THEORY OF NUMBERS AND OF EQUATIONS.

Introduction a l'Étude de la Théorie des Nombres et de l'Algèbre Supérieure. Par ÉMILE BOREL et JULES DRACH, d'apres des Conférences faites a l'École Normale Supérieure, par M. JULES TANNERY. Pp. ix. + 350. Paris, Nony, 1895.

THE beginning of the present century was signalized by epoch-making advances in two of the most interesting and beautiful fields of mathematics. Gauss crystallized the theory of numbers into a well-formed and polished science in that masterpiece of mathematical genius, the *Disquisitiones Arithmeticæ*, while the acumen of the brilliant Galois pierced the mist which had enveloped the theory of algebraic equations, and gave to the mathematical world the clue to the path which must be followed in their further treatment. The work of these masters has borne rich fruit, and we have now reached the stage at which the researches so inspired, have been carried far enough to admit of their results being collated from the journals and other scattered places of original publication, reduced to a common notation, unified, sometimes simplified, and the whole presented in an orderly systematic form. The need of such work has been felt less in the theory of numbers because the treatment of Gauss was so thorough and complete, his results so clearly put and so well arranged that the *Disquisitiones Arithmeticæ* might well serve a beginner as introduction to the subject, and that along the principal lines worked out by Gauss, there was left possible for subsequent writers little more than to paraphrase and illustrate the results which he had reached, to carry out the suggestions which he had not elaborated, and to extend his methods into new fields wherever he had left this possible. We have for some time, therefore, been in possession of several works on the theory of numbers which are more or less close paraphrases of Gauss, and which give an excellent survey of the subject, and in addition, there have been begun within the past few years two treatises (those of Bachmann and Mathews) which set themselves the task, to outline, in connection with a presentation of Gauss's theory, what has been done since Gauss. In the theory of algebraic equations, on the other hand, the great abstractness of the subject itself, heightened by the form of presentation adopted by Galois and by some other workers in this field, notably Kronecker, have combined to make an introductory treatise presenting the theory of equations as it has taken shape during the present century, sketching clearly the fundamental principles and the most important results, a great desideratum. It is to be hoped that the forthcoming elaboration of Kronecker's lectures on the theory of algebraic equations, and likewise Weber's "Lehrbuch der Algebra," of which the first volume has but recently appeared, will do much to meet this need.* The work before us may also be ranked as a noteworthy step in this direction. It was inspired by the lectures of Mr. Jules Tannery, given during the academic year 1891–92, on the subjects of Arithmetic and Algebra. Two of his pupils undertook to edit and publish their notes of his lectures. In carrying out this undertaking, the lectures were so completely recast, that the work appeared under the title given above, with the names of Messrs. Borel and Drach upon the title-page as authors. Mr. Borel prepared the portion relating to the theory of numbers, while Mr. Drach prepared that on algebra.

The task of the reviewer is chiefly to sketch rapidly the contents of the work, so as to give the reader some idea of its nature and scope, and subsidiarily, to indicate the manner in which the work has been executed and to what reader it may prove most useful. We shall endeavor to do these things in turn.

PART I.—THE THEORY OF NUMBERS. Pages 3–156.

CHAPTER I: *General Properties of Congruences.* This chapter discusses in condensed form the leading elementary properties of congruences, Fermat's theorem and Wilson's theorem, with their generalizations and the properties of the function $\varphi(m)$. Of the subjects taken up, the last named is treated the most extensively, and in that connection is given a neat proof, due to Kronecker, of the following formula due to Euler:

$$\left[\sum_{n=1}^{\infty} \frac{1}{n^z}\right] \left[\prod_{(p)} \left(1-\frac{1}{p^z}\right)\right] = 1.$$

where p is an integer taking all prime values and z is any number whose real part is greater than unity. No reference is given to Kronecker's original publication of this proof.

