mathematicians share in common with all mankind to wish to know something of the personality and the surroundings of the men whose works have influenced the progress of civilization. In accordance with this general wish we find more and more the collected works of great mathematicians accompanied by biographical notices except when they form special works accessible to all, as in the case of Abel or Hamilton. Who has not read with pleasure the touching Lebenslauf of Riemann which Dedekind wrote for the Gesammelte Werke, or the biographical sketches attached to the Works of Henry Smith and the Lectures and Essays of Clifford?

For a long time the details of Galois's life were surrounded with mystery. Like a blazing comet his genius appeared to the mathematical world only to disappear with equal suddenness into the gloom of impenetrable obscurity. The introductory remarks of Liouville to his works and possibly the affecting sketch by his friend Chevalier in the *Revue Encyclopédique* comprised all that most of us knew of him. Today, thanks to the patient and loving labors of M. Paul Dupuy, the details of his life are quite well known. Appearing, however, in the *Annales de l'École Normale Supérieure*, this valuable monograph cannot be purchased separately and so is not readily accessible to all. Would not a short sketch containing the principal features of Galois's life have been a valuable addition?

We close here our suggestions; they have been offered solely in the hope that when a new edition becomes necessary it may be enlarged along the lines here indicated and so spread still more widely the appreciation of Galois's genius and the usefulness of his theories.

JAMES PIERPONT.

YALE UNIVERSITY,
November, 1898.

THREE MEMOIRS ON GEOMETRY.*

Thèse sur la géométrie non-euclidienne. Par M. L. GÉRARD, Professeur au Lycée Ampère (Lyon). Paris, Gauthier-Villars et fils, 1892. 4to, 110 pp.

Lezioni di Geometria intrinseca. Per ERNESTO CESÀRO, Professore ordinario della R. Università di Napoli. Naples, Alvano, 1896. 8vo, 264 pp.

* The order in which these memoirs appear above is that given in the official announcement and presumably determined by lot, since the regulations of the Lobatschewsky Prize specify that precedence shall be so determined in case of a tie.

L'hyperespace à $(n-1)$ *dimensions. Propriétés métriques de la corrélation générale.* Par G. FONTENÉ, Professeur au Collège Rollin. Paris, Gauthier-Villars et fils, 1892. 8vo, 133 pp.

AT the anniversary meeting of the Physico-Mathematical Society of the University of Kasan, held November 3, 1897, the first award of the Lobatschewsky Prize was made to Professor Sophus Lie, of the University of Christiania, for the third volume of his classic work on the Theory of Transformation Groups. The three volumes above were accorded each an honorable mention. There were nine works placed in competition for the prize. The gold medal for distinguished services in assisting the Society in examining the works submitted was awarded to Professor Felix Klein, of the University of Göttingen, for his report† on the work of Professor Lie. The prize is worth five hundred rubles in paper money and it is to be given every three years. The next award will be made November 3, 1900; memoirs and treatises published within six years preceding this date in Russian, French, German, English, Italian, or Latin are eligible and will be received up to November 3, 1899.

I. GÉRARD'S THESIS.

Gérard's memoir consists of three parts. In the first part there is presented a new method for establishing the fundamental formulæ of the non-euclidean geometry; this method originates in the theorem: If we designate by L an arbitrary length, there exists a corresponding number λ such that, in every quadrilateral $ABCD$ which has three right angles A, B, C, we have

$$\tfrac{1}{2}\left(e^{\lambda\frac{BC}{L}}+e^{-\lambda\frac{BC}{L}}\right)<\frac{CD}{AB}<\tfrac{1}{2}\left(e^{\lambda\frac{AD}{L}}+e^{-\lambda\frac{AD}{L}}\right);$$

from this we deduce immediately the relation which exists between the three sides of a right triangle, and all the other fundamental formulæ.

The author makes no use of the infinitesimal calculus and deals only with figures whose construction is possible. The latter condition is Euclid's inflexible rule; its observance is desirable, in that the number of postulates introduced without demonstration is thereby reduced to a minimum. The possibility of constructions which can be effected with

†Klein, "Zur ersten Verteilung des Lobatschewsky-Preises: Gutachten, betreffend den dritten Band der Theorie der Transformationsgruppen von S. Lie."

ruler and dividers rests uniquely on four perfectly definite hypotheses, which Gérard formulates as follows: 1° Through two points a straight line can be passed; 2° A circle can be drawn having a given point as center and a given length as radius; 3° A straight line which intersects the perimeter of a polygon in a point other than one of its vertices intersects it again in one or more points; 4° Two straight lines, or two circles, or a straight line and a circle, intersect, if there are points of one of these two lines situated on both sides of the other.

