ON CAYLEY'S THEORY OF THE ABSOLUTE.

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In the following pages I attempt to show, as a matter of purely pedagogic interest, how simply and naturally Cayley's theory of the Absolute follows from a small number of very elementary geometrical conceptions, without any appeal to analytical geometry. Where assumptions are made, the fact is frankly stated; the few points where more advanced mathematical reasoning is needed for the actual proof are clearly indicated; my contention is not that every step in the rigorous proof can be presented under the guise of elementary mathematics, but that it is quite possible to develop the theory so as to be intelligible and interesting to average students at a much earlier stage than is customary.

Let any simple diagram be drawn on a sheet of paper, *e. g.*, a circle with a straight line cutting it in two points. Let this sheet of paper be held at some distance away, in such a position as to be slightly oblique to the line of sight. A difference will now present itself, of such a nature as to suggest that the properties of the figure are of two distinct kinds. It will be as evident as before that there is a straight line, and a curve cut by the line in two points; but it will not be perfectly evident that the curve is a circle, it will appear as an oval curve. Similarly, if we have two straight lines intersecting at right angles, the fact that there are two intersecting straight lines will be evident under whatever aspect the figure may be viewed, but the angle between them will not appear to be a right angle. The same effect will be observed if, keeping the diagram fixed, the position of the eye be changed. Thus we see that the properties of a plane figure are of two distinct kinds; some are purely relative, dependent on the point of view; others are more intimately connected with the figure itself, they have no relation to the point of view. The effect of changing the point of view is considered in the mathematical theory of projection, which must now be briefly explained.

Given any figure in a plane (1), and a point V not in this plane, let V, the centre of projection, be joined to all the points of the given figure, A, B, C, etc., and let the points in which these joining lines cut a second plane (2) be denoted by A', B', C', etc. To an eye at V, with no power of determining the actual position of a distant plane, the two figures would be indistinguishable; figure (2), in its proper place, might be substituted for figure (1) without the change being discovered. And yet in many ways the two figures if drawn side by side on one sheet of paper would appear very different; one might contain a circle, a pair of straight lines at right angles, a number of parallel lines, while in the other figure no one of these things could be found. It will be seen that the properties in which any difference is to be observed in the two figures all depend on measurement of lines or angles; these properties are called metric; the properties that are unaffected by the change from one figure to the other, unaffected, that is, by projection, are called projective.

Starting with any figure, (1), we may project from any selected centre, thus obtaining a projection, (2); taking any new centre of projection, and from this projecting (2) on to any plane we obtain a figure (3). This process can be repeated as often as we please, and after any number of projections we arrive at a figure which, while not strictly a projection of the first, in the sense just given to the word, is still in a projective relation to it; the projective properties, being unaffected by every projection separately, are unaffected by the series of projections, and thus the initial and final figures agree as regards all projective properties.

By the definition of projection a point becomes a point. Also a straight line becomes a straight line; for if points A, B, C, etc., lie on a straight line, a simple application of Solid Geometry shows that their projections lie on a straight line. Moreover, straight lines that meet in a point project into straight lines that meet in a point. These laws may be formulated as follows:—points and straight lines become points and straight lines, and properties of collinearity and concurrence are unaltered by projection. These are the general laws; to these we admit no exceptions whatever.

Certain apparent exceptions must be investigated. Let the plane through the centre V parallel to the plane (2) cut the plane (1) in the line YZ. The line joining V to any point X on YZ is parallel to the plane (2), and accordingly, in the Euclidean phraseology, never meets it; that is, the point X has no projection. This however being at variance with our general laws, is not an admissible form of speech; we have to find some other mode of expressing the idea. Take in the plane (1) any line through X; let a point A move along this line, approaching X; the projection of A recedes more and more, and when A moves up to X the point A' is certainly not at any finite distance; we express this by saying that the projection of the point A is at infinity. This we may regard as a conventional mode of speech expressing the same idea as the Euclidean "never meet"; geometrically we have postulated the existence of a special set of points, the points at infinity, for the sake of making our general laws hold without exception. As regards these points we have at once the theorem, all points at infinity lie on a straight line, the line at infinity; for the points at infinity are by definition simply and solely the projections of the points on the line YZ, and by our general laws the projection of this straight line, like that of any other, is a straight line.