CHAPTER II: *On Congruences with Prime Moduli.* The anal-

* Since the above was written the second volume has also appeared.

ogy between the theory of congruences and the theory of algebraic equations is a very evident one, and the latter subject can thus furnish a guide to the method and order of treatment of the former, and in the first study of the theory of congruences, which has almost always been preceded by some study of the general theory of algebraic equations, it is interesting and suggestive to make prominent the points of similarity and of difference between the two subjects. The work before us is the first one to point out and explicitly to emphasize and utilize this resemblance. This is a point of considerable merit which conduces both to clearness and to brevity in the demonstrations, since it often suffices to indicate the general methods and to omit details except in points of divergence from the corresponding algebraic proofs. The theory of roots of congruences as usually treated would find its precise counterpart in the theory of algebraic equations if in the latter only rational values were admitted. It is only when irrational and complex quantities are admissible as roots, that the fundamental theorem that every algebraic equation has a root can be established. The introduction of the so-called "Galois-imaginaries" allows an analogous generalization to be made in the theory of congruences, and a brief account of the Galois-imaginaries is therefore given. Though succinct and confined to the first properties, the account is clear and serves well as an introduction to the fuller treatment of these quantities given elsewhere. As our authors give no detailed reference either to original sources or to other presentations, it may not be amiss to add here that several such references may be found in Klein-Fricke, Elliptische Modulfunktionen, vol. I., p. 420.

CHAPTER III: *On Binomial Congruences.* Primitive roots and indices are first treated. Only the case of prime moduli is taken up, and for it the principal properties of primitive roots and indices are developed in the ordinary way. Then follows an extension of the theory to Galois-imaginaries, and then, returning to real numbers, the consideration of the congruence

$$x^n \equiv D \pmod{p}$$

is taken up in quite a little detail, after which the facts concerning primitive roots of composite moduli are briefly stated (in two pages).

CHAPTER IV: *Quadratic Residues; The Law of Reciprocity.* A first section discusses the number of solutions of

$$x^2 \equiv D \pmod m$$

for the various forms of m, and thereupon follow the usual definitions and proofs regarding quadratic residues. Jacobi's generalization of Legendre's symbol is given, without alluding to Jacobi, however, and some of its properties deduced according to Kronecker. The notation is misleading, in that p, which previously had denoted a prime, and usually does so, now denotes a composite number. This is the more misleading in that the property

$$\left(\frac{N}{P}\right)\left(\frac{N'}{P}\right)=\left(\frac{NN'}{P}\right)$$

is proved, while the other property

$$\left(\frac{N}{P}\right)\left(\frac{N}{Q}\right)=\left(\frac{N}{PQ}\right)$$

is not even mentioned.

The Gaussian Lemma is developed, and the Law of Reciprocity is proved according to a proof of Kronecker. As usual, our author gives not the slightest clue to aid one who wishes to find the proof as published by Kronecker himself. Slight allusion is made to the host of other proofs of the Law of Reciprocity.

CHAPTER V: *Decomposition of Numbers into Squares.* The general theory of quadratic forms is not taken up, but the proposition that every integer can be expressed as the sum of four or fewer squares is considered, and besides a proof along the usual lines, a proof by Smith is given which is based on the properties of continued fractions and makes use of determinants. The curious reader who might wish to find out how Mr. Smith himself expressed this proof, and what other things he said in the same connection is not given even the initials of Mr. Smith's name, to aid in differentiating the author of the proof from the non-mathematical persons bearing the same surname. A few pages are now devoted to the consideration of complex integers of the ordinary type, $a+bi$, (quite distinct from the Galois imaginaries, of course), and the fundamental theorem is established that in order that a complex integer be prime, it is necessary and sufficient that its norm be a real prime. Four pages, largely taken up with definitions, are given up to quadratic forms, and therewith the part of the book relating to the theory of numbers is brought to a close.

PART II.—HIGHER ALGEBRA. Pages 157–336.

CHAPTER I: *Elementary Algebra.* The author does not take up the ordinary theory of algebraic equations as it is presented in the current works on that subject, but the modern theory of equations as founded by Gauss, Abel and especially Galois, is the subject of discussion.

The properties of positive integers are first considered and the associative, the commutative, and the distributive laws of combination of these quantities by addition and by multiplication are set forth. Negative integers are next defined and considered from the purely logical point of view, and their various properties established. The totality of positive integers form what is known technically as a "group," when the law of combination is addition as well as when it is multiplication. This fact furnishes occasion for the introduction at this point of the group-concept, and an interesting illustrative discussion of the two groups of numbers which have just been mentioned. These sections on positive and negative integers would make admirable reading for the teacher of elementary algebra who finds that the concept of *negative numbers* is difficult of grasp to the beginner. Of course, it is not meant that the strict logic of our author is suitable for exact reproduction to one who has never before heard of negative numbers, but it is a clear and succinct presentation of the theory of negative numbers, which would make, in the teacher's mind, the skeleton of the work.

