THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 29, 1898. Thirty-six persons were in attendance at the two sessions, including the following twenty-nine members of the Society:—Professor Maxime Bôcher, Professor A. S. Chessin, Dr. J. B. Chittenden, Professor F. N. Cole, Professor T. S. Fiske, Mr. G. B. Germann, Miss Ida Griffiths, Dr. G. W. Hill, Professor Harold Jacoby, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Professor James McMahon, Mr. James Maclay, Professor E. H. Moore, Professor Frank Morley, Professor Simon Newcomb, Mr. J. C. Pfister, Professor James Pierpont, Professor M. I. Pupin, Professor J. K. Rees, Dr. Frank Schlesinger, Professor C. A. Scott, Mr. W. M. Strong, Professor H. D. Thompson, Dr. Jacob Westlund, Professor M. W. Whitney, Miss E. C. Williams, and Professor R. S. Woodward.

The President of the Society, Professor Simon Newcomb, occupied the chair during the two sessions. The Council announced the election of the following persons to membership in the Society:—Mr. Edward B. Escott, Grand Rapids, Mich.; Dr. Loring B. Mullen, Central High School, Cleveland, Ohio; Professor James Mills Peirce, Harvard University, Cambridge, Mass.; Professor Alexander Pell, University of South Dakota, Vermillion, S. D.; Professor Arthur Ranum, University of Washington, Seattle, Wash.; Mr. Alfred North Whitehead, Trinity College, Cambridge, Eng.; Mr. Walter C. Wright, Medford, Mass. Five applications for membership were received.

The following papers were presented:

(1) Professor F. MORLEY: "A regular configuration of ten line pairs in hyperbolic space."

(2) Professor R. S. WOODWARD: "The mutual gravitational attraction of two bodies whose mass distributions are symmetrical with respect to the same axis."

(3) Professor E. D. ROE: "On symmetric functions."

(4) Professor A. S. CHESSIN: "Note on the problem of three bodies."

(5) Professor MAXIME BÔCHER: "On singular points of linear differential equations with real coefficients."

(6) Professor E. O. LOVETT: "Contact transformations of developable surfaces."

(7) Dr. L. E. DICKSON: "The largest linear homogeneous group with an invariant Pfaffian."

In the absence of the authors, the papers of Professor Roe, Professor Lovett, and Dr. Dickson were read by title.

Abstracts of those papers which are not intended for publication in the BULLETIN are given below.

Professor Woodward's paper deals with certain problems in the theory of attraction which, although fairly accessible to treatment, seem to have been overlooked hitherto. Let m and m' denote any two masses having mass distributions symmetrical with respect to the same axis. Their gravitational potential is*

$$V = k\int\int\frac{dmdm'}{s} \tag{1}$$

where k is the gravitational constant and s is the distance between dm and dm'. Let the rectangular coördinates of dm and dm' be x, y, z and x', y', z' respectively, and the polar coördinates r, φ, λ and r', φ', λ' respectively, φ, φ' denoting latitudes measured from the xy-plane, and λ, λ' longitudes measured from the zx-plane. Then

$$\begin{aligned} s^2 &= (x - x')^2 + (y - y')^2 + (z - z')^2 \\ &= r^2 + r'^2 - 2rr'\cos\gamma, \qquad (2) \\ \cos\gamma &= \sin\varphi\sin\varphi' + \cos\varphi\cos\varphi'\cos(\lambda - \lambda'). \end{aligned}$$

Let the axis of z be the axis of symmetry and let the desired attraction be denoted by A. Then

$$\begin{aligned} A = \frac{\partial V}{\partial z} &= \frac{\partial V}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial V}{\partial \varphi}\frac{\partial \varphi}{\partial z} \\ &= \frac{\partial V}{\partial r}\sin\varphi + \frac{\partial V}{\partial \varphi}\frac{\cos\varphi}{r} = -\frac{\partial V}{\partial z'} \qquad (3) \\ &= -\left(\frac{\partial V}{\partial r'}\sin\varphi' + \frac{\partial V}{\partial \varphi'}\frac{\cos\varphi'}{r'}\right), \end{aligned}$$

wherein the derivatives with respect to r, φ, r', φ' are to be found by operating on $1/s$.

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* If potential is defined as in (1), then the usual form, $V = k\int dm/s$, should be called potential per unit mass.