Now consider any two or more lines, AX, BX, that intersect at X, on the line YZ. The point X projects to infinity, and thus the lines in the projection have their common point at infinity; now the planes VAX, VBX, etc., all contain the line VX, parallel to the plane (2), and consequently the lines $A'X'$, $B'X'$, etc., in which they are cut by this plane (2) are parallel. We are thus led to the conclusion that parallel lines meet at infinity, and this again is merely another way of stating the idea involved in the Euclidean phrase "never meet". Thus we see that while the law, concurrent lines project into concurrent lines, is not interfered with, the concurrent lines may happen to be parallel. The property of concurrence is projective; the property of parallelism is metric. The first is unalterable by projection; the second may be destroyed.

We now consider the effect of projection on the separate points that lie on any one line. These being A, B, C, etc., we require only the lines VA, VB, etc., and the line in which the plane of these is cut by the plane (2); hence the whole diagram now lies in the plane $VABC$; we may even speak of projecting from the one line on to the other. It is at once apparent that by suitably chosing the position of V, the distance between two points can be altered to any extent; and further, taking three points A, B, C, we can project so that the distances apart shall become anything we please; for, taking any line through A, and measuring on that the distances Ab, bc, that we wish to be assumed by AB, BC, all that is necessary is to use the intersection of Bb, Cc, as the centre of projection. This is usually stated in the form:— any three collinear points can be projected into any three collinear points. To project A, B, C into A', B', C', join AC', use any point on the line CC' as centre of projection, so obtaining on the auxiliary line the points AbC'; now use the intersection of bB', AA' as a new centre of projection, and project AbC' on to the second line. Thus the three points A, B, C are projectively related to the three A', B', C'; in the more extended sense of the term, the one figure is a projection of the other. Our conclusion is therefore that three points on a line have no projective relation other than that of collinearity.

Passing on to consider four points on a line, we are led to the important result that four points have a projective relation. If we take four points A, B, C, D on a line, and four points A', B', C', D' on another line, and attempt to connect them projectively, we proceed as before for the three points A, B, C, but then the point D on the first line gives us a definite point on the auxiliary line, and this gives a point D'' on the second line; it can be shown that D'' will be the same however the arbitrary elements in the construction may be varied; we have then no control over the position of this point D'', we have no way of making it come at D'. The truth is more obvious if the two triads A, B, C and A', B', C' are in projective position; every point D on the first line gives as its correspondent on the second line a perfectly determinate point D'. Hence four points on a line have some relation that is unalterable by projection. For our present purpose it is not essential that we should know the precise nature of this relation; it is enough to assure ourselves that it must be expressible by some number. It cannot be any length, nor any product of lengths, for these would certainly be altered by projection from a line on to any parallel line; it is consequently some number that can be obtained without measurement, and this number is not even a simple ratio of lengths, for we have seen that two lengths can be made to assume any desired magnitude.

There is one particular arrangement of the four points that is of special importance. Let four lines be taken, then every one intersects every other, thus giving six points of intersection, these falling into three pairs that are not joined; draw the three joining lines. The figure thus constructed is a complete quadrilateral; the four lines by which it is determined are the sides, the six intersections of these lines are the vertices, and the three lines drawn to complete the joining of the vertices are the diagonals. On any diagonal we have two pairs of points, viz., two vertices and two intersections with the remaining diagonals; these two pairs are said to be harmonic. Thus two pairs of points on a line are harmonic when a quadrilateral can be described with vertices at one pair, and diagonals through the other pair. The whole diagram, depending simply on points and straight lines, with properties of collinearity and concurrence, is plainly projective; hence, if one of the two pairs be given, and one point of the other pair, the fourth point, the harmonic conjugate to P with respect to A, B is determined. Moreover, if A, B are the pair at which the vertices lie, and P, Q the pair through which the diagonals pass, it is easy to construct a quadrilateral having vertices at P, Q and diagonals through A, B; this proves that the harmonic relation involves the two pairs symmetrically, whatever relation the pair A, B has to the pair P, Q, that same relation has the pair P, Q to the pair A, B. Not only this, but also the two points of either pair are involved symmetrically; this can be shown in the usual manner, by showing that $ABPQ$ can be projected into $BAPQ$. Hence when two pairs of points are known to be harmonic, not only the order of the pairs, but also the order of the points in a pair, is a matter of absolute indifference. Similarly, two pairs of concurrent lines may be harmonic, this being the case if the pairs of points in which they are cut by any transversal are harmonic.

