ON THE NUMBER OF ROOTS OF THE HYPERGEOMETRIC SERIES BETWEEN ZERO AND ONE.

BY MR. M. B. PORTER.

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IN 1890 Klein* published his solution of the problem of enumerating the roots of the Hypergeometric Series between 0 and 1.

His method depending on the conformal property of the Schwarzian s-function, finally turns on a discussion of the shape of the circular triangles on which the x-halfplane is mapped.

Solutions by Hurwitz† and Gegenbauer‡ appeared soon after, both depending on the determination of a chain of contiguous hypergeometric functions which could be employed as a set of Sturmian functions.

Klein's method, while it makes use only of the differential equation and yields the desired result in an exceedingly neat form, does not lead to this result so directly or naturally as certain methods of Sturm (Tom. 1, *Liouville's Journal*).

It is the object of this paper to apply two well known theorems of the above mentioned memoir of Sturm to the solution of the problem in hand.

The theorems referred to are:

A. Let x_1 and x_2 be two regular singular points of the differential equation

$$\frac{d^2\bar{y}}{dx^2} = \varphi(x, a)\,\bar{y} \tag{1}$$

If there be no singular point between x_1 and x_2 and all the magnitudes involved be supposed real, the real roots of $\bar{y}_1$ between x_1 and x_2, $\bar{y}_1$ being the solution corresponding to the larger exponent of x_1, will move toward the point x_1, if, for all values of x between x_1 and x_2, $\varphi(x, a)$ decrease with the decrease (increase) of a; *i. e.*, $\bar{y}_1$ is gaining or *at most* not losing roots between x_1 and x_2.

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* *Math. Ann.*, vol. 37.

† *Math. Ann.*, vol. 38.

‡ *Wiener Sitzungsberichte*, vol. 100[2a].

B. In the interval $x_1\ x_2$, between any two consecutive real roots of a solution of (1) lies one and but one root of a linearly independent solution. A more general form of this theorem is the following: If $\bar{y}_1$ denote the solution corresponding to the larger exponent of x_1 and x_0 the root in the interval $x_1\ x_2$ nearest x_1, any solution linearly independent of $\bar{y}_1$ will have one root between each pair of consecutive roots of $\bar{y}_1$ and one root between x_1 and x_0.*

The singular points of the hypergeometric equation:

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0$$

are	0	∞	1
the exponents,	0	α	0
	$1-\gamma$	β	$\gamma-\alpha-\beta$

the exponent-differences being

$$\lambda=\gamma-1,\qquad \mu=\alpha-\beta,\qquad \nu=\alpha+\beta-\gamma.$$

The hypergeometric equation is reduced to the binomial form,

$$\frac{d^2\bar{y}}{dx^2}=\varphi(\lambda,\mu,\nu;\ x)\bar{y}$$

$$=-\tfrac{1}{4}\frac{x^2(1-\mu^2)+x(\mu^2-\nu^2+\lambda^2-1)+1-\lambda^2}{x^2(1-x)^2}\bar{y}$$

by the change of dependent variable

$$y=\bar{y}\,x^{-\frac{1+\lambda}{2}}(1-x)^{-\frac{1+\nu}{2}}$$

The method to be explained serves primarily, as does the method used by Klein, to determine the number X of real roots of the solution corresponding to the larger exponent of $x=0$ in the interval 0–1. The word root will always be used to denote a real root lying between 0 and 1. In the case where $\lambda<0$ the determination of the number of roots of the hypergeometric series which is then the solution corresponding to the lesser exponent of $x=0$ is effected by means of B. The treatment of the so-called exceptional cases is that of Klein but in the method here employed appears as a special case of the general discussion.

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* These slightly generalized forms of Sturm's theorems can be deduced from the analysis given by Sturm, for other proofs see Professor Bôcher's paper in the March BULLETIN.

Since $$F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$$
there is no loss of generality in assuming

$$\mu = \alpha - \beta \geqq 0.$$

Consequently

$$\frac{\partial \varphi(\lambda, \mu, \nu, x)}{\partial \mu} = -\tfrac{1}{2} \frac{\mu}{x(1-x)} < 0, \; 0 < x < 1.$$

Further, since

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\beta, \gamma-\alpha, \gamma, x) \quad (2)$$
$$F(\gamma-\beta, \gamma-\alpha, \gamma, x)$$

has the same roots as $F(\alpha, \beta, \gamma, x)$

and as it satisfies a differential equation with the exponent differences λ, μ, $-\nu$; we may suppose $\nu \geqq, 0$.

By A., the solution $y,$ $(y,)$ corresponding to the larger exponent of $x = 0$ is gaining roots as μ increases. Two cases present themselvès:—I. $\lambda > 0$, II. $\lambda < 0$.

I.