On the other hand, if we consider with Lobatschewsky and Bolyai, figures which we do not know how to construct, for example, parallels,* horicycles, horispheres, et cetera, we are obliged either to admit the existence of all these figures as additional hypotheses, or to demonstrate their possibility by invoking the aid of the principle of continuity, which, at best, is poorly defined—both as to precision necessary to a basis of reasoning, and as to generality sufficient to include all cases; accordingly, the author proposes to establish the relations which exist between the elements of a triangle by founding them upon the four hypotheses framed above. This direct determination for the single case of the relation between the two sides of a right triangle and one of the acute angles was made by Battaglini,† but his demonstration is not free from all flaws, as Gérard points out.

The second part of the memoir introduces the elements of non-euclidean analytical geometry by defining the position of a point in the plan by means of three coördinates, X, Y, Z, connected by the relation

$$X^2 + Y^2 - Z^2 + 1 = 0.$$

This system of coördinates gives formulæ quite analogous to those of spherical geometry; in fact, by the notion of *ideal* points we distinguish between points on the front and on the back of the plane in precisely the same manner as between the points diametrically opposite of a sphere; this distinction is necessary to the generality of certain theories, for example, the theorem of Menelaus, and the power of a point with regard to a circle.

The author reviews the various constructions of the non-

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* The author does not ignore Bolyai's construction for parallels, but remarks in a footnote that Bolyai indicated the means of effecting the construction only after the fundamental formulæ had been established.

† Battaglini, "Sulla geometria immaginaria di Lobatschewsky," *Giornale di Matematiche* (1867); this demonstration is reproduced in the Traité de géometrie of Rouché and de Comberousse.

euclidean geometry which can be made by means of ruler and dividers, and demonstrates the impossibility of constructing the length $\frac{1}{\lambda}L$, defined above, which Bolyai has called the natural unit of length.

Gérard shows in the third part of the memoir that the measure of the areas of polygons can be presented in a very simple manner by discarding every postulate and considering two polygons as equivalent only in the case where they are composed of parts superposable each to each; to this end it is necessary to demonstrate: 1° That any two polygons A and B are comparable, that is we can always either decompose A and B into parts superposable each to each; or decompose A into two parts, one of which will be equivalent to B; or decompose B into two parts, one of which will be equivalent to A; 2° That these three cases of relation between the two polygons A and B are incompatible. The first of these two theorems is indispensable from every point of view; as to the second, it might be set up as an axiom were it incapable of demonstration, but Gérard finds it is susceptible of a simple demonstration both for euclidean and non-euclidean* geometry.

II. Cesàro's Intrinsic Geometry.

Professor Cesàro publishes in this volume a course of lectures which he delivered at the University of Naples in 1895. Its material is drawn almost exclusively from the writings†

* The author's list of works consulted in the preparation of the memoir exhibits a useful bibliography of non-Euclidean geometry for the general reader:

Lobatschewsky, "Études géométriques sur la théorie des parallèles." —Pangeometria, Italian translation, *Giorn. di Mat.*, 1867.

Bolyai, "Appendix scientiam spatii absolute veram exhibens," Italian translation, *Ibid.*

Battaglini, "Sulla geometria immaginaria di Lobatschewsky," *Ibid.*

Beltrami, "Saggio di interpretazione della geometria non-euclidea," *Ibid.*, 1868.

Klein, "Ueber die sogenannte nichteuklidische Geometrie," *Math. Annalen*, 1871.

Flye Sainte-Marie, "Études analytiques sur la théorie des parallèles."

Cassini, Geometria rigorosa.

De Tilly, "Essai sur les principes fondamentaux de la géométrie et de la mécanique," *Mémoires de Bordeaux*, 1880.

Rouché and de Comberousse, Traité de géométrie, Note 2 of the sixth edition.

† The list of Cesàro's memoirs bearing directly or indirectly on this, one of his favorite fields of research, is an extensive one; we have space only to mention the following which contain some of the more striking

of the author and many a familiar figure of differential geometry appears here in new form. The work aims to collect and coördinate the fundamental formulæ of the intrinsic analysis of geometric forms; to present an elementary and temperate exposition of the more simple applications of the method with the purpose of bringing into light the power of this method and of maintaining its superiority over all the other processes by which the infinitesimal analysis is used to study the geometric facts of space; to exhibit the method as a uniform and expressive unifying principle whereby the developments of infinitesimal geometry are effected with elegance and agility; finally to awaken in the reader a zeal for researches similar to those which have been of such lively interest to the writer.

In all but the second of these objects the author has achieved unqualified success; the exposition is marked by the author's characteristic clearness and vigor, and by a masterful moderation in the presentation of his material made possible only by a complete mastery of the subject; but in the face of Darboux's development of the powerful kinematical method in differential geometry and of Ricci's* recently proposed absolute differential calculus in the theory of surfaces, there may be those to question the claims of any one method to superiority over all others in the field.