With the help of the idea of parallelism, the harmonic relation leads at once to the idea of the bisection of a line. For let the quadrilateral be a parallelogram, then seeing that one of the three diagonals is entirely at infinity, the point Q on one of the remaining diagonals is at infinity. But in ordinary geometry the two diagonals of a parallelogram are said to bisect one another, that is, AB is bisected at P; hence in the modified phraseology here adopted, instead of saying that AB is bisected at P, we say that AB is divided harmonically at P and infinity.

This conception enables us to mark a scale of measurement on any line. Let two points O, Z, be arbitrarily assumed, and assign to these any arbitrarily selected numbers, *e. g.*, 0 and ∞; take any point A, and assign to this the number 1; take the harmonic conjugate to O with respect to AZ, call it B, and assign to it the number 2; the harmonic conjugate to A with respect to BZ is to be called C, and marked with the number 3, and so on. The harmonic conjugate to Z with respect to OA is the point $\frac{1}{2}$, the harmonic conjugate to Z with respect to this and O is $\frac{1}{4}$, and so on. Thus every point has a numerical magnitude associated with it; when the point Z is actually the point at infinity on the line, the points $0, 1, 2, 3$, etc., are all at equal distances apart; the number associated with a point measures its distance from O, the length OA being the unit of measurement.

But though this gives a scale on the line, that is, a means of measuring on the one line, it gives no way of comparing measures on different lines; for even if we take the intersection of the two as the point O, and suppose the line at infinity, and consequently the points at infinity on the two lines, to be given, we have still no way of comparing the selected units, OA and OA', on the two lines.

Similarly with the help of the idea of perpendicularity, the harmonic relation gives us the idea of equality of angles about a point, and if we can draw at a point a line perpendicular to any line through the point, then we can mark off a scale of angular measurement about that point. For let there be given two lines o and a; through their intersection draw a line a' perpendicular to a, and let b be the harmonic conjugate to o with respect to the pair a, a'; then the angle oa is equal to the angle ab, as can be seen at once by drawing any transversal perpendicular to a. We can then draw through the point a line that shall make the same angle with b, and so on, and thus construct a scale of angular measurement about the point.

But just as before, this gives no way of comparing angles about two different points.*

We cannot get any further in measurement unless we take some postulate, *e. g.*, that of the existence and possible construction of a circle. For we cannot obtain this as the locus of points at equal distances from a fixed point unless we define equal distances; nor as the locus of the vertex of a right angle whose sides pass through fixed points unless we assume that angles between parallel lines are equal, and so assume that we can determine the equality of angles about different points. If however the existence and construction of a circle with any centre and any radius be postulated, we can use this as a means of defining equality of lines in different directions, and, if we choose, of defining right angles. We can then deduce all properties and constructions for perpendicular and parallel lines—all the constructions of metric geometry, in fact—with very little alteration of the arrangement of the propositions in Euclid's elements, which is most literally the geometry of the straight line and circle.

Now the circle projects into a curve that is not a circle,

* One caution should be given here. It might be imagined that we could compare angles about different points by means of parallel lines, and lines in different directions by means of diagonals of a rectangle; but it must be remembered that we have not yet got so much as a definition of equality for lengths on different lines or angles about different points.

though agreeing with it in the properties that a straight line can be drawn to meet it in two points but not in more, and that from a point it may be possible to draw two tangents, but not more; the special properties of a circle are all metric. Thus all metric properties and constructions depend on the circle, though some are more simply expressed with reference to the ideas of parallelism and perpendicularity.

