Chicago Congress through their generous contribution to the Lobachèvsky memorial fund, contains the promise of a still closer union between the mathematicians of America and Russia, and proves the solidarity of scientific interests among all nationalities.

KAZÀN, *March* 7, 1894.

MACFARLANE'S ALGEBRA OF PHYSICS.

Principles of the Algebra of Physics. By A. MACFARLANE, Professor of Physics in the University of Texas, Austin, Texas. *Proceedings of the American Association for the Advancement of Science,* vol. 40, 1891, 53 pp.

On the Imaginary of Algebra. By A. MACFARLANE. *Ibid.,* vol. 41, 1892, 23 pp.

THE purpose of the first of the articles which are to form the subject of this review may most properly be stated in the author's own words: "The guiding idea in this paper is generalization. What is sought for is an algebra which will apply directly to physical quantities, will include and unify the several branches of analysis, and when specialized will become ordinary algebra."

A student who sets out to use Grassmann's algebra in geometrical work finds that it applies beautifully to projective problems in curves and surfaces of no higher order than the second, but beyond them he is confronted and stopped by difficulties which can be overcome only by the study of the ordinary theory of algebraic forms. In the same way quaternions work out many metrical properties of curves and surfaces with facility and grace, but I think every student who has tried to go far with them finds that he is at last brought back to the study of the equations and functions of ordinary analysis. There seems to be no way around the difficulties of the theories of forms and functions, and even when results have been attained by methods which appear to avoid them the mind is seldom convinced of their validity. As we shall see, Professor Macfarlane derives the formulas of trigonometry with great facility, but it seems almost certain that no analyst would dare to use them if they had no other foundation.

Passing by considerations of this kind which seem to make it doubtful whether or not any system of analysis other than the ordinary one can do much to advance mathematical science, we come to the author's first objection to quaternions as Hamilton left them. Hamilton made the square of a vector negative; in fact he wrote

$$ij = k$$

and

$$i.ij = ik = -j;$$

so that he had to take $i^2 = -1$ if he made

$$i(ij) = (ii)j = i^2 j;$$

that is, if he preserved the association law in multiplication. The result of this is that the scalar parts of Hamilton's products are taken negatively. Professor Macfarlane prefers to make them positive even at the expense of the associative law. He finds

(1) That problems in vector analysis can be worked out without the minus;

(2) That the expressions so obtained are more consistent with those of algebra;

(3) That the square of a directed quantity in algebra is positive;

(4) That the quaternion rule throws the system out of harmony with determinants;

(5) That it makes the operator ∇^2 negative.

As to the third of these points the rule stated is certainly very limited; while $(-3)^2 = 9$, it is also true that $(3\sqrt{-1})^2 = -9$, and $3\sqrt{-1}$ is just as much a directed quantity as -3; but, granting them all, one is hardly convinced that Hamilton did not choose wisely in preferring the conservation of the associative law. Professor Macfarlane's own ingenious equations show how inconvenient analysis becomes without it. To be sure, each equation means more, but the question is whether it is better to manipulate two or three facile equations or a single "anfractuous" one.

Proceeding with his argument for making the square of a vector positive, he finds what amounts to a contradiction in Hamilton's system when vectors are taken to represent physical entities. The units i, j, k admit of a double interpretation: (1) each causes rotation through the angle $\frac{1}{2}\pi$ in the plane to which it is at right angles; (2) they are simply directed lines. In the first case we may find the meaning of ij by adding the quadrants which represent their successive effects, and it comes out easily that ij must equal k, while ii represents the rotation π, or simply -1. Now if i is a rotator and j a line, we find as before that $ij = k$, where k is a line; while

ii, where the first i is a rotator and the second a line, ought to leave the line i unchanged. As a matter of fact in quaternions $ii = -1$ always. There is certainly what seems an "insuperable objection," but from Hamilton's point of view it is no objection at all. Hamilton lays down a consistent body of symbols and combinatory laws, and then makes such physical applications as he can. The fact that the symbols fail of certain representative powers is no objection to them as a logical system. The truth is that a vector symbol will not bear interpretation as a rotator except when it is multiplied into a vector perpendicular to itself, and then it will bear that interpretation. This is a limitation, but it is not an objection. The laws of multiplication are first laid down and then we ask for their physical meaning; the other course might be taken and Professor Macfarlane actually takes it, but it is hard to see why one of them is more right or wrong than the other. All through his discussion of Hamilton's rules the author seems to identify the rotator i (or $i^{\frac{\pi}{2}}$) with the quadrant corresponding; this introduces difficulties which are perhaps unnecessary. Hamilton looked upon the arc rather as the logarithm of the rotator, and these arcs have in fact a remarkable resemblance to ordinary logarithms. Their addition is not commutative, but the addition of the logarithms of rotators should not be.

