

and

$$\begin{array}{rcccl}
 1 & ab . cd . ef & ghi & ab . cd . ef . ghi & \\
 & & gih & ab . cd . ef . gih &
 \end{array}$$

Jordan gives an enumeration of the primitive groups through degree 17 in *Comptes Rendus*, vol. 75, in which the number of primitive groups of degree 9 (excluding the groups that contain the alternating group) is given as eight, while Professor Cole's list contains nine such groups. The group omitted is that of order 1512, as may be learned from Jordan's article on the classification of primitive groups in volume 73 of the same journal.

By these additions the number of known groups of degree 8 becomes 200 instead of 199 as stated in my former note, and the number of groups of degree 9 becomes 258.

UNIVERSITY OF MICHIGAN, *May*, 1894.

FOURIER'S SERIES AND HARMONIC FUNCTIONS.

An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, with applications to problems in Mathematical Physics. By WILLIAM ELWOOD BYERLY, Professor of Mathematics in Harvard University. Boston: Ginn & Co., 1893. 8vo, xii and 288 pp.

THIS book has recently been made the subject of a rather singular review in a leading New York paper, in which a number of curious statements are made. The reviewer begins with the statement that, "notwithstanding its name, so redolent of Helicon, there is mighty little poetry in spherical harmonics." He then, apparently overlooking the greater part of the contents of the book, and even of the title, goes on to give a rather restricted description of the use of spherical harmonics, ending up with the statement that the subject "is of great utility, and, like other utility-mathematics, is tedious, difficult, disagreeable, and unbeautiful." This is rather discouraging to one intending to read Professor Byerly's book, and, at the risk of being thought rash, I shall venture to disagree somewhat from the learned reviewer. No doubt the interest and beauty of a mathematical subject is largely a matter of personal taste, and one may profess a dislike for any subject involving the necessity of developments in infinite series, as he may to the employment of irrationals. But in regard to the subject of partial differential equations, to which this subject properly belongs, the opinions of many would be different from that above cited. The present

writer has always considered this subject, upon which nearly all problems in mathematical physics depend, one of the most interesting in the range of analysis. He must even admit that, in case of fire in his library, the first books he would seize to rescue would be Kirchhoff's *Mechanik* and Riemann-Hattendorff's *Partielle Differentialgleichungen*.

The partial differential equations of mathematical physics are nearly all linear, with constant coefficients. By all odds the most important ones are, as a matter of fact, particular forms of the equation

$$a \frac{\partial^2 u}{\partial t^2} + b \frac{\partial u}{\partial t} + c \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right\} = 0.$$

This includes the equations for the propagation of waves in elastic solids and fluids, and of electromagnetic disturbances, the equation for the conduction of heat and electricity, for diffusion of liquids and gases, for the vibrations of strings, membranes, and air in musical instruments, for the propagation of telegraphic and telephonic signals, and many others. The mathematical problem is in all cases of the same nature. The arbitrary functions that arise in the integration of a partial differential equation are to be determined by imposing, for a certain value of t , or at a certain surface, or both, given values upon the dependent variable u , and, if possible, upon one of its derivatives. Cauchy gave a general method of effecting the determination, depending upon a particular method used by Fourier, but it is not always convenient. The first half of Professor Byerly's book deals with Fourier's method of solution.

One of the earliest, and the most famous solution of the class here referred to, was that given by Laplace of what has since received the name of Dirichlet's Problem. If in the above equation a and b are zero, we are to determine u as a uniform continuous function within or without a certain surface by giving its values at every point of the surface. The method of Laplace, applicable when the surface is a sphere, and generalized by Lamé, was to transform the differential equation into one involving three space parameters as independent variables, of such a nature that for the surface in question one of the parameters is constant, and u is given as a function of the other two. u is then developed in a convergent series of functions of the three parameters, of such a nature that each term satisfies the differential equation, and is besides a product of three functions, each involving only one of the three variable parameters. The partial differential equation then resolves itself into three ordinary differential equations, so that the problem is much simplified. The

general question, then, involves particular solutions of certain ordinary differential equations, and the development of arbitrarily given functions of one or two variables in series involving them.