We now pass on to see whether it is possible to construct any system of measurement that shall not have any dependence on these ideas, that is, any truly projective system. This we consider in the first place as relating only to points on a line.

We have seen that we cannot get any projective relation unless we use four points; in order to obtain any relation between O, P, we require other two points A, B, and then we have some number belonging to the set of four points (OP, AB). But here there is a certain lack of definiteness. If there be a number belonging to the four points, then any function of the number is itself a number equally belonging to the points, *e. g.*, the logarithm or the square of the first mentioned number; some one of this set of numbers must be selected. Also, the points A, B, C, D, are not projectively the same as B, A, C, D, hence the number must depend in some way on the order of the points.

Supposing for the moment that these two things, the form of the function and its dependence on the order of the points, have been decided upon, then if three points O, A, B, are arbitrarily chosen, the position of any point P on the line is absolutely determined by the value of this numerical mark. Bearing in mind that what we ordinarily do with points on a line is to measure their distances apart, position on the line being assigned by distance from any selected point, *e. g.*, the point O, we see that if we adopt the order O, P, A, B, in determining the value for the point P, we must adopt the order O, Q, A, B, in determining the value for the point Q. Moreover, if we are to obtain a result that can be interpreted as referring to the distance OP, even by a convention, we must regard O, P as points whose relation with respect to A, B has to be given. Hence we shall write the function as $f(OP, AB)$. Thus the mark which determines the position of the point P with regard to O, with respect to the points A, B, is $f(OP, AB)$; and similarly the mark for Q is $f(OQ, AB)$. By parity of reasoning the mark which determines the position of Q with regard to P is $f(PQ, AB)$.

The most important property of ordinary distances is that expressed by the relation $OQ = OP + PQ$; in order that the marks now under discussion may be susceptible of simple interpretation in terms of distances, we must therefore have $f(OQ, AB) = f(OP, AB) + f(PQ, AB)$. This is found to be enough to determine, except as to a multiplier, which particular number out of all possible ones is to be selected; assuming this to be the case, we have arrived at the conclusion:—Instead of measuring the distance of a point P from a point O in the usual way, this giving a relation of the points that has no permanence in projection, we can assign a numerical mark, entirely unalterable by projection, expressing the relation of P to O with respect to an arbitrarily selected pair of points on the line. These two points being once selected, the mark belonging to any point P on the line with reference to O is called the generalized distance OP, and is denoted by the symbol $\overline{OP}$. Generalized distances on a line obey the same law as ordinary distances, viz, $\overline{PQ} + \overline{QR} = \overline{PR}$.

Similarly we can measure angles about a point if we have a pair of lines a, b through the point as a standard of reference. The generalized angular distance from any line p to any line q, that is, the generalized measure of the angle made by q with p, is $f(pq, ab)$, *i. e.*, $\overline{pq}$. And just as in the case of the linear measurements, these angles are subject to the law expressed by the equation $\overline{pq} + \overline{qr} = \overline{pr}$.

Thus we can construct a system of measurement on a line if we have on that line a certain absolute configuration, two fixed points; and a system of measurement about any point if we have an absolute configuration, two fixed lines through the point. In attempting to apply this to a plane, we require an absolute configuration that shall give us two points on every line, and two lines through every point. It is necessary that these be given by some configuration, for the number of lines in the plane being indefinitely great, it would not be possible to assign the two desired points separately for every line, and similarly as regards the points. Now the only configurations that can exist contain either a finite number of points and lines—which would not supply us with the necessary two points on every line, two lines through every point—or an indefinite number of points and lines. This last is therefore the one to be discussed. We confine ourselves for the moment to the points that have to be determined. We have nothing to do with random assemblages of points, for we require a configuration that can be *given*; hence the points must be given in one of two ways; as lying on some line, straight or curved, or as lying in some area. This last is plainly inapplicable, for it would give an indefinite number of points on any line passing through the area; hence the desired configuration must be a line, straight or curved. The only essential property of this line is that it be met by a straight line in two points; this suggests the circle, or rather a projection of a circle. Now it can be proved that the only curve possessing this property of meeting a straight line in not more than two points is a projection of a circle; that is, a conic section, or a conic.