Now it is well known that

$$\frac{1}{s} = \frac{1}{r}\left(P_0 + P_1\frac{r'}{r} + P_2\frac{r'^2}{r^2} + \cdots\right) \text{ for } r > r',$$
$$\frac{1}{s} = \frac{1}{r'}\left(P_0 + P_1\frac{r}{r'} + P_2\frac{r^2}{r'^2} + \cdots\right) \text{ for } r < r', \qquad (4)$$

in which P_0, P_1, $\cdots$ are Legendre's polynomials, or spherical harmonics. It is well known also that P_n has the form $Q_n Q_n' +$ terms multiplied by cosines of multiples of $(\lambda - \lambda')$. Hence, for the case of symmetry here considered, the only part of P_n that need be retained is $Q_n Q_n'$, since terms multiplied by cosines of $(\lambda - \lambda')$ will vanish in the integrations of (3) with respect to λ and λ', each of which has the limits 0 and 2π. Q_n and Q_n' are zonal harmonic functions of φ and φ'. They are most conveniently expressed by the formula of Rodrigues. Thus, writing for brevity

$$\xi = \sin\varphi, \quad \xi' = \sin\varphi', \qquad (5)$$

Q_n is given by

$$Q_n = \frac{1}{2^n n!}\frac{d^n}{d\xi^n}(\xi^2 - 1)^n. \qquad (6)$$

Operating on the first of (4) as specified by (3), there results

$$A = -k\int\int\frac{dmdm'}{r^2}\Sigma\, Q_n'\left\{(n+1)\, Q_n \sin\varphi - \frac{dQ_n}{d\varphi}\cos\varphi\right\}\left(\frac{r'}{r}\right)^n$$
$$= -k\int\int\frac{dmdm'}{r^2}\Sigma\, Q_n\left\{n Q_n' \sin\varphi' + \frac{dQ_n'}{d\varphi'}\cos\varphi'\right\}\left(\frac{r'}{r}\right)^n.$$

These expressions are simplified by the following relations, which are easily proved by means of (6), namely :*

$$(n+1)\, Q_n \sin\varphi - \frac{dQ_n}{d\varphi}\cos\varphi = (n+1)\, Q_{n+1},$$
$$n Q_n \sin\varphi + \frac{dQ_n}{d\varphi}\cos\varphi = n Q_{n-1}. \qquad (7)$$

* The first of these equations is given by Todhunter, History of the Theories of Attraction and Figure of the Earth, vol. 2, p. 102. The sum of (7) is a familiar expression.

Making use of these relations the attraction becomes

$$A = -k \int\int \frac{dmdm'}{r^2} \sum_{n=0}^{n=\infty} (n+1) Q_{n+1} Q_n' \left(\frac{r'}{r}\right)^n. \quad (8)$$

Denoting the densities of the element masses by ρ and ρ' respectively,

$$dm = \rho r^2 dr \cos\varphi d\varphi d\lambda,$$

$$dm' = \rho' r'^2 dr' \cos\varphi' d\varphi' d\lambda'.$$

Introducing these values and making use of the abbreviations (5), equation (8) becomes

$$A = -\frac{4}{3} k\pi^2 \int\int\int\int \Big(Q_1 Q_0' + 2 Q_2 Q_1' \frac{r'}{r} + 3 Q_3 Q_2' \frac{r'^2}{r^2} + \cdots \Big) \rho\rho' dr dr'^3 d\xi d\xi'. \quad (9)$$

The corresponding expression coming from the second of (4) is found by interchanging r with r' and Q_n with Q'_{n-1} in (9). The problem is thus reduced to one of integration with respect to the four variables r, r', ξ, ξ', the densities ρ and ρ' being in general functions of those variables.

In Professor Roe's paper symmetric functions are considered from two quite different standpoints. Part I deals with symmetric functions as a whole, and is critical and expository. Two new methods for finding symmetric functions as a whole are given, one of which depends upon the use of Aronhold's operator applied to the resultant of two binary forms, and the solution of the identical equations resulting therefrom; the other is based upon the theorem of corresponding matrices, and was stated to the author by Professor Gordan, of Erlangen. Part II treats of the isolation of the individual terms of a symmetric function together with their coefficients. The coefficient of

$$a_0^{\lambda_0} a_1^{\lambda_1} \cdots a_n^{\lambda_n} \text{ in } a_0^m \sum a_1^{\kappa_1} a_2^{\kappa_2} \cdots a_n^{\kappa_n}$$

is denoted by

$$\begin{pmatrix} 0^{\lambda_0} 1^{\lambda_1} \cdots n^{\lambda_n} \\ 0^m \kappa_1 \kappa_2 \cdots \kappa_n \end{pmatrix}.$$

