THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Saturday, February 26. As has now become the established rule the meeting extended through a morning and an afternoon session. This arrangement is not only necessary for the adequate presentation of the large and increasing number of papers offered, but is also generally commended as affording to those present an agreeable opportunity for scientific and social intercourse.

The attendance at the meeting included the following twenty-six members of the Society: Professor E. W. Brown, Professor F. N. Cole, Professor A. M. Ely, Professor Irving Fisher, Professor T. S. Fiske, Mr. G. B. Germann, Professor James Harkness, Mr. P. R. Heyl, Professor Harold Jacoby, Mr. S. A. Joffe, Mr. C. J. Keyser, Professor Pomeroy Ladue, Professor Gustave Legras, Dr. Emory McClintock, Mr. James Maclay, Professor W. F. Osgood, Professor A. W. Phillips, Professor James Pierpont, Professor J. K. Rees, Mr. P. L. Saurel, Professor C. A. Scott, Mr. W. M. Strong, Professor H. D. Thompson, Professor J. M. Van Vleck, Miss E. C. Williams, and Professor R. S. Woodward.

The First Vice-President of the Society, Professor R. S. Woodward, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. Henry Frederick Baker, St. John's College, Cambridge, England; Mr. W. C. Breuke, University of Illinois, Champaign, Ill.; Mr. E. H. Comstock, Cornell University, Ithaca, N. Y.; Professor Edwin Bailey Elliott, Magdalen College, Oxford, England; Professor C. H. Hinton, University of Minnesota, Minneapolis, Minn.; Professor W. E. Stilson, McKendree College, Lebanon, Ill. Six applications for membership were received. The report of the auditing committee was presented and accepted.

The following papers were presented:

(1) Mr. P. L. SAUREL: "On integrating factors."

(2) Professor MAXIME BÔCHER: "The theorems of oscillation of Sturm and Klein (second paper)."

(3) Dr. J. I. HUTCHINSON: "On the tetrahedroid."

(4) Professor W. F. OSGOOD: "A new proof of the existence of a solution of the differential equation $dy/dx = f(x, y)$, the Cauchy-Lipschitz condition not being imposed."

(5) Dr. M. B. PORTER : "On the roots of Bessel's functions."

(6) Professor JAMES PIERPONT : "The early history of the Galoisian theory of equations."

(7) Mr. P. R. HEYL : "The measure of the bluntness of the regular figures in four dimensional space."

(8) Mr. JAMES MACLAY : "Certain double minimal surfaces."

(9) Dr. G. A. MILLER : "On an extension of Sylow's theorem."

(10) Professor H. S. WHITE : "Inflexional lines, triplets, and triangles associated with the plane cubic curve."

(11) Professor W. H. METZLER : "On the excess of the number of combinations in a set which have an even number of inversions over those which have an odd number."

(12) Professor R. S. WOODWARD : "On the significance of the product of the gravitational constant and the mean density of the earth."

In the absence of the authors, the papers of Professor Bôcher, Dr. Hutchinson, Dr. Miller, Professor White, and Professor Metzler were read by title. Dr. Porter's paper was presented by Professor Osgood.

The papers of Dr. Porter and Professor White have already appeared in the March number of the BULLETIN. Those of Mr. Saurel, Dr. Hutchinson, Professor Pierpont and Dr. Miller are contained in the present issue. Those of Professor Bôcher and Mr. Maclay will appear in subsequent numbers. Abstracts of the other papers are given below.

Professor Osgood's paper deals with the fundamental existence theorem of the theory of differential equations. Riemann's definition of the definite integral led to the proof of the existence of a function

$$\varphi(x) = b + \int_a^x f(x)dx$$

which, in case $f(x)$ is continuous—the most general case that concerns us here—satisfies the differential equation $dy/dx = f(x)$ and assumes for $x = a$ the arbitrarily preassigned value b. There exists only one such function. Cauchy showed that the differential equation $dy/dx = f(x, y)$, where $f(x, y)$ is continuous in the neighborhood of the point $x = a$, $y = b$, admits a solution, $y = \varphi(x)$, which assumes the value b when $x = a$, provided $\frac{\partial f(x,y)}{\partial x}$ is finite in the neighborhood of the point (a, b). The latter condition was generalized by Lipschitz so that it reads: $|f(x, y') - f(x, y)| < k|y' - y|$, k being a constant. In the present paper Professor Osgood shows that by extending what is in substance Riemann's method of proof of the existence of the limit in question to the case of the more general differential equation $dy/dx = f(x, y)$, only the continuity of $f(x, y)$ being assumed, a proof can be obtained that this differential equation has at least one solution taking on the value b when $x = a$. In case more than one such solution exists, any solution $\varphi(x)$ will lie between a certain pair of solutions—the "maximum solution" $\varphi_1(x)$ and the "minimum solution" $\varphi_2(x)$, *i. e.*, $\varphi_2(x) \leqq \varphi(x) \leqq \varphi_1(x)$, and the belt between $\varphi_1(x)$ and $\varphi_2(x)$ will be completely filled with solutions. Finally, a sufficient condition is obtained that $\varphi_1(x)$ and $\varphi_2(x)$ coincide, *i. e.*, that only one solution exists, which is more general than the Cauchy-Lipschitz condition. The condition is as follows; Let $\varphi(u)$ be any continuous function of u such that $\varphi(u) = 0$; $\varphi(u) > 0$ if $u > 0$; $\varphi(-u) = \varphi(u)$; and

$$\lim_{e=0} \int_e^{u_0} \frac{du}{\varphi(u)} = \infty, \quad u_0 > 0, e > 0.$$

For example, $\varphi(u) = k|u|$, or $k|u| \log \frac{1}{|u|}$, or $k|u| \log \frac{1}{|u|} \log \log \frac{1}{|u|}$, k being a constant. Then the Cauchy-Lipschitz condition, which consists in assuming $\varphi(u) = k|u|$, may be replaced by the following: $|f(x, y) - f(x, y')| < \varphi(y - y')$. The method employed in this paper was geometric, but the final proofs have all been reduced to arithmetic form.

