period. If for a point there should be a fifth circuit of any period then there will be an infinite number of circuits of the same period. That such points exist for any particular period appears at once from a study of the number of conditions of contact and the number of parameters involved in them. The process here employed is adequate to produce the locus of points which admit circuits of any period, but for periods higher than four the eliminations become exceedingly complicated.

Many interesting phases of this problem appear by making certain transformations of the plane. For instance, a projection of the plane will convert our configuration into the more general one consisting of lines through a point and an equal number of conics through four points, each line tangent to two conics, and each conic touched by two lines. An inversion would convert the lines of the circuit into circles of a coaxial system, leaving the circles of the circuit still circles of a coaxial system. Thus our configuration would come to consist of a given number of circles of one coaxial system and an equal number of a second coaxial system, each circle of either set touching two of the other set, the whole forming a continuous chain. Our locus would then become the locus of one of the four intersection points of the two systems of circles, which moves, the other three remaining fixed, so as always to make such a chain of given period possible.

NORTHWESTERN UNIVERSITY,
July, 1898.

RECIPROCAL TRANSFORMATIONS OF PROJECTIVE COÖRDINATES AND THE THEOREMS OF CEVA AND MENELAOS.

BY PROFESSOR ARNOLD EMCH.

1. Among the great number of correspondences between certain configurations of the plane and space it is interesting and valuable to consider relations of the triangle in connection with certain surfaces. It will be seen that propositions of plane geometry interpreted in Cartesian space lead to geometrical questions of a more general character. In this paper we shall confine ourselves to the theorems of

Ceva and *Menelaos* and their connection with certain transformations of plane and space.

I. *On the theorem of Ceva.*

2. The projective coördinates* x_1, x_2, x_3 of a point P with regard to a triangle $A_1A_2A_3$ and its center of gravity E, (Fig. 1), as the unit point are defined by the ratios

$$\frac{A_3P_1}{A_2P_1}=\frac{x_2}{x_3},\quad \frac{A_1P_2}{A_3P_2}=\frac{x_3}{x_1},\quad \frac{A_2P_3}{A_1P_3}=\frac{x_1}{x_2}. \tag{1}$$

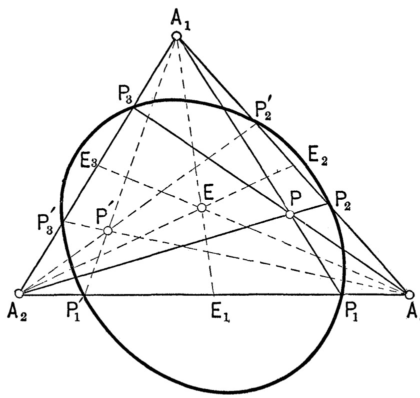

FIG. 1.

By the reciprocal transformation

$$x_1':x_2':x_3'=\frac{1}{x_1}:\frac{1}{x_2}:\frac{1}{x_3}$$

the point P is transformed into a point P' so that its coördinates are expressed by the ratios

$$\frac{x_2'}{x_3'}=\frac{x_3}{x_2},\quad \frac{x_3'}{x_1'}=\frac{x_1}{x_3},\quad \frac{x_1'}{x_2'}=\frac{x_2}{x_1}. \tag{2}$$

As this transformation is well known it will be sufficient for our purpose to state some of its most important properties.

A straight line

$$a_1x_1'+a_2x_2'+a_3x_3'=0 \tag{3}$$

* Fiedler, Geometrie der Lage, vol. 3, p. 74.

is transformed into the conic

$$a_1x_2x_3 + a_2x_3x_1 + a_3x_1x_2 = 0 \qquad (4)$$

which passes through the vertices of the triangle $A_1A_2A_3$. To the line at infinity corresponds a conic K, and it is easily seen that the conics corresponding to straight lines are hyperbolas, parabolas, or ellipses, according as these straight lines respectively intersect, touch, or do not intersect the conic K. To every straight line through a vertex corresponds a straight line through the same vertex. The six rays

$$x_1 \pm x_2 = 0, \quad x_2 \pm x_3 = 0, \quad x_3 \pm x_1 = 0, \qquad (5)$$

including the medians of the triangle, and their four points of intersection, including the center of gravity, form the invariant elements of the reciprocal transformation. In general a curve of the nth order is transformed into a curve of the $2n$th order. If a curve of the nth order passes through the vertices A_1, A_2, A_3, the corresponding curve consists of a curve of the order $2n - 3$ and the three sides of the fundamental triangle. Thus a curve of the third order