The book consists of seventeen chapters supplemented by three notes. I. The intrinsic discussion of a plane curve—tangent, normal, curvature, intrinsic equation, flexional points, cusps, asymptotes, asymptotic points, and circles, in a word the study by this method of the geometry of a plane curve at a point, copiously illustrated by examples. II. The fundamental formulæ for the intrinsic analysis of plane curves—Cesàro so designates the expressions

$$\frac{\delta x}{ds} = \frac{dx}{ds} - \frac{y}{\rho} + 1, \quad \frac{\delta y}{ds} = \frac{dy}{ds} + \frac{x}{\rho},$$

where the axes are respectively the tangent and normal to the curve at a point M, (x, y) the coördinates of a point P

applications of the method of intrinsic analysis: "Sulla geometria intrinseca degli spazii curvi," *Atti della R. Accademia di Napoli*, 1894; "Sulla geometria intrinseca delle congruenze," *Rendiconti della R. Accademia di Napoli*, 1894; "Teoria intrinseca delle deformazione infinitesime," *Ibid.*; "Sulla trattazione intrinseca delle questioni baricentri," *Rivista di Matematica*, vol. 1; "Formole fondamentali per l'analisi intrinseca delle curve," *Rendiconti dei Lincei*, 4th series, vol. 1; "I numeri di Grassmann in geometria intrinseca," *Ibid.*, 5th series, vol. 3. The author has omitted all reference to his own memoirs in the volume.

* Ricci, Sulla teoria delle superficie, Padova, Fratelli Drucker, 1898.

movable with M, $(x + \delta x, y + \delta y)$ the coördinates of P', the new position of P when M advances to the point M' on the curve, referred to the axes through M. These fundamental formulæ are used by the author to determine geometrical loci, the equations and properties of envelopes, parallel curves, evolutes and involutes; there is a large list of useful examples scattered through the text of this chapter. III. Important plane curves—here the method developed in the preceding chapter is applied to conics, determining asymptotes, centers, diameters, vertices, axes, foci, and curvature; the derivation of the intrinsic equation of a conic reveals incidentally Maclaurin's celebrated construction for the center of curvature of the envelope of a conic; Cassini's ovals, the curves of Ribaucour, and sinusoidal spirals receive similar treatment. IV. The theory of contact and osculation is presented in this chapter. V. The theory of roulettes is susceptible of an elegant exposition by Cesàro's method; it is amply illustrated in the fifth chapter. VI. The sixth chapter presents the theory of the barycenter of plane curves. VII. The notion of centroid which was elaborated in the preceding chapter has served as the basis of a useful method of geometrical analysis which cannot be completely expounded without entering the field of the pure intrinsic geometry; the seventh chapter develops the method of barycentric analysis and applies it to conics, to the triangular symmetric curves of La Gournerie, and to the anharmonic curves of Halphen. VIII. In this chapter devoted to systems of plane curves, the author introduces the notions, curvilinear coördinates and differential parameters, determines the relations of Lamé, derives the formula of Bonnet, and applies the method to the problems of trajectories, orthogonal systems and isothermal systems in the plane. IX. The theory of space curves and ruled surfaces is subjected to this method in the ninth chapter; this theory naturally lends itself readily to the treatment. X. Remarkable space curves are studied in the tenth chapter; among these are the spherical curves, the cylindrical helix, the geodesics of a conical surface, space circles, the curves of Bertrand, et cetera. XI. The general theory of surfaces in all its classic details is capable of a precise presentation by the method of the intrinsic geometry; the theorems of Meusnier, Bonnet, Euler, Dupin, Laguerre and Darboux, are demonstrated by Cesàro with great facility in the eleventh chapter. XII. The study undertaken in the preceding chapter is continued in this one which bears the caption exercises on surfaces; this most interesting collection of examples constitutes one of the most valuable chapters of the book. Besides a number of classic special problems the author studies in general surfaces of revolution, surfaces of constant total curvature, surfaces of constant mean curvature, quadrics and Weingarten's surfaces; the figures illustrating this chapter are especially instructive. XIII. The thirteenth chapter attacks the problem of the infinitesimal deformation of surfaces. XIV. The principal theorems relative to congruences are deduced here. XV. The fifteenth chapter studies properties of spaces of three dimensions. XVI. The method of the intrinsic analysis is extended in this chapter to the investigation of curves in hyperspace. XVII. This the last chapter of the book applies the method to determine the ordinary properties of hyperspace. The three supplementary notes are on the use of Grassmann's numbers, on the equilibrium of an inextensible flexible string, and on the equation of elasticity in hyperspace; they furnish three important applications of Cesàro's method to Grassmann's analysis and to mechanics, which could not be included in the text.

