J' on $m-1$ indices (a simple group if $m-1>2$) by the simple icosahedral group of degree 60.

The lowest orders of the simple groups J and J_1 are seen to be as follows:

$$\Omega_2' = 60, \quad \Omega_3' = 2^6 \cdot 3^4 \cdot 5,$$
$$\Omega_4' = 2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17, \quad \Omega_5' = 2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17,$$
$$\Omega_{3,1} = 2^6 \cdot 3^2 \cdot 5 \cdot 7 = \tfrac{1}{2}8!, \quad \Omega_{4,1} = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7,$$
$$\Omega_{5,1} = 2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31, \quad \Omega_{3,2} = 2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17,$$
$$\Omega_{4,1} = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2, \quad \Omega_{3,3} = 2^{18} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 13 \cdot 73.$$

Denote by (m, n, p) the order of the simple* group of linear fractional substitutions of determinant unity on $m-1$ indices in the $GF[p^n]$. We thus find

$$\Omega_{3,1} = (4, 1, 2) = (3, 2, 2), \quad \Omega_{3,2} = (4, 2, 2).$$

UNIVERSITY OF CALIFORNIA,
March 8, 1898.

ON THE HAMILTON GROUPS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

ACCORDING to Dedekind a Hamilton group is a non-Abelian group all of whose subgroups are self-conjugate.† If the order of such a group is $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots$ ($p_1, p_2, p_3, \cdots$ being prime numbers) it must be the direct product of its subgroups of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, p_3^{\alpha_3}, \cdots$ since each of these subgroups is self-conjugate and no two of them can have any common operator except identity.‡ Each of these subgroups is either Abelian or Hamiltonian. We proceed to prove that one of the given prime numbers must be 2 and that every subgroup whose order is a power of any other prime number must be Abelian.

Suppose that G represents a Hamilton group of order p^n, p being an odd prime number. We may evidently select a in such a manner that all the operators of G whose orders

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* Dickson: *Annals of Mathematics*, 1897, p. 136.
† *Mathematische Annalen*, vol. 48 (1897), p. 549.
‡ Cf. Dyck: *Mathematische Annalen*, vol. 22 (1883), p. 97.

do not exceed $p^{\alpha-1}$ ($\alpha > 1$) are commutative to every operator of G but that there are some operators of order p^α which do not possess this property. Let s_1 be one of the latter operators and let s_2 be an operator of G that is not commutative to s_1, S_1 and S_2 representing the cyclical groups generated by these operators respectively. We shall first suppose that s_2 is also of order p^α; S_1 and S_2 must contain at least p common operators since s_1 and s_2 are not commutative.

S_1 and the subgroup of order $p^{\alpha-1}$ which is contained in S_2 generate an Abelian group whose order is equal to the order divided by p of the group generated by s_1, s_2. This Abelian group must therefore contain all the operators of the latter group whose orders are less than p^α. By hypothesis $s_1s_2s_1^{-1} = s_2^\beta$, where β differs from unity. Hence

$$(s_2s_1)^{p^{\alpha-1}} = s_2s_2^\beta s_2^{\beta^2}s_2^{\beta^3} \cdots s_2^{\beta^{p^{\alpha-1}}-1} s_1^{p^{\alpha-1}} = s_2^{\frac{\beta^{p^{\alpha-1}}-1}{\beta-1}} s_1^{p^{\alpha-}}$$

$\beta = xp^{\alpha-1} + 1$ where x is one of the $p-1$ numbers, 1, 2, 3, $\cdots$, $p-1$.
Hence

$$\frac{\beta^{p^{\alpha-1}} - 1}{\beta - 1} = \frac{(1 + xp^{\alpha-1})^{p^{\alpha-1}} - 1}{xp^{\alpha-1}} \equiv p^{\alpha-1} \mod p^\alpha.$$

From this it follows that if s_1 remains fixed while s_2 takes all its possible values in S_2 the orders of some of the operators of the form s_2s_1 must be less than p^α. As this is impossible every operator of order p^α must be commutative to every other operator of this order.