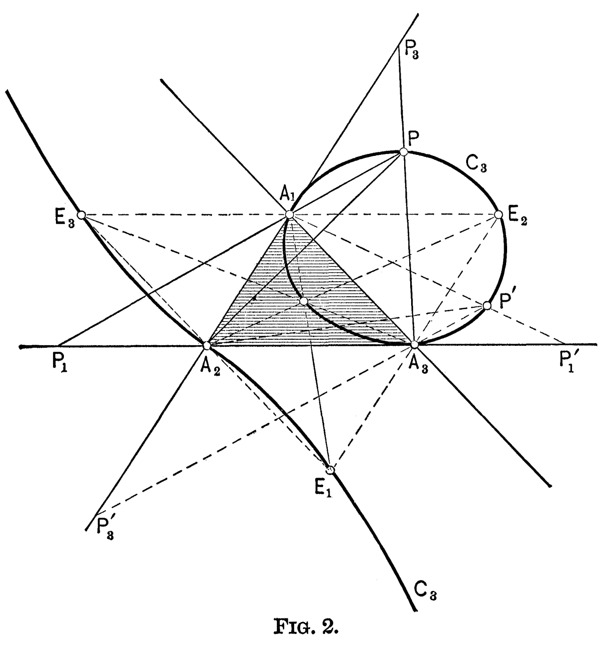

FIG. 2.

passing through the points A_1, A_2, A_3 is transformed into a curve of the third order through the same points. From the equation

$$a_1x_1(x_2^2 - x_3^2) + a_2x_2(x_3^2 - x_1^2) + a_3x_3(x_1^2 - x_2^2) = 0 \quad (6)$$

it is seen that all curves of the third order passing through the vertices A_1, A_2, A_3 and the invariant points E, E_1, E_2, E_3 (Fig. 2) are invariant. Every point P of the curve is transformed into a point P' of the same curve.

3. The relation between the plane of the triangle $A_1A_2A_3$ and space is now established by introducing the new variables $A_2P_1 = x$, $A_3P_2 = y$, $A_1P_3 = z$, and $A_3P_1 = a - x$, $A_1P_2 = b - y$, $A_2P_3 = c - z$. Considering x, y, z as the Cartesian coördinates of a point Q in space, it is evident that to every point P of the plane with the projective coördinates x_1, x_2, x_3 corresponds a point Q in space with the coördinates x, y, z. But according to the *theorem of Ceva*,* these quantities are subject to the condition (fig. 1)

$$A_2P_1 \cdot A_3P_2 \cdot A_1P_3 = A_3P_1 \cdot A_1P_2 \cdot A_2P_3.$$

Thus the theorem may be stated:

To every point of the plane corresponds a point of the surface of the third order

$$xyz = (a - x)(b - y)(c - z), \quad (7)$$

and vice versa.

From (1), the formulas expressing the transformation of the points of the plane into the points of this surface are obtained

$$x = \frac{ax_3}{x_3 + x_2}, \quad y = \frac{bx_1}{x_1 + x_3}, \quad z = \frac{cx_2}{x_2 + x_1}. \quad (8)$$

In the reciprocal transformation of the plane, to a point $P(x_1, x_2, x_3)$ corresponds a point

$$P'\left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right).$$

On the surface (7), to the point P' corresponds a point Q' with the coördinates $x' = a - x$, $y' = b - y$, $z' = c - z$, or

$$x' = \frac{ax_2}{x_3 + x_2}, \quad y' = \frac{bx_3}{x_1 + x_3}, \quad z' = \frac{cx_1}{x_2 + x_1}. \quad (9)$$

* Ceva, De lineis se invicem secantibus. Milan, 1678.

