The cylinder and cone may be treated by similar methods.

The preceding theorems suffice to show the validity of Euclid's constructions. It remains to show that the quadratic space is the least space which will suffice for these constructions, and that the restriction of AC and BC to quadratic values is not arbitrary. The rational operations are all required in the addition and subtraction of lines, and the division of lines into any number of equal parts; the operation of extracting any square root is equivalent to finding a mean proportional and is therefore essential. Moreover any point of the space can be reached by a finite number of operations. The reason that AC and BC should be restricted to quadratic values is: the function of C is to determine a plane through AB, while AB is the fundamental distance; if C is assumed to be a point of the space we ought to be able to reach C by Euclid's constructions, starting from A and B.

The antithesis of the quadratic space is a space consisting of the points whose rectangular coördinates, in a certain system, are transcendental quantities. It has an infinitely greater number of points than the quadratic space, their multiplicity being of the order of the continuum. Moreover every line or surface of continuous space, except certain cases of a plane, or lines in a plane, parallel to one of the coördinate planes is also a line or surface of this space. We might therefore expect all Euclid's constructions to be possible in it; such, however, is not the case, as we have already mentioned. Two striking peculiarities of this space are: two circumferences may intersect in a single point; a circumference may have no center.

YALE UNIVERSITY,
April, 1898.

NIEWENGLOWSKI'S GEOMETRY.

Cours de Géométrie analytique, par B. NIEWENGLOWSKI; avec une *Note sur les transformations en géométrie*, par EMILE BOREL. Paris, Gauthier-Villars, 1894–96. Vol. I., 483 pp; Vol. II., 292 pp; Vol. III., 572 pp.

THE work before us, of which the first two volumes deal with plane analytic geometry, the third with analytic geometry of three dimensions, is intended for boys in the *Classes de mathematiques speciales* at the French *Lycées*, at one of the best known of which (the Lycée Louis-le-

Grand) the author was professor at the time he published the first two volumes. The work in mathematics done by these boys covers in a general way, although much more thoroughly, the same ground as is covered by an American boy who is devoting a good deal of attention to mathematics in the first two or three years of his college course. The geometrical portion of this work is included in the three volumes before us together with a good deal of matter not required in the examinations for which these boys are preparing themselves; the subjects of trilinear point and line coördinates for instance being briefly treated.

The book resembles in a general way in regard both to the subjects taken up and to the methods of treatment the smaller text-book of Briot and Bouquet. The elementary nature of the treatment throughout the whole of the three volumes is perhaps their most characteristic feature. Indeed we doubt if an equally extensive treatise on analytic geometry can be named which is so accessible in all its parts to the comparatively immature student. The somewhat different training in algebra and the calculus which French boys receive tends at times to obscure the fact just referred to, but the use which is constantly made of the notation of partial differentiation and the frequent use of the theorems concerning the representation of a quadratic form as a sum of squares (to mention two examples) really involve, as here applied, nothing beyond the most elementary principles.

It may be asked how the author succeeds in filling 1260 pages if he confines himself to subjects of this elementary character. The answer is that he considers a very large number of subjects usually left aside as of minor importance, while in the case of many of the more important questions two or even three methods of treatment are given. Moreover some subjects of importance which are usually postponed until later are introduced here in an elementary manner; for example unicursal curves* are treated in such a way that the young student can get a real grasp of the ideas involved; while such subjects as surfaces of translation (vol. III., p. 159) and anallagmatic curves (vol. I., p. 369, vol. II., p. 136 and p. 161) are touched upon.

* By an unfortunate mistake the author defines (vol. II., p. 99) a plane curve as unicursal if the coördinates (x, y) of its points can be expressed as *single valued* functions of a parameter. According to a remarkable theorem of Poincaré not only every algebraic but every analytic curve satisfies this definition. It should have been required that x and y be *rational* functions of the parameter. In point of fact the author restricts himself to this case throughout the chapter.

Throughout this treatise theorems and methods are from time to time referred to their discoverers. It would be unfair to expect that in a book of this grade much space should be devoted to such historical references. It seems, however, doubtful whether much is gained by the mere mention of the name of the discoverer of a theorem without any reference to the place where the theorem was published * and this is all that is usually given except in the case of the more elementary French mathematical journals. This method of vague references is not peculiar to the present work, but is far too much in vogue in many otherwise admirable recent French treatises. It strikes one as strange to find that in a chapter on quaternions, even though it contain only two pages, Hamilton's name is not mentioned.

The subject of invariants is repeatedly touched upon ; both invariants with regard to the general projective group and invariants with regard to certain subgroups of the projective group being considered. Of these last the invariants with regard to rigid motions (or if one prefers invariants with regard to changes from one system of rectangular coördinates to another) form such an admirable introduction to the subject of invariants that we could wish that a more extended treatment of them had been given. Covariants can also be readily introduced in the same way. Thus, for instance. the first member of the equation of the principal axes of the general conic is a covariant with regard to rigid motions. Moreover the identical vanishing of this covariant gives the conditions that the conic is a circle. Thus we get an elementary illustration of the fact that two or more conditions are usually expressed not by the vanishing of two or more invariants, but by the identical vanishing of a covariant.

