he easily concludes again that $A\alpha^{\frac{\pi}{2}}$ is the logarithm of α^A. The formal similarity of the powers of α to those of e^i being complete it can be seen at once how expressions of the most general kind can be obtained for the logarithms of negative numbers. By multiplying together α^A and β^B and taking the real and imaginary parts the fundamental equations of spherical trigonometry are reached almost instantaneously as in the former paper.

A section on circular spirals closes with the differentiation of α^A and other quaternion expressions, and with some remarks on hyperbolic trigonometry and hyperbolic spirals the article terminates. The hyperbolic trigonometry is founded on the equation

$$h\alpha^A = \cosh A + \alpha^{\frac{\pi}{2}} \sinh A,$$

where the symbols on the right have their usual meaning, and A is the area of a hyperbolic sector.

C. H. CHAPMAN.

UNIVERSITY OF OREGON, *May* 14, 1894.

NOTE ON THE SUBSTITUTION GROUPS OF EIGHT AND NINE LETTERS.

BY G. A. MILLER, PH.D.

IN calculating the possible groups of a given degree it is very helpful to have an accurate list of the groups of the lower degrees. An error in the lower groups is apt to give rise to numerous errors in the higher groups. On this account I have calculated all the possible groups through degree 9 and compared my results with the published lists. No complete lists of the groups beyond degree 9 have yet been published.

In the April number of this journal I noted several errors and one omission in the lists of the groups of eight letters. The following forms a supplement to this note.

There is a primitive group of degree 8 and order 1344, which is not given in the lists referred to in my note in the April number of this journal. The existence of a transitive group of this order and degree can be proved as follows:

$$A(abcdefgh)_8$$

consists of these substitutions:

$$\begin{array}{ll} 1 & ab \, . \, cd \, . \, ef \, . \, gh \\ & ac \, . \, bd \, . \, eg \, . \, fh \\ & ad \, . \, bc \, . \, eh \, . \, fg \\ & ae \, . \, bf \, . \, cg \, . \, dh \\ & af \, . \, be \, . \, ch \, . \, dg \\ & ag \, . \, bh \, . \, ce \, . \, df \\ & ah \, . \, bg \, . \, cf \, . \, de \end{array}$$

This group evidently contains as factors of its substitutions all the cycles whose degree and order are two in the symmetric group of eight letters. We can prove that there are thirty such groups in this symmetric group. For we may suppose the first column of cycles to be fixed for all these groups; the rest of the first substitution may then be formed in

$$\frac{\dfrac{6\,.\,5}{2} \times \dfrac{4\,.\,3}{2} \times \dfrac{2\,.\,1}{2}}{6} = 15$$

ways. Since the first two substitutions of this group are contained in the group obtained by a (1, 1) correspondence between two four-groups (Vierergruppe), the second substitution can be completed in two ways after the first has been chosen. These two substitutions with the given column of cycles determine the group. There are therefore thirty such groups in the symmetric group, and each is transformed into itself by $9!/30 = 1344$ substitutions, which constitute the transitive group in question.

Having proved the existence of the group it is necessary to show that there is only one such group and to find its generating substitutions. The group must have the group of order 168 and degree 7 as a subgroup. It therefore has 112 groups of the form

$$(abe \, . \, cdf) \text{ cyc}$$

and $\dfrac{1344}{112} = 12$ positive substitutions which transform this last group into itself. Since six such substitutions, viz.,

$$\begin{array}{lll} 1 & ab \, . \, cd & abe \, . \, cdf \\ & ae \, . \, cf & aeb \, . \, cfd \\ & be \, . \, df & \end{array}$$

are in the given group of order 168, the other six must be of degree 8 and must transform these six into themselves. The only substitutions which satisfy the requirements are

$ac \,.\, bd \,.\, ef \,.\, gh$	$adecbf \,.\, gh$
$ac \,.\, bf \,.\, de \,.\, gh$	$afbced \,.\, gh$
$ad \,.\, bc \,.\, ef \,.\, gh$	
$af \,.\, bd \,.\, ce \,.\, gh$	

These must therefore be in the group. By multiplying $ac \,.\, bd \,.\, ef \,.\, gh$ into $abcefdg$ we have

$$aedcbgh.$$

Since the powers of this substitution cannot be in the given 168 there are at least 168×7 common substitutions in all groups of order 1344 and degree 8. Hence there can be only one such group.