Considering the coördinates of the points Q and Q', it is easily seen that the surface is symmetrical with regard to a center S which lies on the surface and whose coördinates are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$. We have, therefore, the theorem:

The reciprocal transformation of the projective coördinates of the plane may be interpreted as a symmetrical transformation of a surface of the third order with regard to a center which lies on the surface. The surface itself is defined by the analytical expression of the theorem of Ceva concerning the transversals of a triangle.

4. The character of this transformation and the surface appears from a few simple facts. In the reciprocal transformation of the plane, to a straight line g corresponds a conic g'. In the transformations (8) and (9) of the plane into the surface, the lines g and g' are transformed into two curves of the third order on the surface. These curves evidently are symmetrical with regard to the center S and lie therefore on a conical surface whose vertex coincides with S. As the intersection of this cone and the surface consists of two curves of the third order it necessarily follows that the cone is of the second order.

Its equation has the form

$$\alpha\left(y-\frac{b}{2}\right)\left(z-\frac{c}{2}\right)+\beta\left(z-\frac{c}{2}\right)\left(x-\frac{a}{2}\right) + \gamma\left(x-\frac{a}{2}\right)\left(y-\frac{b}{2}\right)=0 \tag{10}$$

and shows that the cone passes through the lines drawn through the center S parallel to the axes X, Y, Z. *There are* ∞^2 *such cones, corresponding to the* ∞^2 *straight lines of the plane.*

In the reciprocal transformation of the plane the curve of the third order as defined by equation (6) is invariant. Its equation may also be written

$$a_1\left(\frac{x_2}{x_3}-\frac{x_3}{x_2}\right)+a_2\left(\frac{x_3}{x_1}-\frac{x_1}{x_3}\right)+a_3\left(\frac{x_1}{x_2}-\frac{x_2}{x_1}\right)=0. \tag{11}$$

Transforming this equation by means of formulas (1), the new equation results

$$a_1\left(\frac{a-x}{x}-\frac{x}{a-x}\right)+a_2\left(\frac{b-y}{y}-\frac{y}{b-y}\right) + a_3\left(\frac{c-z}{z}-\frac{z}{c-z}\right)=0. \tag{12}$$

It is the equation of a cone which intersects the surface in a curve that corresponds to the invariant curve of the third order in the plane. Excluding the singular elements due to the vertices of the fundamental triangle, this cone reduces to a plane whose equation is easily found as

$$\frac{x}{a}(a_2 + a_3) + \frac{y}{b}(a_3 + a_1) + \frac{z}{c}(a_1 + a_2) - (a_1 + a_2 + a_3) = 0. \tag{13}$$

It passes through the center of the surface, and as there are ∞^2 such planes the theorem follows:

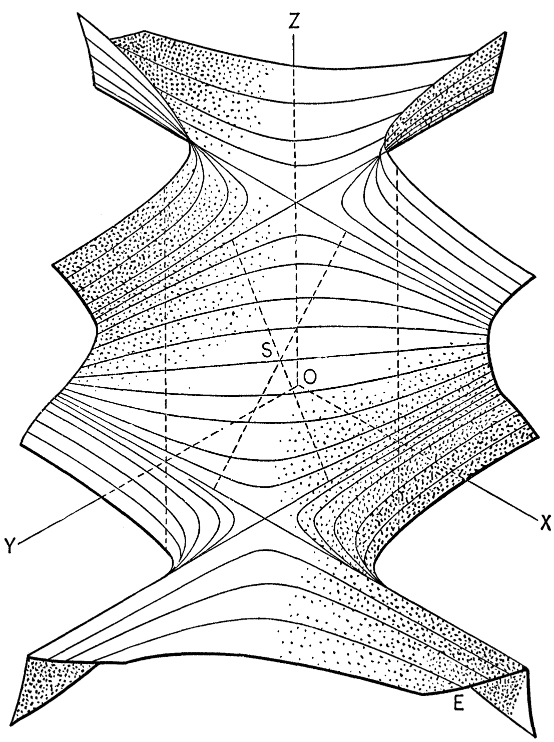

FIG. 3.