Leaving this interesting subject, Professor Macfarlane asks next, "Are the principles of the method of quaternions consistent with the theory of dimensions . . . ?" If i and j denote lengths, then in the equation

$$ij = k$$

k ought to denote an area. This is undoubtedly true, and in any equation where i and j were interpreted as lines k would have to be interpreted as an area; and to this interpretation the rules of quaternions offer no hindrance. Directed areas are a part of the system. The difficulty about dimensions is purely psychological and comes from confusing in thought the symbol with what it may, but need not invariably, stand for. The symbols i, j, k denote imaginary numbers; for these numbers there is an arithmetic, and one of the rules of this arithmetic is that when taken abstractly ij shall give k, where k is the same kind of a thing as i and j; just as 3 times 2 gives 6. The dimension theory is not in question. If the equations are to represent truths of geometry they must be homogeneous, but the homogeneity takes care of itself generally. It is perfectly useless to try to take actual lines and manipulate them mathematically; between the symbol and the reality there is an impassable gulf. The equation cited by Professor Macfarlane from the "Directional Calculus"

$$p_2 = p_1 + \epsilon,$$

where p_2 and p_1 denote points and ϵ a line, is necessarily a pseud-equation as he implies, and results apparently obtained from it can only be delusive unless the so-called points are really lines; but quaternions present no such anomalies. It seems to be unnecessary to try to make i, j, k mean merely lines or merely directions, or to lay down such a law as that "in such an expression as xi it is more philosophical and correct to consider x as embodying the unit, while i denotes simply the axis." It is surely not a defect of Hamilton's system that x may or may not be taken as a pure tensor according to circumstances, and that the number i may be of one or two dimensions in the unit of length like any other number, or that it need not contain the unit of length at all but may denote a certain rotation or a certain axis of rotation.

The essential difference between Professor Macfarlane's system of algebra and that of Hamilton is that he makes $i^2 = j^2 = k^2 = +1$ instead of -1. This destroys the associative character of the multiplication, but it allows an extension to manifoldnesses which quaternions cannot reach. With this rule the product of the imaginary numbers $A = ai + bj + ck$ and $B = li + mj + nk$ easily takes the form

$$AB = \cos AB + \text{Sin}\, AB,$$

where the meaning of the symbols cos and Sin is pretty obvious. $\cos AB = -\text{S}.\, AB$ in the language of quaternions and $\text{Sin}\, AB = \text{V}.\, AB$. The analytical meanings of cosine and sine are so well established that one would be tempted to wish the author had not made this use of them, especially as his reasons for discarding Hamilton's expressive symbols would apply quite as well to his own.

The formulas for the products of three and four of these numbers grow complex and would seem difficult to employ. Since $ABCD$ may mean various things according to the mode of associating the letters, it is indispensable to use the greatest care at every step. The author selects from the five possible products the two $((AB)C)D$ and $(AB)(CD)$ as the most important. The form $(AB)(CD)$ differs from the product of two quaternions on account of the squares of the units being positive.

A quaternion is represented by the symbol $a\alpha^A$, where a is the tensor, α the axis, and A the angle. The ordinary representation is $a\alpha^{\frac{2A}{\pi}}$, so that here α^A is the same as the familiar

$\alpha^{\frac{2A}{\pi}}$, and it comes out that α^{π} is the old α^2 or -1. Of course Professor Macfarlane's α^2 denotes a rotation through the angle 2 in circular measure, and there is no difficulty in distinguishing between this symbol and the α whose square is $+1$ in his system. Remembering that α^A turns a line lying in a plane at right angles to α through the angle $\frac{\pi}{2}$ it is evident that

$$a\alpha^A = a\ (\cos A + \sin A \,.\, \alpha^{\frac{\pi}{2}}),$$

as the author asserts. This is Hamilton's equation :