But it may be argued that this does not necessarily give two points on every line, for some lines do not meet the curve at all. But just as in algebra it is found convenient to say that a quadratic equation has always two roots, though these may be equal or imaginary, this convenience presenting itself in the general consistency thereby obtained in the results, so in geometry it is convenient and legitimate to say that a straight line always meets a circle in the same plane with it in two points, though these may be coincident (when the line is a tangent), or imaginary (when the line does not visibly intersect the circle). That is, just as in algebra we postulate the existence of imaginary quantities, so in geometry we postulate the existence of imaginary points.* Hence any projection of a circle does give us the necessary two points on every line to serve as a standard of reference for a system of measurement on that line.

Furthermore, from any point there can be drawn two tangents to a circle; even if the point be within the circle, so that no real tangents can be drawn, yet we say that there are two, the existence of the imaginary ones being postulated exactly as in the case of the points. The same thing holds therefore of any projection of a circle; and we are led to the conclusion that if any conic be given in a plane, the points and lines (tangents) provide a framework, a universal standard of reference, with respect to which metric relations can be projectively formulated. This conic is called the Absolute.

Since all metric properties are expressible as relations of the figure to the Absolute, the properties of parallelism and

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* By the number of solutions of a geometrical problem is to be understood the greatest number that can be obtained when the data are arranged at pleasure in accordance with the given conditions; if a different arrangement of the data gives apparently a smaller number, the difference gives the number of imaginary solutions. Thus imaginary points and lines are postulated; they are the missing solutions of geometrical problems; the justification for introducing them is the generality and consistency thereby gained.

perpendicularity can be thus expressed. There are two particular relations of straight lines with respect to the Absolute that call for investigation, and it is found that one of these leads to the conception of parallelism, the other to that of perpendicularity. In the first place, two lines p, q, may meet on the Absolute. In this case, the tangents that can ordinarily be drawn from the point of intersection of the two lines coincide; and it is found that the particular function* adopted for the expression of the numerical mark has in this case the value zero. In metric geometry, when the angle between two lines is zero, the lines are said to be parallel; accordingly the system we are constructing will agree with ordinary geometry if we define parallel lines as lines that meet on the Absolute. The other relation of two straight lines is that expressed by the term conjugate; it is shown in elementary geometry that taking lines through a point P, cutting a circle in U, V, the locus of Q, the harmonic conjugate to P with respect to U, V, is a straight line, the polar of P; and that this is the chord of contact of tangents from P. Harmonic properties being projective, this holds for conics. The defining property of polars shows that if the polar of P pass through Q, then the polar of Q passes through P; the poles P, Q, as also their polars p, q, are said to be conjugate; and it is at once seen that conjugate lines are harmonic conjugates with respect to the two tangents a, b, that can be drawn from their intersection, and hence that (pq, ab) is projectively the same as (qp, ab). Applying this, we have $f\ (pq, ab) = f\ (qp, ab)$, *i. e.*, $\overline{pq} = \overline{qp}$, which states that the angle from q to p is equal to the angle from p to q. If we were speaking of angles in the ordinary sense, this would mean that the adjacent angles are equal, that is, the lines p and q would be said to be perpendicular; hence in the present system we define perpendicular lines as lines that are conjugate with respect to the Absolute. Hence one perpendicular to any given line can be drawn through any point T, by joining T to the pole of the line.

It has now been shown that with the help of the Absolute, we can construct a system of metric geometry; and the question arises, is there any way of choosing the conic, the Absolute, so that this system shall be the ordinary Euclidean geometry?

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* The function for (PQ, AB) is $k \log \left(\frac{AP}{PB} \middle| \frac{AQ}{QB}\right)$; and log 1 is known to be $= 0$.