To the ∞^2 invariant curves of the third order passing through the vertices and the invariant points of the fundamental triangle in the plane correspond one by one the ∞^2 curves of the third order *formed by the intersection of all planes passing through the center S of the surface* (7).

Without entering into further details we shall confine ourselves to giving a picture of the surface in question (Fig. 3). It is obtained by tracing the hyperbolas which correspond to the straight lines in the plane passing through the vertex A_3 of the fundamental triangle.

II. *On the Theorem of Menelaos.*

5. Assuming again a fundamental triangle $A_1A_2A_3$ we can also consider the straight line as the element of the plane. Let the straight line p intersect the sides of the fundamental triangle in the points P_1, P_2, P_3 and put $A_2P_1 = x$, $A_3P_2 = y$, $A_1P_3 = z$, $A_1A_2 = a$, $A_2A_3 = b$, $A_3A_1 = c$, then, according to the theorem of Menelaos,*

$$xyz = -(a-x)(b-y)(c-z). \qquad (14)$$

If x, y, z are again considered as the coördinates of a point Q in space, it is clear that to the ∞^2 straight lines of the plane correspond ∞^2 points in space which form a surface with (14) as its equation.

This equation may also be written in the form

$$ayz + bzx + cxy - bcx - cay - abz + abc = 0, \qquad (15)$$

and represents a hyperboloid of one sheet. If (x, y, z) is a point of the surface, the same is true of the point $(a-x, b-y, c-z)$. Thus, as in the previous case, the point S with the coördinates $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ is a center of symmetry of the surface, viz., the center of the hyperboloid. In the plane of the fundamental triangle this fact is expressed by the reciprocal transformation of the projective coördinates of the straight line

$$\xi_1' : \xi_2' : \xi_3' = \frac{1}{\xi_1} : \frac{1}{\xi_2} : \frac{1}{\xi_3}. \qquad (16)$$

By this transformation, to a point of the plane corresponds a conic which touches the sides of the fundamental triangle. The line at infinity is invariant. *Thus, to a point at infinity corresponds a parabola which is inscribed in the fundamental triangle.*

The transformation of the plane into the hyperboloid is

* Found by Menelaos about 100 A. D.

made in a similar way as in the interpretation of the theorem of Ceva. It is easily found that *to a point and its corresponding conic in the plane correspond two curves of the third order on the hyperboloid which are symmetrical with regard to the center S.*

To the infinite points of the plane (pencils of parallel rays) and the parabolas inscribed to the fundamental triangle correspond the rectilinear elements of the hyperboloid.

6. From the above statement it is evident that the properties concerning the interpretation of the theorems of Ceva and Menelaos might be multiplied. In a similar manner we might interpret other relations of the triangle and thus obtain a special kind of surfaces whose properties are connected with the triangle. Such surfaces have been found by Rosace* and Steiner. Considered from this point of view, the investigations of Tucker, Taylor, McCay, Lemoine, Brocard, and others in the so-called "modern geometry of the triangle" would probably also lead to a number of valuable facts.†

BIEL, SWITZERLAND,
June, 1898.

NOTES.

A NEW list of members of the AMERICAN MATHEMATICAL SOCIETY will be published in January. Blank forms for furnishing necessary information have been sent to each member, and a full and prompt response is requested.

THE annual meeting of the London Mathematical Society was held on the evening of November 10th. LORD KELVIN consented to be nominated for the presidency, and Professors E. B. ELLIOTT and H. LAMB and Lieut.-Col. A. J. C. CUNNINGHAM for the vice-presidencies. The retiring members of the Council are Messrs. M. JENKINS and G. B. MATHEWS.

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* Rosace, p. 170 of *L'intermédiaire des mathématiciens*, No. 8, vol. 4. The equation of this reference is

$$R^2(x+y+z)(x+y-z)(x-y+z)(-x+y+z)=x^2y^2z^2.$$

The equation of Steiner's surface mentioned above is

$$m(x^2+2xy+y^2)=xy(x+y+z)(x+y-z).$$

† See "Die moderne Dreiecksgeometrie," by Dr. E. Wölffing. *Umschau*, No. 45, vol. 1.