It remains to consider the case where the order of s_2 exceeds p^α. In this case S_2 and the subgroup of order $p^{\alpha-1}$ which is contained in S_1 generate an Abelian group. The order of each of the operators of the group generated by s_1, s_2 which is not contained in the Abelian group must exceed $p^{\alpha-1}$. Since the subgroup of order p^α which is contained in S_2 has at least p operators in common with S_1 it follows that the product of s_1 into some operator of order p^α contained in S_2 must be of an order which does not exceed $p^{\alpha-1}$. Hence every operator of order p^α that is contained in G is commutative to every operator of G provided every operator of order $p^{\alpha-1}$ has this property, *i. e., there is no Hamilton group of order p^n, p being an odd prime number.* *

* This result could have been derived directly from the fact that the order of the commutator of two operators of a Hamilton group cannot exceed 2, which has been proved by Dedekind, loc. cit., p. 557. It seemed desirable to give an independent proof of it since this proof prepares the way for what follows.

Hamilton groups of order 2^α.

By definition every operator of order 2 that is contained in such a group H is self-conjugate. We proceed to prove that every operator of order 4 that is contained in H is commutative to every operator whose order exceeds 4. If this is not the case we select an operator s_1 of order 4 that is not commutative to some operator s_2 of order 2^β $(\beta > 2)$ and we let S_1, S_2 represent the cyclical groups which these two operators generate respectively. S_1 and S_2 have two common operators, otherwise each operator of the one would be commutative to every operator of the other. Hence the order of the subgroup generated by s_1, s_2 is $2^{\beta+1}$. For convenience we shall call S_2 the head of this subgroup and its remaining 2^β substitutions we shall call its tail. Since the head is an Abelian group each of its operators must be either commutative to every operator of the tail or it must transform all the operators of the tail according to some regular substitution.

If s_1 and $s_2^{2^{\beta-2}}$ were commutative there would be some operators of order 2 in the tail of the given group. This is impossible since none of these operators is commutative to s_2. Hence s_2 must transform all the operators of this tail according to a regular substitution of order $2^{\beta-1}$. As this tail includes s_1 the order of this substitution cannot exceed 2. Hence this is impossible, *i. e.*, an operator of order 4 that is contained in H must be commutative to every operator whose order exceeds 4 if such operators can occur at all.

Suppose H contains two non-commutative operators s_3, s_4 of order 2^β $(\beta > 2)$ and that all the operators of this order are commutative to every operator of a lower order. S_3, S_4 will be used to represent the groups generated by s_3, s_4 respectively. These groups have at least two common operators. The Abelian group generated by S_4 and s_3^2 will be called the head of the group generated by s_3, s_4 and its remaining operators will be called the tail. This tail can evidently contain no operator whose order is less than 2^β. We may suppose that $s_3 s_4 s_3^{-1} = s_4^\gamma$ where γ differs from unity. Hence

$$(s_4 s_3)^{2^{\beta-1}} = s_4 s_4^\gamma s_4^{\gamma^2} \cdots s_4^{\gamma^{2^{\beta-1}-1}} s_3^{2^{\beta-1}} = s_4^{\frac{\gamma^{2^{\beta-1}}-1}{\gamma-1}} s_3^{2^{\beta-1}}$$

where

$$\gamma = 2^{\beta-1} + 1$$

$$\frac{\gamma^{2^{\beta-1}}-1}{\gamma-1}=\frac{(1+2^{\beta-1})^{2^{\beta-1}}-1}{2^{\beta-1}}\equiv 2^{\beta-1} \mod 2^{\beta}.$$

Hence the order of s_4s_3 does not exceed $2^{\beta-1}$. As this operator occurs in the tail S_3 and S_4 must be commutative if they occur in H.

Suppose that s_3 is non-commutative to an operator s_5 whose order exceeds 2^{β}. The Abelian group generated by s_5 and s_3^2 will be called the head of the group generated by s_3, s_5. Since s_3 is commutative to the subgroup of order 2^{β} contained in the group generated by s_5 and since the $2^{\beta-1}$ power of s_3 is contained in this subgroup the tail of the group generated by s_3, s_5 would contain operators whose order is less than 2^{β}. Hence we see that if H contains any operators whose order exceeds 4 each of these operators is commutative to every operator of H.

H must, therefore, contain some operator s_6 of order 4 that is not commutative to each one of its operators. The operators which are commutative to s_6 will be called the head of H and the rest of its operators will be called its tail. All the operators of the tail are of order 4 and the square of each one is equal to s_6^2, otherwise such an operator would be commutative to s_6*. The given head of H contains some operators that are commutative to every operator of H. These form a subgroup which includes all the operators of order 2. We proceed to prove that this subgroup does not include any operator of order 4.