Forms having

$$\lambda_0 > 0, \quad \lambda_n > 0, \quad \kappa_1 = m, \quad \kappa_n = 0,$$

are called normal forms; all other forms, the completely reducible forms excepted, may be reduced to such as are normal forms having a lower m or n. Formulas are given for the four kinds of reduction for passing from higher to lower forms, and also for the four kinds of derivation for passing from lower to higher forms. The completely reducible form has the general formula

$$\begin{pmatrix} 1^{\kappa_1-\kappa_2}2^{\kappa_2-\kappa_3}\cdots n^{\kappa_n} \\ 0^{\kappa_1}\chi_1\chi_2\cdots\chi_n \end{pmatrix}$$

and is equal to $(-1)^{\kappa_1+\kappa_2+\cdots+\kappa_n}$,

while the normal form

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\cdots n^{\lambda_n} \\ 0^m\ m^{\mu_0}\ (m-1)^{\mu_1}\cdots 0^{\mu_m} \end{pmatrix}$$

is given as the sum of simpler forms, by the formula,

$$\mu_0 x\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\cdots n^{\lambda_n} \\ 0^m\ m^{\mu_0}(m-1)^{\mu_1}\cdots 0^{\mu_m} \end{pmatrix} =$$

$$-\sum_{r=1}^{r=n}(1+\mu_r)x\begin{pmatrix} 0^{\lambda_0+1}1^{\lambda_1}2^{\lambda_2}\cdots r^{\lambda_r-1}\cdots n^{\lambda^n} \\ 0^m\ m^{\mu_0-1}(m-1)^{\mu_1}\cdots(m-r)^{\mu_r+1}\cdots 0^{\mu_m} \end{pmatrix}$$

In a paper presented at the summer meeting of the AMERICAN MATHEMATICAL SOCIETY, Professor Chessin showed that relative motion of a system may be considered as if it were absolute motion disturbed by space motion, and the author gave several expressions for what he called the *perturbative function of relative motion*. In Professor Chessin's present paper this theory is applied to a case of the problem of three bodies, m_1, m_2, m_3, namely when the mass of one of them (m_3) is infinitely small compared to the masses of m_1 and m_2, while the eccentricity of the orbits of m_1 and m_2 is zero, so that m_1 and m_2 move uniformly in concentric circles about their common center of gravity. Such is very nearly the case of a small planet in the presence of the sun and Jupiter. Such is, in particular, very approximately the case of satellites. For the case considered the undisturbed motion is that of a body (m_3) attracted to two fixed points (m_1 and m_2). This problem can be solved by means of elliptic functions. If λ and μ denote the elliptic coördinates of m_3 and φ the angle which the plane through m_1, m_2, and m_3 forms with its initial position, then we can express λ, μ, φ, and t as functions of one parameter (θ). The disturbed motion is obtained by assuming that the masses m_1 and m_2 are now revolving about their common center of gravity with the uniform angular velocity (ω). The perturbative function of relative motion is

$$\Omega = \omega \cos \varphi \left\{ f \frac{d\Delta}{dt} + \left(\frac{\lambda\mu}{c}\right)^2 \frac{d}{dt}\left(\frac{c\Delta}{\lambda\mu}\right) \right\}$$

where c denotes half the distance between m_1 and m_2; Δ the distance of m_3 from the line joining m_1 and m_2

$$\Delta = \frac{1}{c}\sqrt{(\lambda^2 - c^2)(c^2 - \mu^2)};$$

and f is a coefficient depending on the masses m_1 and m_2 and reducing to zero when these are equal. While the problem can not be solved rigorously this method affords a solution by approximation where the convergence of all the developments is established *a priori*.

In Lie's theory of contact transformations we have the theorem demonstrated by Lie, Darboux, and Mayer that the relations

$$[X_i, X_j] = [X_i, Z] = [P_i, P_j] = 0,$$

$$[P_i, Z] = \rho P_i, \qquad [P_i, X_j] \begin{matrix} = 0 \\ = \rho \end{matrix} \begin{pmatrix} i \neq j \\ i = j, \quad \rho \neq 0 \end{pmatrix}$$

where $$[F_i, F_j] = \sum_1^n {}^k \left(\frac{\partial F_i}{\partial p_k}\frac{dF_j}{dx_k} - \frac{\partial F_j}{\partial p_k}\frac{dF_i}{dx_k} \right),$$

$$\frac{dF_j}{dx_k} = \frac{\partial F_j}{\partial x_k} + p_k \frac{\partial F_j}{\partial z} = (j, k)$$

are the necessary and sufficient conditions that the $2n + 1$ equations

$$z' = Z(z, x_1, \cdots, x_n, p_1, \cdots, p_n), \; x_i' = X_i, p_i' = P_i \;\; (i = 1, \cdots, n)$$

define a contact transformation. The problem of Professor Lovett's paper is to determine the contact transformations which leave invariant the partial differential equation