From what has been said it follows directly that the group is generated by

$$(+\, abcdef\,)_{24},\ abcefdg,\ aedcbgh.$$

A more convenient notation is

$$+\, [\{(ab)(cd)(eh)(fg)\} \text{ pos } (ABCD) \text{ all}](abcefdg) \text{ cyc.}$$

After having worked out this group I found it in Kirkman s list of transitive groups through degree 10, *Proceedings of the literary and philosophical society of Manchester,* vol. 3, and in Jordan's classification of primitive groups, *Comptes Rendus,* vol. 73. As very little is given in regard to the group in either of these places I have not hesitated to publish this proof of its existence which at the same time establishes the important property that it is the largest group of degree 8 which contains $A(abcdefgh)_8$ as a self-conjugate subgroup and exhibits it in the notation employed by Professors Cayley and Cole in their late lists. It needs scarcely be added that this group is threefold transitive.

The only complete list of the groups of degree 9 is that given by Professor Cole in the *Quarterly Journal of Mathematics,* vol. 26. In this list I found only two omissions, viz., the two intransitive groups of order 6, $\{(abcdef)$ cyc (ghi) cyc$\}$ tris and $(ab \,.\, cd \,.\, ef\,)(ghi)$ cyc. The substitutions are

1	$ad \,.\, eb \,.\, cf$	$abcdef \,.\, ghi$	$ace \,.\, bdf \,.\, gih$
		$afedcb \,.\, gih$	$aec \,.\, bfd \,.\, ghi$

and

1	$ab \,.\, cd \,.\, ef$	ghi	$ab \,.\, cd \,.\, ef \,.\, ghi$
		gih	$ab \,.\, cd \,.\, ef \,.\, gih$

Jordan gives an enumeration of the primitive groups through degree 17 in *Comptes Rendus,* vol. 75, in which the number of primitive groups of degree 9 (excluding the groups that contain the alternating group) is given as eight, while Professor Cole's list contains nine such groups. The group omitted is that of order 1512, as may be learned from Jordan's article on the classification of primitive groups in volume 73 of the same journal.

By these additions the number of known groups of degree 8 becomes 200 instead of 199 as stated in my former note, and the number of groups of degree 9 becomes 258.

UNIVERSITY OF MICHIGAN, *May,* 1894.

FOURIER'S SERIES AND HARMONIC FUNCTIONS.

An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, with applications to problems in Mathematical Physics. By WILLIAM ELWOOD BYERLY, Professor of Mathematics in Harvard University. Boston: Ginn & Co., 1893. 8vo, xii and 288 pp.

THIS book has recently been made the subject of a rather singular review in a leading New York paper, in which a number of curious statements are made. The reviewer begins with the statement that, "notwithstanding its name, so redolent of Helicon, there is mighty little poetry in spherical harmonics." He then, apparently overlooking the greater part of the contents of the book, and even of the title, goes on to give a rather restricted description of the use of spherical harmonics, ending up with the statement that the subject "is of great utility, and, like other utility-mathematics, is tedious, difficult, disagreeable, and unbeautiful." This is rather discouraging to one intending to read Professor Byerly's book, and, at the risk of being thought rash, I shall venture to disagree somewhat from the learned reviewer. No doubt the interest and beauty of a mathematical subject is largely a matter of personal taste, and one may profess a dislike for any subject involving the necessity of developments in infinite series, as he may to the employment of irrationals. But in regard to the subject of partial differential equations, to which this subject properly belongs, the opinions of many would be different from that above cited. The present