1 - \frac{u}{\alpha_{\kappa}}\right) e^{\frac{u}{\alpha_{\kappa}} + \frac{1}{2}\left(\frac{u}{\alpha_{\kappa}}\right)^2}}{\prod_{\kappa=1}^m \prod_{\beta_{\kappa}} \left(1 - \frac{u}{\beta_{\kappa}}\right) e^{\frac{u}{\beta_{\kappa}} + \frac{1}{2}\left(\frac{u}{\beta_{\kappa}}\right)^2}},$$

and now, making use of the fact that the second logarithmic derivatives of  $\prod_{\alpha_{\kappa}}$ ,  $\prod_{\beta_{\kappa}}$  are doubly periodic, we show in the customary manner that  $g''(u)$  is a constant and therefore that  $g(u)$  is a polynomial of second degree while  $\prod_{\alpha_{\kappa}}$ ,  $\prod_{\beta_{\kappa}}$  are of the form

$$\tau(u | a, p, q, r) = e^{p+qu+ru^2} \sigma(u-a).$$

Hence the most general elliptic function of order  $m$  is the quotient of two integral transcendental functions of the type

$$(4) \quad T^{(m)}(u) = \prod_{\kappa}^{1, m} \tau(u | a_{\kappa}, p_{\kappa}, q_{\kappa}, r_{\kappa}).$$

These functions satisfy the conditions (1), (2), (3); and conversely, a one valued integral transcendental function satisfying these conditions is a function of the type (4). We have thus reached our goal.

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