$$| p_{11}, p_{22}, \cdots, p_{nn} | = 0.$$

By putting

$$F^{(i)} = F_{x_i} + p_i F_z + \sum_1^n {}^j p_{ij} F_{p_j}, \; p_i = \frac{\partial z}{\partial x_i}, \; p_{ij} = \frac{\partial^2 z}{\partial x_i \partial x_j},$$
$$(i = 1, \cdots, n)$$

and $\Phi(X_1, X_2, X_3, \cdots, X_n) = | X_1^{(1)}, X_2^{(2)}, \cdots, X_n^{(n)} |,$

we find $| P_{11}, P_{22}, \cdots, P_{nn} | = \dfrac{\Phi(P_1, P_2, \cdots, P_n)}{\Phi(X_1, X_2, \cdots, X_n)},$

in virtue of the equations

$$dZ = \sum_1^n {}^i P_i dX_i, \quad dP_i = \sum_1^n {}^i P_{ij} dX_j.$$

By developing $\Phi(P_1, \cdots, P_n)$ and equating the coefficients of $p_{11}, p_{12}, \cdots$ to zero, after an easy reduction the initial conditions give the complete systems

$$(i, 1) = 0, \quad (i, 2) = 0, \cdots, \quad (i, n) = 0; \quad (i = 1, \cdots, n).$$

The application of the method of Mayer to these systems gives

$$P_i = \varphi_i \left(\sum_1^n {}^j p_j x_j - z, p_1, \cdots, p_n\right) = \varphi_i (\zeta, p_1, \cdots, p_n)$$
$$(i = 1, \cdots, n)$$

where the functions φ_i are arbitrary. We verify easily that

$$Z = \sum_1^n {}^i \varphi_i X_i - \varphi_0 (\zeta, p_1, \cdots, p_n)$$

in which φ_0 is arbitrary. Letting

$$M_1 = (\psi_1, \cdots, \psi_n), \; \cdots, M_{n+1} = (\psi_1, \cdots, \psi_n)_{n+1}$$

be the minors of the m's in the determinant

$$\begin{vmatrix} \psi_{1\zeta} & \psi_{2\zeta} & \cdots, & \psi_{n\zeta} & m_1 \\ \psi_{1p_1} & \psi_{2p_1} & \cdots, & \psi_{np_1} & m_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{1p_n} & \psi_{2p_n} & \cdots, & \psi_{np_n} & m_{n+1} \end{vmatrix}$$

we have finally

$$X_i = \frac{(\varphi_{i+1}, \cdots, \varphi_n, \varphi_0, \cdots, \varphi_{i-1})_1 - \sum_1^n {}^j x_j (\varphi_{i+1} \cdots, \varphi_n, \varphi_0, \cdots, \varphi_{i-1})_{j+1}}{\sum_1^n {}^j x_j (\varphi_1, \varphi_2, \cdots, \varphi_n)_{j+1} - (\varphi_1, \varphi_2, \cdots, \varphi_n)_1}.$$

If C be the contact transformation whose defining functions are the above X_i, P_i, Z; Q an arbitrary point transformation; and L the transformation of Legendre as generalized by Lie it may be shown analytically and geometrically that

$$C = LQL.$$

In case the contact transformations degenerate into point transformations, Q must be projective. Among the results of the note are complete generalizations of those of a memoir of G. Vivanti, *Rend. del circ. mat. di Palermo*, vol. 5 (1891). F. N. COLE.

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CONCERNING A LINEAR HOMOGENEOUS GROUP IN $C_{m,q}$ VARIABLES ISOMORPHIC TO THE GENERAL LINEAR HOMOGENEOUS GROUP IN m VARIABLES.

BY DR. L. E. DICKSON.

(Read before the American Mathematical Society at its Fifth Summer Meeting, Boston, Mass., August 20, 1898.)

1. While the present paper is concerned chiefly with continuous groups, its results may be readily utilized for discontinuous groups.* Indeed, the finite form of the general transformation of the group is known *ab initio*. Further, the method is applicable to the construction of a linear $C_{m,q}$-ary group isomorphic to an arbitrary m-ary linear group.

2. The formula of composition of m-ary linear homogeneous substitutions

$$(a_{ij}): \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (j = 1, \cdots, m)$$

is as follows, where the matrix (a_{ij}') operates first:

$$(a_{ij}'') = (a_{ij})(a_{ij}'),$$

where

$$a_{ij}'' = \sum_{k=1}^{m} a_{ik}a_{kj}' \qquad (i, j = 1, \cdots, m).$$

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* An analogous isomorphism between certain linear groups in the Galois field of order p^n has heen discussed by the writer in an article presented to the London Mathematical Society.