Mr. Heyl obtained numerical values for the magnitude of the angles of the regular figures in four dimensional space. As the bluntness of a regular solid is the measure of the solid angle at its corner, so the bluntness of a regular four-fold figure is the measure of the four-fold angle at its corner. To measure a four-fold angle formed by several solids meeting at a point in four-space, take that point as a center and describe a four-fold sphere of unit radius. The solid sides of the four-fold angle will cut from out the solid boundary of the four-fold sphere a curved solid figure of some sort, whose volume may be taken as the measure of the four-fold angle.

The bluntness of three of the six regular four fold figures is known from geometrical considerations. Complete bluntness (*i. e.*, the volume of one half the total solid boundary of a four-fold sphere) being $\pi^2 = 9.8696$ we find that the bluntness of the 8-hedroid is 1.234; of the 16-hedroid, 0.8225; and of the 24-hedroid, 2.468. The determination of the bluntness of the three others may be reduced to the problem of finding the volume of a curved isosceles tetrahedron of regular base, when its edges are known. The required formula is obtained by integrating in four-space polar coördinates, and is reduced from a triple integral to a single integral which is numerically evaluated by taking the means of a large number of equally spaced ordinates and multiplying by the difference of the limits of the integral.

This formula is tested upon figures whose bluntness is known, and is found to give correct results. Applying it to the three figures whose bluntness is yet to be determined we obtain the following values, correct to all the figures given: 5-Hedroid, 0.193; 120-Hedroid, 6.28; 600-Hedroid, 5.45.

Professor R. S. Woodward presented an explanation of the significance of the product of the mean density of the earth and the gravitational constant. Referring to a paper "On the gravitational constant and the mean density of the earth," which he had published in the *Astronomical Journal* (No. 424, January, 1898), he stated that he had shown this product to be a quantity of the same kind as the square of an angular velocity, without being able to refer it to any familiar phenomenon. The explanation offered connects this product with the square of the periodic time of an infinitesimal satellite moving without resistance close to the earth along the equator, supposing the earth to be centrobaric. Calling the equatorial radius of the earth r, the mean density of the earth ρ, the gravitational constant k, and the periodic time of the satellite T, the attraction of the earth will equal the centrifugal force of the satellite; that is,

$$\tfrac{4}{3}\pi k\rho r^3/r^2 = r(2\pi/T)^2, \text{ whence } k\rho = 3\pi/T^2, \text{ or } k\rho T^2 = 3\pi.$$

In the paper cited above the numerical value of $k\rho$ in C. G. S. units is found to be $k\rho = 36797 \times 10^{-11}/$ (second)^2. Hence $T = 5061^s = 1^h\, 24^m\, 21^s$.

Professor Metzler's paper is in summary as follows: If we are given any combination of n numbers m at a time the combination of the remaining $n-m$ numbers is termed the complementary with respect to n of the given combination. Suppose now we form the n_m combinations of any n numbers m at a time and consider the set of combinations formed by combining each of these n_m combinations with their complementaries in such a way that the numbers in any combination arranged in their natural order are immediately followed by the numbers in the complementary combination arranged in the same way. The paper then gives an expression for the number of inversions in any combination of this set and also the excess of the number of combinations in the set which have an even number of inversions over those which have an odd number.

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THE THEOREMS OF OSCILLATION OF STURM AND KLEIN. (FIRST PAPER.)

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of December 29, 1897.)

IN the first volume of *Liouville's Journal* (1836) Sturm has deduced certain properties of the real solutions of linear differential equations of the second order which are of fundamental importance both in pure and in applied mathematics. The opinion has been expressed* that Sturm's work cannot be regarded as rigorous and that other methods must be substituted for his, for instance the method of successive approximations recently employed by Picard for establishing some of these theorems. In one sense it is true that Sturm's work is not rigorous, as hardly any work in analysis done during the first half of the present century shows an appreciation of the difficulties connected with the conception of continuity. The work of Sturm may, however, be made perfectly rigorous without serious trouble and with no real modification of method. In the first two sections of the present paper I have proved such of Sturm's results as are necessary to establish his theorem of oscillation.† In doing this I have departed somewhat from his

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* Cf. the first paragraph of Picard's note in the *Comptes Rendus* for February, 1894, and also Klein, Lineare Differentialgleichungen der zweiten Ordnung (lithographed 1894) p. 266: "In der That genügen die Existenzbeweise, wie sie Sturm und Liouville führen, keineswegs den heutigen Anforderungen d r Strenge. Man wird verlangen alle die von ihnen gegebenen Entwickelungen in neuer Weise abzuleiten."

† This name is due to Klein.