Consider those values of w that yield values of z for which $F(z)$ is defined, and for which then $F(z)$ is a function of w. These values of $F(z)$ do not constitute an analytic function of w; for the domain of values of w consists of two separate continua. Thus the theorem, unrestricted, would be false in this case. *

HARVARD UNIVERSITY,
April, 1898.

NOTE ON POISSON'S INTEGRAL.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

THE following treatment of Poisson's integral in two dimensions seems to the writer to have at least one advantage over the treatments ordinarily given; viz., that it involves no artifice.

Given a function $V(x, y)$ which within and upon the circumference of a certain circle C is a continuous function of (x, y) and within C is harmonic (*i. e.*, has continuous first and second derivatives and satisfies Laplace's equation). By a well-known theorem of Gauss the value of V at the centre (x_0, y_0) of C is the arithmetic mean of its values on the circumference.† That is, if we denote by V_c the values of V on the circumference and by φ the angle at the centre,

$$(1) \qquad V(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} V_c d\varphi.$$

This theorem may be immediately generalized by the method of inversion, if we remember on the one hand that a harmonic function remains harmonic after inversion, and on the other hand that angles are unchanged by inversion and that circles invert into circles. We thus get the theorem:

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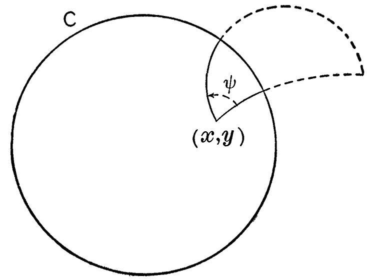
* Burkhardt has given simple examples of multiple-valued functions for which the unrestricted theorem is false. See his book: "Einführung in die Theorie der analytischen Functionen einer complexen Veränderlichen," vol. 1, Leipzig, 1897; p. 198.

† An elementary proof of this theorem will be found in a paper by the writer on p. 206 of the BULLETIN for May, 1895.

If (x, y) *is any point within the circle* C,

$$(2) \qquad V(x, y) = \frac{1}{2\pi}\int_0^{2\pi} V_c d\psi,$$

where ψ *is the angle measured from a fixed circle through* (x, y) *which cuts* C *orthogonally to a variable circle of the same sort.*



The formula (2) gives the value of V at any point within C in terms of the values on the circumference, and is, in fact, nothing but Poisson's integral in a somewhat unfamiliar form. To reduce it to the ordinary forms we will introduce the element of arc ds of C and write (2) in the form:

$$(3) \qquad V(x, y) = \frac{1}{2\pi}\int V_c \frac{\partial \psi}{\partial s} ds.$$

ψ may be regarded as a function of the coördinates (x', y') of a point on the variable circle above referred to. Thus regarded it is an (infinitely multiple valued) solution of Laplace's equation whose conjugate function is $\log \rho_1 - \log \rho$ where ρ is the distance from (x', y') to (x, y) and ρ_1 the distance from (x', y') to the inverse of (x, y) with regard to C. If then n denote the interior normal of C we have:

$$(4) \qquad V(x, y) = \frac{1}{2\pi}\int V_c \frac{\partial [\log \rho_1 - \log \rho]}{\partial n} ds.*$$

Other forms can be obtained by computing $\frac{\partial \psi}{\partial s}$ directly as a function of (x, y, x', y').

We proceed now to the theorem:

Given a continuous function V_c *upon the circumference of the circle* C; *the function* $V(x, y)$ *defined by* (2) *throughout the interior of* C *is harmonic throughout* C *and joins on continuously to the values* V_c *on the circumference.*

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* Cf. Picard: Traité d'Analyse, vol. 2, p. 16.

That $V(x, y)$ thus defined is harmonic follows at once from (3) since $\frac{\partial \psi}{\partial s}$ is easily seen, either by direct computation or from its value:

$$\frac{\partial}{\partial n}[\log \rho_1 - \log \rho]$$

to be a harmonic function of (x, y).

For the proof of the second part of the theorem formula (2) is particularly adapted. We have here to prove that if (x, y) approaches a point P on the circumference $V(x, y)$ approaches as its limit the value of V_c at P. The idea upon which this proof rests is that when (x, y) is near to P a small arc including P corresponds to a large range of values of ψ and, therefore, when we take the arithmetic mean as indicated in (2) the value of V_c at P will predominate.* The exact proof based upon the idea just stated merely requires the writing down of a few inequalities.

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ON THE POLYNOMIALS OF STIELTJES.

BY PROFESSOR E. B. VAN VLECK.

(Read before the American Mathematical Society at the Meeting of April 30, 1898.)

BY a Stieltjes polynomial will here be understood any polynomial satisfying a regular linear differential equation of the second order

$$\frac{d^2y}{dx^2} + \left(\frac{1-\lambda_1}{x-e_1} + \cdots + \frac{1-\lambda_r}{x-e_r}\right)\frac{dy}{dx} + \frac{\varphi(x) = A_0x^{r-2} + A_1x^{r-3} + \cdots + A_{r-2}}{(x-e_1)\cdots(x-e_r)}y = 0 \tag{I}$$

in which the singular points $e_1, \cdots, e_r, \infty$ are real and in which also r exponent-differences $\lambda_1, \cdots, \lambda_r$ are (algebraically) less than unity. We shall here for the most part confine our

* It will be seen that this idea is similar to that suggested by Schwarz. (Ges. Werke, vol. 2, p. 360. See also Klein-Fricke: Modulfunctionen, vol. 1, p. 512.) We avoid, however, the artificiality of Schwarz's method.