Fractional numbers are introduced into the system by the same logical processes as negative numbers, and then we pass to polynomials in one variable, with rational coefficients; addition, multiplication and decomposition into rational linear factors and decomposition into rational factors generally are considered, bringing out strongly the analogy with the decomposition of integers into prime factors.

This chapter on elementary algebra is clearly, concisely and logically written, and well repays reading.

CHAPTER II: *Algebraic Numbers.* Algebraic numbers, *i. e.*, the roots of algebraic equations with rational coefficients, are now defined and added to the system of rational numbers already on hand, and their properties are investigated in a manner analogous to that in which rational numbers were treated in the preceding chapter. Thus several of the well-known laws of the decomposition of algebraic polynomials into factors are deduced, and the fundamental problem of the theory is formulated:

Considering the totality of algebraic numbers, to determine what ones it is necessary and sufficient to introduce into the calculus, that is to say, to represent by explicit symbols, in order that all other algebraic numbers can be rationally expressed in terms of these.

The nature of this problem is illustrated by an analogous case from the theory of rational numbers, the problem itself, however, is not considered here, but constitutes the nucleus of what follows, especially of the fourth chapter. Algebraic functions, the theorem of D'Alembert algebraically interpreted, and symmetric functions with their properties and modes of calculation, are each the subject of a short section. Considering an irreducible algebraic polynomial of degree n, and with integral coefficients, the idea of the adjunction of its root, $\xi_1, \xi_2, \cdots\cdots \xi_n$, to the totality of numbers heretofore in use, viz., the rational numbers, is now developed, and the properties of rational functions of the elements of this new "domain," and the question of reductibility therein are taken up. Defining algebraic *integers* to be the roots of irreducible algebraic equations with integers as coefficients, and with unity as the coefficient of the highest power of x, it appears readily that it is sufficient to consider in a given domain, only equations whose roots, so far as they lie in the domain, are integers of that domain.

CHAPTER III: *Systems of Equations*. Under the caption of linear equations, the theorem of Rouché concerning the solution of p equations linear in n variables $p \gtreqless n$, is concisely given, but without any reference to the place of publication or even to the author of the theorem. The resultant of two polynomials in one variable is taken up, the form of elimination known as Sylvester's is explained, the resulting determinant is written in a slightly different form from that which is ordinarily used, and a determinant criterion is deduced for the existence of a common divisor of order q as follows:

If R denote the resultant expressed in Sylvester's method as a determinant of order $m+n$, and if R_i denote the determinant of order $m+n-2i$ formed by striking out of R the i first and the i last rows and columns, then the necessary and sufficient condition that the polynomials have a common divisor of order q is

$$R = R_1 = R_2 = \cdots\cdots = R_{q-1} = 0.$$

Other methods of forming the resultant are not considered, but the principal properties of the resultant are briefly deduced. Systems of two equations in two variables are discussed at some length, and the general properties of their resolvent explained, with a sketch of the extension to the case of p equations in n variables.

CHAPTER IV: *The Calculus of Algebraic Integers.* This chapter treats further of the formation of an algebraic "domain" and of calculation therein, as begun in Chapter II., of the nature of the process of "adjunction" introduced by Galois, and some of the properties of the "Galois-resolvent" are clearly set forth. An important step is made in the solution of the fundamental problem announced in the second chapter. Defining as *normal equation* every irreducible equation with integral coefficients which has the property that all its roots are integral functions, with integral coefficients, of one of them, and defining as *normal* algebraic integer, every algebraic integer which satisfies a normal algebraic equation it is proved that:

Every algebraic integer is an integral function with integral coefficients of a normal algebraic integer.

The field of research in our fundamental problem, which began with the totality of all algebraic numbers, is now narrowed to that of all normal algebraic integers.