In his introductory chapter Professor Byerly introduces the various equations to be treated, together with their transformations in terms of the various variables required, showing how the particular solutions, such as sines and cosines, Legendre's and Bessel's functions, arise, and illustrating the use made of them. The remainder of the first half of the book is devoted to problems solved by Fourier, where the functions used in the developments are sines and cosines of multiples of a single parameter. An elementary treatment of Fourier's series is given, which is clear enough to the student, but suffers under the disadvantage of not being rigorous. Chapter III is devoted to the subject of Fourier's series, although the frank statement is made that the subject is altogether too large to be completely handled in an elementary treatise. This seems to the writer a great pity if it is true. Professor Byerly's book is the result of a course of lectures delivered at Harvard, and the writer has scanned the Harvard catalogue in vain for an indication of a course in which the gap here left would be filled up. Dirichlet's and Riemann's work would of course be referred to, but treatments of Fourier's series in English generally leave much, or rather everything, to be desired on the score of rigor. The subject of convergency of series is one that seems to receive little attention in the colleges in this country, and the ignorance of students regarding it is often abysmal. The important question of uniform convergence is hardly hinted at in any English book, unless it be one of the recent treatises on theory of functions or Chrystal's algebra. This is certainly not as it should be, and it is to be hoped that this lack will soon be filled. Professor Byerly gives a proof of convergency in a special case which could without much trouble have been made general. He gives a page of cuts showing the gradual approximation to the representation of several functions, as terms are successively added to the series, which are very interesting.

The second part of the book is devoted to spherical harmonics, and to Bessel's and Lamé's functions. All the principal properties of spherical harmonics are given, while what is unnecessary is omitted. Here we again find an instructive plate giving graphical representations of the first seven Legendre's polynomials, or zonal harmonics, which will be appreciated by the student. Similar figures for Bessel's functions would have been instructive. There are also figures representing incomplete developments which are interesting by comparison with those previously mentioned.

In the chapter on Lamé's functions it is to be regretted that symmetry has not been preserved in relation to the three ellipsoidal co-ordinates, as has been most elegantly done in Halphen's "Traité des Fonctions Elliptiques."

The book is just what it purports to be. The preface states that the first part is based on Riemann-Hattendorff, and it includes besides a great deal not there treated. It is a clear, compact treatment of its subject-matter, and will be of great value to students of mathematical physics and to all persons who have to perform calculations of the kind considered. It contains those things that the "business" mathematical physicist wants to know, so arranged that he can find them at once. It is in addition much more interesting than such books have generally been. Heine's and Thomson and Tait's have been the standard treatises on spherical harmonics, but no one could pretend that Heine's was an attractive book to read, or Thomson and Tait's easy. Byerly's book is crowded with physical problems of all sorts, mostly worked out in detail. A good opportunity is also given the student to exercise himself in real numerical calculation by which he may get a tangible idea of the processes involved. A series of valuable tables of the values of the various functions is also given. Last, and not least in value, is to be mentioned the historical summary contributed by Dr. Maxime Bôcher, giving an admirable sketch of the whole subject, with a bibliography.

The book is well and clearly printed, and attractive in appearance (to one, as was stated at the beginning, who likes that sort of thing). Misprints are rare. On page 91 Ångström appears as Ängstrom, which spoils the pronunciation.

It may be mentioned that the historical essay on trigonometric series mentioned on page 61 is to be found in the *Bulletin des Sciences mathématiques* for 1880.

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NOTE ON SMITH'S REVIEW OF CAJORI.

BY PROF. GEORGE BRUCE HALSTED.

THE review, in the May BULLETIN, of Cajori's History of Mathematics by Professor David Eugene Smith produces an unfair impression. The facts upon which he says he bases his "harsh statement" do not justify it; and what he states as his "facts" are in large part not facts, but specimens of Professor Smith's *petitio principii*.