If it included an. operator of order 4, the group of order 16 generated by s_6 and this operator of order 4 would contain operators of order 4 that are not commutative to the operators of the tail of H and whose squares would differ from the squares of s_6. This is clearly impossible. Hence *H does not include any operator whose order exceeds* 4 *nor does it contain any operator of order* 4 *that is commutative to all its operators.*

We have seen that all the operators in the given tail of H have the same square. From what has just been proved each operator of order 4 may occur in a tail. Hence we see that *the square of every operator of order* 4 *that is contained in H is equal to the square of a given one, i. e.*, all of these operators have the same square. This operator generates the commutator subgroup of H.

If we multiply s_6 by the subgroup which contains all the

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* Dyck : loc. cit.

operators of order 2 we obtain an Abelian group which contains all the operators that are commutative to s_6. If this were not the case let s_7 be some other operator that is commutative to s_6. Since s_7 has the same square as s_6 its product into s_6 gives an operator of order 2. As this is impossible, the order of the given subgroup which contains all the operators of H that are commutative to every one of its operators must be $2^{\alpha-2}$. Hence H is completely determined when its order is given. If we add an operator of order 2 that is commutative to every operator of a Hamilton group of order 2^α to such a group we clearly obtain a Hamilton group of order $2^{\alpha+1}$. Hence there is one and only one Hamilton group of order 2^α whenever α exceeds 2. It is the direct product of the Abelian group of order $2^{\alpha-3}$ which contains no operator whose order exceeds 2 and the quaternion group.

Summary.

1. The order of every Hamilton group is even.

2. A Hamilton group of order $2^\alpha p_1^{\alpha} p_2^{\alpha_2} \cdots$ ($p_1, p_2, \cdots$ being prime numbers) is the direct product of its subgroups of orders 2^α, $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $\cdots$. The first one of these subgroups is Hamiltonian and all the others are Abelian. The entire group may be represented as an intransitive substitution group in which each of these subgroups is represented by a distinct set of letters.

3. There is one and only one Hamilton group of order 2^α ($\alpha > 2$). This group contains $2^{\alpha-2}$ operators of order 2. The rest of its operators are of order 4 and the squares of all of these operators are equal to each other. Every operator of order 4 is commutative to just one-half of all the operators of the group. The group is isomorphic to the 4-group with respect to the given subgroup of order $2^{\alpha-2}$.

4. If we multiply a Hamilton group of order 2^α by any Abelian group of an odd order all of whose operators are commutative to every operator of the given Hamilton group, the product will be a Hamilton group and every possible Hamilton group can be constructed in this manner.

5. If two operators of a Hamilton group are not commutative they may be represented as two substitutions in which the cycles whose orders are divisible by 4 are of order 4, transform each other into their 3d powers, and belong to the subgroup of order 2^α. The letters which are found in these cycles do not occur in any of the remaining cycles of these two substitutions. The remaining cycles of these

substitutions are commutative. Hence the commutator of two such operators is of order 2.*

6. A Hamilton group of order 2^{α} contains $2^{2\alpha-6}$ quaternion groups as subgroups. All of these have the commutator group of the entire group in common.†

CORNELL UNIVERSITY,
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NOTE ON THE INFINITESIMAL PROJECTIVE TRANSFORMATION.

BY PROFESSOR EDGAR ODELL LOVETT.

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It is proposed here to find the form of the most general infinitesimal projective transformation‡ of ordinary space directly from its simplest characteristic geometric property. Geometrically, infinitesimal projective transformations of space are those infinitesimal point transformations which transform a plane into a plane, *i e.*, which leave invariant the family of ∞^3 planes of ordinary space. Analytically, then, the most general infinitesimal projective transformation is the point transformation

$$Uf \equiv \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z} \quad (1)$$

which leaves invariant the partial differential equations

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* Cf. Dedekind: loc. cit.

† Cf. Miller: *Comptes Rendus*, vol. 126 (1898), pp. 1406–1408.

‡ In a note on the general projective transformation, *Annals of Mathematics*, vol. 10, No. 1, the forms of the finite projective transformations of ordinary space and those of n-dimensional space are found directly from the conditions for the invariance of the equations $y'' = 0$, $z'' = 0$, which expresses the geometric property that straight line is changed into straight line by these transformations. The form of the general infinitesimal projective transformation of ordinary space is deduced from the finite transformation by the method of Lie. In this derivation three steps are made to intervene, two of which are removed and the other replaced by a simpler one by the method of the present note: 1° two intersecting planes producing the straight line and its property of invariance; 2° the ordinary differential equations of the straight line and the conditions for their invariance; 3° the finite forms of the transformation.