Special equations are next taken up, but only sufficiently to permit of the introduction of the idea of their "group," and this, in turn, calls for the definition of "substitutions" and a brief consideration of their laws of combination considered as abstract operations, and of isomorphism, thus preparing the way for fuller treatment in the next chapter.

CHAPTER V: *Groups of Substitutions.* In twenty-three pages this chapter treats briefly the effect on rational functions of all possible substitutions on n letters, and also the general properties of groups and sub-groups and of Lagrange's resolvent.

CHAPTER VI: *Composite Groups.* This chapter takes up briefly in turn the series of composition of composite groups, the criterion of Galois for the solvability of algebraic equations, the proof that equations of degree greater than four are not algebraically solvable, and closes with a few words concerning the icosahedron group.

CHAPTER VII: *Applications; Conclusion.* This chapter treats of normal equations, and here the last step is made in the treatment of the fundamental problem by showing that, if we define as a *simple* equation an equation whose group is simple:

Every normal algebraic integer can be obtained by successively introducing into the calculus normal algebraic integers which are roots of simple normal equations.

The variety of normal equations known as Abelian equations, receives brief separate treatment and a few concluding remarks, followed by a note on Fermat's theorem, and one on the Galois imaginaries, close the volume.

This work may be characterized either as elementary or as advanced, if the sense in which the terms are used is explained, but either predicate would be misleading if used alone without explanation. The work is perhaps best described as "elementary advanced." It has been indicated at various points in the preceding sketch of the contents of the work, that it touches upon recent researches in the subjects under consideration, researches of which a first presentation could not take cognizance with profit to the beginner; in this sense the work is *advanced;* but only the most general outlines of the advanced topics are given; in this sense the work is *elementary.* It is not a work for beginners, though no specific acquisitions in these subjects seem to be presupposed. The part on the theory of numbers, while to some extent included in the preceding characterization, may with propriety be called elementary, and indeed might well be read by a beginner, but it does not cover a field sufficiently wide to make an adequate introduction to the subject. The part on algebra is an elementary introduction to the advanced region of the subject, and presupposes considerable training in abstract mathematical reasoning. There are very few illustrative examples and no exercises whatever, to which the reader may apply for himself the principles which have been set forth. This fact alone, even if the ground covered were quite appropriate to a beginner, would to the mind of the reviewer, make the book seem unsuited for a first course. The value of the work lies in its clear, concise outlines of general principles stripped of illustrations and amplification; such a presentation, even though it technically presupposes nothing, yet is of full value only to him who is quite familiar with the details which are suppressed, and with the point of view from which they are ordinarily seen. The general trend of the work has been well indicated by Mr. Tannery in the first paragraph of his Preface:

"During the scholastic year 1891–2, I gave some lectures on Arithmetic and Algebra to the students of the third year in the Normal School. I had in no sense the purpose of setting forth dogmatically portions of the science which have, indeed, always aroused a lively interest in me, but which I have not studied in detail; I wish solely to direct the curiosity of my hearers towards problems which are among the most beautiful of those which mathematics presents, and towards methods which have been found or perfected by men of a genuis singularly rare and penetrating. The incessant advance of Analysis and of Geometry should cause neither these problems nor these methods to be forgotten. It is well enough known besides, that the different branches of mathematics are not independent and that those properties of integers which are the subjects of study in Arithmetic and Algebra, appear also in many questions arising in Analysis. Finally, it must be remembered that the teaching of the most elementary parts of Arithmetic and Algebra, of those parts which are taught in the *lycées* presupposes that the professor, if he wishes really to dominate his subject, has had at least some glimpses of the more advanced part of the science. My intention was simply to induce my hearers to look in this direction."

To say that this end has been well attained, that the clear logic, the well ordered arrangement, the attractive style, the keen analysis of fundamental principles combine to make this work an interesting introduction to the borderland of algebra such as Mr. Tannery purposed to make it, is to give the work but its due meed of praise, and leaves little more to be said. It is not a book of reference, it is a book to be read, and the scanty table of contents offers but little assistance to one, who, without having read the book, or at least having familiarized himself with the details as to its contents, wishes to consult its pages on a specific question.