$\lambda > 0$. Here 0 is the larger exponent and the corresponding solution is

$$y = F(\alpha, \beta, \gamma, x) = F\left(\frac{1+\lambda+\mu+\nu}{2}, \frac{1+\lambda-\mu+\nu}{2}, 1+\lambda, x\right)$$
$$= a F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$$
$$+ b(1-x)^{-\nu} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-x)$$

where $$a = \frac{\Gamma(1+\lambda)\,\Gamma(-\nu)}{\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right) \Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right)}$$

$$b = \frac{\Gamma(1+\lambda)\,\Gamma(2+\nu)}{\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right)}$$

Since $\nu > 0$, it is obvious from the series itself that y can have no positive roots unless $\beta = (1+\lambda-\mu+\nu)/2 < 0$. If then μ, starting with a value a little less than $1+\lambda+\nu$, increase, β will decrease. For $\beta = 0, -1, -2$, etc., we get a series of polynomials of degree 0, 1, 2, etc. Whenever β

passes through zero or a negative integer, b changes sign, $\Gamma(\beta)$ becoming infinite. Since $\nu > 0$, the sign of y for values of x a little less than 1, depends only on b and therefore only on $\Gamma(\beta)$. When $\beta = 0$, $y = 1$, which has no vanishing point, however, immediately *after* β decreases through 0, b changes sign and consequently $y_{x=1}$ changes sign; so that as $y = 0$ is gaining roots as μ increases, an odd number of roots must have been gained immediately after β passed through 0,—a sudden dropping of the whole curve $y = F(\alpha, \beta, \gamma, x)$ below the x-axis is impossible since $y_{x=0}=1$. There could have been a gain of but *one* root when β decreased through 0, for as β keeps on decreasing no roots are lost and when $\beta = -1$, $y = F(\alpha, -1, \gamma, x) = 1 - \frac{\alpha}{\gamma} x$ which has but one root.

In the same way immediately after β decreases through -1, b changes sign and one root is gained for when $\beta = -2$

$$y = F(\alpha, -2, \gamma, x) = 1 - = \frac{2\alpha}{\gamma} x + \frac{2\alpha(\alpha+1)}{2\gamma(\gamma+1)} x^2$$

which has not more than two roots. X, the number of roots, is, therefore, equal to the number of changes of sign of b as β decreases from a small positive value to the required negative value, *i. e.*

$$X = E(1-\beta) = E\left(\frac{\mu - \lambda - \nu + 1}{2}\right)$$

which is the number of roots of

$F(\alpha, \beta, \gamma, x) = 0$ when $\lambda > 0$, $\nu > 0$.

I follow Klein in using $E(n)$ to denote the largest positive integer less than n, so that $E(n) = 0$ if $n \leqq 1$. When

$$\nu < 0, \text{ by 2. } \quad X = E(1 - (\gamma - \alpha))$$
$$= E\left(\frac{\mu - \lambda + \nu + 1}{2}\right)$$

II.

When $\lambda < 0$, $1 - \gamma$ is the larger exponent of 0 and the corresponding solution is

$$y = x^{-\lambda} F\left(\frac{1 - \lambda + \mu + \nu}{2}, \frac{1 - \lambda - \mu + \nu}{2}, 1 - \lambda, x\right)$$
$$= x^{-\lambda} F(\alpha', \beta', \gamma', x)$$

The hypergeometric series $F(\alpha', \beta', \gamma', x)$ satisfies a differential equation whose exponent differences are $-\lambda, \mu, \nu$, therefore, by I. it has $X = E(1-\beta')$ roots between 0 and 1 if $\nu > 0$ or if $\nu < 0$ has $X = E\big(1-(\gamma'-\alpha')\big)$ roots.

All four results can be written in one formula

$$X = E\left(\frac{\mu - |\lambda| - |\nu| + 1}{2}\right).$$

In case II. we have determined the number of roots of a solution linearly independent of the hypergeometric series. By B. $F(\alpha, \beta, \gamma, x)$ has when $\lambda < 0$ $N = X$ or $X+1$ roots, the even or odd value of N being chosen according as $y_{x=1}$ is positive or negative. There are thus two cases.

1°. $\nu > 0$. Here, according as b is $>$ or < 0, *i. e.*, according as

$$\Gamma(1+\lambda)\,\Gamma\left(\frac{1+\lambda+\mu+\nu}{2}\right)\;\Gamma\left(\frac{1+\lambda-\mu+\nu}{2}\right) > \text{or} < 0$$

the sign of $y_{x=1}$ is $>$ or < 0.

2°. $\nu < 0$. Here according as a is $>$ or < 0, *i. e.*, according as

$$\Gamma(1+\lambda)\,\Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)\;\Gamma\left(\frac{1+\lambda+\mu-\nu}{2}\right) > \text{or} < 0$$

$y_{x=1}$, is $>$ or < 0.

If in 1° $b = 0$, *i. e.*, if $(1+\lambda+\mu+\nu)/2$ or $(1+\lambda-\mu+\nu)/2$ is zero or a negative integer, the hypergeometric series ceases to be linearly independent of $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-x)$ which when $\nu > 0$ is the solution corresponding to the larger exponent of $x = 1$; so that by B. in this case the hypergeometric series has X roots. For the same reason in 2°, when $a = 0$, *i. e.*, when $(1+\lambda-\mu-\nu)/2$ or $(1+\lambda+\mu-\nu)/2$ is zero or a negative integer the hypergeometric series has X roots.

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