It is to be regretted that the scope of this review forbids a detailed analysis of the method and material of the chapters; such a digest, were it attempted, would be unsatisfactory. The book must be read to be appreciated, and a very readable book it is; the reader will lay the volume down with a desire to become better acquainted with the other works of this Italian mathematician, whose treatises have been succeeding each other with wonderful regularity during the last few years: the well known Corso di analisi algebrica, in 1894; the Teoria matematica della elasticita, in 1894; the Lezioni di geometria intrinseca, in 1896; the Elementi di calcolo infinitesimale* in 1899; a Manuale di aritmetica assintotica is in the press, and a volume on the Teoria matematica delle probabilita is in preparation.

III. Fontené's Memoir.

There are few better illustrations of the breadth of geometry and the specialization of it within itself than the diversity of subjects and methods exhibited by these three memoirs. Fontené's memoir eludes any detailed digest, because of its elaborate nomenclature and the complicated

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* A review of this treatise, supplemented by a short account of its predecessor and companion volume on algebraic analysis, is in preparation for the BULLETIN.

typographical character of its symbolism; however, the difficulties of the latter have been admirably met by the publishers, Gauthier-Villars et Fils. The nine chapters are as follows: I. The polytrope of reference. II. The Δ of a point and of a trope, etc. The γ of two points; the Γ of two tropes. III. Point functions and tangential functions. IV. The elements M_p^q; moments and comoments. V. Pseudo-distance, etc. VI. Intersections, etc. VII. Equation of the Δ^2 principals. VIII. Geometry about an element M_α in an element M^β. IX. Correlation; reciprocal polarity. Magnitude of figures.

The memoir is a didactic work; the reader is supposed to be in possession of no preliminary notions relative to hyperspace and the algebraic entities which form the elements of the study are concisely defined. All descriptive properties are ignored and the investigation is concerned with metric properties alone. It is interesting to remark in passing that the author has succeeded in extending rigorously to hyperspace the notion of positive sense of progression on a right line and that of positive rotation in a plane.

In real space, it is well known that in the euclidean geometry, metric properties of figures are related to the circle at infinity; in non-euclidean geometry, the circle at infinity is replaced by a sphere at infinity; this sphere at infinity is the quadric directrix of a correspondence by reciprocal polars which the author calls the metric correspondence. In place of a metric correspondence by reciprocal polars we can state a general metric correlation; in this case equal figures no longer exist and the displacement of a figure with conservation of its metric elements is no longer possible; but since it is a question of hyperspace, we can accept the notion of a space thus constituted, and proceed to examine its metric properties. The results of the author may be interpreted as giving the essential and fundamental facts relative to the latter problem; in particular his notions parameter of a ray, and parameter of an axis, characterize a space possessing a general metric correlation. But, on the other hand, whatever be the nature of a space, whether it be possessed of a general metric correlation or only of a metric correspondence by reciprocal polars, we can study in this space the metric properties of a general correlation, and these properties are independent of the nature of the space considered; they are, moreover, identical with those of a general metric correlation, and in a space where this correlation was supposed to exist it would result from the notion of homography. But this notion is of no avail when it is a question of a space possessed only of a metric correspondence by reciprocal polars ; this is the case for real space supposed non-euclidean, and it is for this reason, Fontené avers, that the metric properties of a general correlation in real space have not been studied. The author studies them in hyperspace; his theory is readily applicable to real space by introducing the notions parameter of a ray and parameter of an axis. This second interpretation of the theory of the memoir is indicated in the last chapter of the work.

EDGAR ODELL LOVETT.

PRINCETON, NEW JERSEY.

STAHL'S ABELIAN FUNCTIONS.

Theorie der Abel'schen Functionen. Von DR. HERMANN STAHL, Professor of Mathematics in the University of Tübingen. Leipzig, Teubner, 1896. 8vo, 354 pp.

DURING the last few years the literature on abelian functions has been enriched by three important treatises. The extent and scope is different in each case, so that no one of them will supplant another.

In the work under review, the author presupposes a ready knowledge of elliptic, and some familiarity with hyperelliptic functions; and an extensive knowledge of higher plane curves. His aim is to construct a serviceable bridge from the older to the newer parts of the theory, thus filling up a decided gap between the older books and the later memoirs.

The book is divided into two parts of about equal extent; the first deals with the algebraic function and its integral, while the second is concerned with inversion. Each part is then divided into four chapters. The author introduces his subject by giving a brief summary of the problems in elliptic functions, and shows how each has a natural generalization.

The first chapter is devoted to the treatment of the n branches of the algebraic function $F(x, y) = 0$, and a description of the associated Riemann's surface. The functional element is derived in the vicinity of various kinds of points, under the same restriction as was made by Clebsch and Gordan as to the nature of the branch points. Through-