The work is marred by one serious blemish. Almost no references are given either to the original sources of the material used, or as guides to those who wish to study the subject further. The meager references which are given are usually of the most elusive character, either simply a name, or the title of a large work, or of a journal without year or volume. This lack of references, which has been specifically indicated in passing at a number of points in the preceding sketch of the contents of the work, would be a grave defect in any work, but it is doubly so in a book like that before us which simply present the broad outlines and prominent features of a vast subject, gives fundamental principles, but few details, and serves chiefly to arouse interest and to introduce to a fuller study of the questions which have been raised. In such a work references and bibliography should be especially copious and detailed.

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THE UNIVERSITY OF CHICAGO.
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QUATERNIONS.

A Primer of Quaternions. By ARTHUR S. HATHAWAY. New York, The Macmillan Co., 1896. 8vo. x+113 pp. 90c.

In this compact little book the author has compressed much of the theory of quaternions. The treatment is strictly Hamiltonian in method though a little different in form. The author seems to have caught the spirit of the master admirably and as a result the real simplicity and beauty of quaternions breathes from every page. The author rightly lays stress upon the *number* (*i. e.*, *multiple number*) idea of a quaternion. The term *vector* is not used to indicate a *directed length*, but to indicate a *right quaternion*. As a right quaternion a vector must possess an axis and this axis may be represented by a direction, and thus a definite length in a definite direction (*a step*) may *represent* a vector, but it *is* a step and not a vector. "In none of these cases is the concrete quantity an absolute number." This discrimination between step and vector (although the term vector is used differently) is Hamiltonian. It is a move in the right direction, away from the too geometrical and too physical view that has unfortunately always pervaded treatises on quaternions. It does not seem entirely the wisest way to present a subject *through* its applications, for in this way much fog and much recrimination has been brought about. Whatever we may say, we must at last admit that quaternions is an *algebra* and as such has *many* applications, the usual one being only one, which form really no inherent part of the subject *in abstracto*. Many so-called modifications of it are easily perceived to be from this point of view *other algebras*. The point the author insists upon rightly is that a quaternion is a further extension of the idea of number than that of the ordinary complex number, but similar. No one ever confuses the *representation* of a number $a+b\sqrt{-1}$ in the complex plane with the number itself. So a vector and its representation are two different things.

Chapter I. deals with the *step* and its use. We find the usual kind of examples and theorems of what is generally called vector analysis. The examples are of the kind best calculated to stimulate a student—they are suggestive of further developments to him and if he has any originality cannot fail to develop it. Such examples however are not strictly quaternionic.

Chapter II. discusses the rotations of steps and leads up to multiple numbers which are really quaternions, such as

(2, 30°). We meet here the old familiar term of Oliver, Wait, and Jones's algebra—*versitensor*. The use of arcs to represent rotations is developed and the student is set a few examples on spherical conics which are to be used in the following chapter.

Chapter III. introduces the quaternion and the names of its various elements. There is no appeal to the four-unit form in proving the usual introductory theorems—in fact, Cartesians are abandoned entirely throughout the text, a phase which is again thoroughly Hamiltonian. The rotator $q(\ )q^{-1}$ is introduced; powers and roots of quaternions are considered; the products of vectors, conditions of perpendicularity and parallelity, the distributivity of quaternion multiplication and related subjects are handled with skill and clearness. Geometric applications appear here, and some development of trigonometry. The examples were, of course, meant to be illustrated and analyzed by the instructor. Few pupils with the knowledge to be had up to this point would be able to handle them well. They are of the suggestive kind, calculated rather to point a way than to furnish an exercise.

Chapter IV. is perhaps the most compact of all the book. Within a limit of thirty pages are discussed equations of the first degree, their applications to the line and plane, nonions or the linear vector operator, linear homogeneous strain, nullity and vacuity of a nonion, and applications to surfaces of the second order. Almost as much is brought out or hinted at as Tait devotes one hundred and twenty large pages to. Yet we can conceive that under proper teaching and not too hasty progress the work could be done. The work in nonions brings in many ideas which have come from matrices, though easily enough developed by quaternion processes. The chief one of these is the notion of nullity.

The author tells us the work has had the test of the class-room and that it may follow ordinary algebra and geometry, and that he has given it as a substitute for solid analytics. This surely certifies to the feasibility of the use of the text. It is to be hoped that it will find the success it deserves and be followed by its natural sequel.

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