It is to be noted that a conic may be real or imaginary; the real conic, as already stated, has both real and imaginary points; an imaginary conic has, in general, only imaginary points.* Again, a conic may be proper or degenerate; a pair of intersecting straight lines is met by any straight line in two points, and is therefore classed as a conic; and similarly a pair of points, I, J, is classed as a conic, and would serve to determine two lines through any point O, viz. OI, OJ. The easiest way of forming a conception of this mode of degeneration is to consider an ellipse and hyperbola with the same axis AA', defined by the relation $PN^2 : AN.NA' = k$, where for the ellipse k is positive, and for the hyperbola negative. Letting the quantity k diminish, it is seen that the curves become flatter and flatter, approximating to the straight line, while the tangents have a tendency to pass through the points A, A'. When we take k indefinitely small, the tangents become lines through one or other of the two points A, A'; looked at in this way, the conic is said to have degenerated into these two points. But considering the description of the curve by a moving point, we see that a part of the straight line AA' is described twice; if k be positive and indefinitely small, it is the inner part AA' that is so described; if negative, the outer part. If k be regarded as actually zero, there is no such distinction, and we have to regard the line (not any particular part of it) as described twice. Thus the full description of this form of degenerate conic is that it consists of a pair of points, with the line joining them taken twice.

In considering what conic must be chosen in order that the system of measurement may agree with ordinary measurement, we note that one of the fundamental principles of Euclidean geometry is that exactly one parallel can be drawn from a point T to a line. Now a real conic as Absolute may give two real parallels, obtained by joining T to the two points in which the line cuts the Absolute; this gives a perfectly consistent system of geometry, but not the Euclidean; it is a non-Euclidean system, the geometry of Bolyai and Lobatchewsky. An imaginary conic gives no real parallel in any case; this is again a non-Euclidean system, the geometry of Riemann and others. Neither of these will serve; we have therefore to seek among the degenerate conics. The pair of straight lines gives two distinct points on every line, and therefore two parallels; the

* It may, however, have two or four real points.

pair of points, I, J, with the repeated straight line joining them, gives the necessary two lines through every point, and on every line it gives two coincident points, and consequently through a point there can be drawn precisely one parallel to a given straight line. Hence if there be any conic which as Absolute will give ordinary metric geometry, it must be of this type; moreover, the fact that parallel straight lines intersect on the straight line at infinity shows that the repeated straight line which is a part of this degenerate conic can only be the line at infinity. Hence the Absolute that we are in search of must be a pair of points on the line at infinity.

Perpendicular lines, dividing the chord of contact of tangents from their intersection harmonically, are in this case conjugate with respect to the points I, J. Now if the pairs PQ, IJ are harmonic, the quadrilateral diagram shows that as P approaches I indefinitely on one side, Q approaches it indefinitely on the other; hence if P be at I, Q is also at I, and thus any two lines meeting at one of the points I, J, satisfy the condition of perpendicularity. Regarding a circle on AB as diameter as the locus of the intersection of perpendicular lines through A, B, the fact that AI, BI, are perpendicular shows that the circle passes through I, and likewise through J. Hence the points I, J, have the property of lying on every circle; they are the points in which every circle meets the line at infinity, and are obviously imaginary.

The conclusion so far obtained is:—*If it be possible to express the ordinary Euclidean measurement by reference to the Absolute, this Absolute must be a pair of imaginary points; the line joining these points will be the line at infinity, and every circle will pass through the points.* By somewhat refined mathematical reasoning it is shown that the constants involved in the particular functions which give the numerical value for (OP, AB) and (op, ab) can be chosen so that the generalized expressions for the distance and the angle shall, in this case, become the ordinary expressions for the distance and the angle. This having been accomplished, the conditional conclusion stated above follows.

Cayley's theory of the Absolute (in a plane) is therefore that all metric properties can be expressed under the form of projective relations to a given degenerate conic; it applies also to non-Euclidean systems of geometry, these being differentiated by a different choice of the Absolute.

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