It is, perhaps, worth while to call attention to the notation used here and in other French books for the general equation of the second degree, viz :

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

We regret that this notation is not used in English and American books rather than the awkward notation :

$$ax^2 + 2hyx + by^2 + 2gx + 2fy + c = 0.$$

This latter notation is based on the idea that the x^2, the y^2,

* The theorem that if three conics have a common chord their three opposite common chords meet in a point is ascribed on p. 263 of vol. II. to Sturm. Plücker gave this theorem in 1828 in his Analytisch-geometrische Entwickelungen, vol. I., p. 257 and elsewhere. I have not been able to find this theorem in any of the papers of Sturm published before this date.

and the constant terms on the one hand, and the terms in xy, in x, and in y on the other hand, are related. This is true from the point of view of projective geometry, but there it is desirable to use the symmetrical form $\sum a_{ij}x_ix_j = 0$. It is not true from the point of view of elementary, *i. e.*, metrical geometry.

The work before us forms a compendium which will be found useful by any teacher of elementary courses in analytic geometry who is not satisfied to repeat year after year the same course. Such a teacher will find in these volumes material which will make it possible for him to continually change the details of his courses without introducing into them either matter or methods which are beyond his pupils. An index would form a valuable addition to the book.

We come finally to the *Note* on geometric transformations contributed by M. Borel. This note covers the last 78 pages of the third volume. Although it presupposes no more mathematical knowledge than the rest of the book it will be found profitable reading only for the student of distinct mathematical ability. To such a student, however, it gives an inspiring outlook into some parts of modern mathematics. The ideas here developed are more or less intimately associated with Lie's name, some of them being actually due to him. We will close by giving a brief summary of this admirable *Note* reminding our readers merely that it is impossible for M. Borel to do much more than touch upon the various subjects mentioned.

The *Note* is divided into eight sections. In Section I the fundamental ideas concerning transformations and groups of transformations are explained. Section II is devoted to projective transformations in one, two and three dimensions. Some of the principal subgroups are considered and a few examples of invariants with regard both to the general projective groups and to the subgroups are taken up. Even differential invariants are touched upon. Section III, while touching on point transformations in general and birational transformations, is devoted almost entirely to the subject of inversion and the group generated by motions, reflexions in planes, and inversions. Section IV contains an explanation of two important systems of spherical coördinates: Darboux's pentaspheric coördinates which give the most convenient analytical means of studying the group last mentioned, and Lie's more general system which leads up naturally to a group of contact transformations whose consideration is however postponed until Section VII. In Section V the transformation by reciprocal polars is first taken up, then the general correlation is considered and finally the group consisting of all projective transformations and all correlations in space of three dimensions is treated by means of line coördinates. In Section VI the general subject of contact transformations is introduced and in Section VII a special contact transformation due to Lie is discussed. Finally in Section VIII transformations in space of more than three dimensions are considered, considerable attention being paid to the relation between space of five dimensions and line geometry on the one hand and sphere geometry on the other hand.

MAXIME BÔCHER.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

GOURSAT'S PARTIAL DIFFERENTIAL EQUATIONS.

Leçons sur l'intégration des équations aux dérivées partielles du second ordre a deux variables indépendantes. Par E. GOURSAT, Professeur de calcul différentiel et intégral à l'Université de Paris.

Tome I. *Problème de Cauchy.—Caractéristiques.—Intégrales intermédiaires.* Paris, A. Hermann, 1896. 8vo, viii + 226 pp.

Tome II. *La méthode de Laplace.—Les systèmes en involution.—La méthode de M. Darboux.—Les équations de la première classe.—Transformations des équations du second ordre.—Généralisations diverses.* Paris, A. Hermann, 1898. 8vo, i + 344 pp.

THESE two volumes constitute a fitting sequel to the author's volume* and that of Mansion† on partial differential

* Goursat: "Leçons sur l'intégration des équations aux dérivées partielles du premier ordre, rédigées par C. Bourlet." Paris, A. Hermann, 1898. 8vo, 354 pp. The subjects studied and their order of development in this volume may be of interest here in connection with the above volumes of the series. They are as follows: Théorèmes généraux sur l'existence des intégrales.—Équations linéaires. Systèmes complets.—Équations linéaires aux différentielles totales.—Équations de forme quelconque. Généralités. Méthode de Lagrange et Charpit.—Méthode de Cauchy. Caractéristiques.—Définition des expressions (ϕ, ψ) et $[\phi, \psi]$. Première méthode de Jacobi.—Méthode de Jacobi et Mayer.—Méthode de Lie.—Étude géométrique des équations a trois variables. Courbes intégrales. Solutions