$$a\alpha^{\frac{2A}{\pi}} = a\ (\cos A + \sin A \,.\, \alpha),$$

the difference being merely one of notation. This new $\alpha^{\frac{\pi}{2}}$ exactly places the quaternion unit α in a multiplication-table. We may call the part in the parenthesis a versor without prejudice to either system, and by forming the product of two versors and taking its scalar part we get the fundamental formula

$$\cos C = \cos A \cos B - \sin A \sin B \cos C.$$

It is well known that the algebra of quaternions all having the same plane is not different from that of ordinary complex quantities, except in form. Thus we may write with Professor Macfarlane

$$\alpha^A = \cos A + \alpha^{\frac{\pi}{2}} \sin A,$$

or, as usual,

$$e^{iA} = \cos A + i \sin A,$$

and proceed to develop a plane trigonometry. The analyst will be likely to prefer the latter because he can reach it step by step from the multiplication-table he learned at school with no breach in the continuity of his logic, and with no hypothesis as to the geometrical meaning of i. Moreover the second equation is true in the same way that $3 + 2 = 5$ is true. It does indeed define the function e^{iA}, but perhaps in a very different sense from that in which the first equation may be said to define α^A. Changing A to $A + B$ and factoring the right member, we easily conclude with our author that

$$\alpha^{A+B} = \alpha^A \,.\, \alpha^B,$$

with commutative factors, and De Moivre's theorem is a near consequence. We shall have $(\alpha^A)^n = \alpha^{nA}$, but it is by no means true that $(\alpha^A\beta^B)^n = \alpha^{nA}.\beta^{nB}$. The result is much more complicated, for $(\alpha^A\beta^B)^n$ is a complex number of the form $(l + m\gamma^{\frac{\pi}{2}})^n$, where only γ is an imaginary, and the binomial theorem must be used.

Remembering that $\alpha^{\frac{\pi}{2}}$ has all the analytical properties of $\sqrt{-1}$, at least when not combined with other imaginaries, Professor Macfarlane easily shows that $\alpha^A = e^{A\alpha^{\frac{\pi}{2}}}$. Of course he implicitly defines the symbol $e^{A\alpha^{\frac{\pi}{2}}}$ by the series

$$1 + A\alpha^{\frac{\pi}{2}} + \ldots$$

He then concludes without difficulty that the logarithm of the quaternion $r\alpha^A$ is $\log r + \alpha^{\frac{\pi}{2}}A$; an expression which he shows how to generalize.

The products of vectors not being associative, the problem of finding their derivatives with respect to a scalar variable presents some interesting matter; and formulas for $\frac{d}{dt}(A^n)$, where A is a vector, are derived simply and elegantly. Of course the differentiation of a quaternion presents no new principles, and the same may perhaps be said of the author's treatment of matrices; but at any rate his remarks upon the latter subject are very interesting reading. After all, however, there is a certain flavor of genius in what Hamilton and Tait have to say about this particular kind of linear substitutions which nothing more recent seems to replace exactly. They certainly have the advantage in simplicity of language and method, and they are much more easily understood.

Professor Macfarlane dwells at some length on the properties of the operator $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$, which is formally a vector and shows easily that, if u is a scalar quantity,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

whereas in quaternions ∇^2 is negative.

When C is a vector ∇C will contain a scalar and vector part and he writes the solenoidal condition

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

in the concise form cos $\nabla C = 0$; while the condition for no molecular rotation is Sin $\nabla C = 0$. Tait writes these conditions:

$$S. \nabla C = 0; \quad V. \nabla C = 0.$$

Our author prefers to reserve the symbol ∇^2 for Laplace's operator and remarks that, for a vector,

$$\nabla(\nabla C) \neq \nabla^2 C.$$

Quite commonly $\nabla(\nabla o)$ would be written $\nabla^2 o$, and his remark that "$\nabla(\nabla C)$ is not equal to $\nabla^2 C$" can only mean that ∇^2 on a vector gives a result very different in form from ∇^2 on a scalar function. The first paper closes with some interesting examples of the use of ∇ and remarks on the addition of scalar and vector quantities having fixed positions in space.

In his second article Professor Macfarlane proposes "to review the different explanations [of the $\sqrt{-1}$] . . . . which have been contributed, with the hope of finding a theory which will tend to unify them." He points out that the investigation of the subject was started by the controversy about the logarithms of negative numbers and quotes from D'Alembert the following astonishing fallacy: "$e^{\frac{1}{2}} = +\sqrt{e}$ or $-\sqrt{e}$; but the logarithm of e is $\frac{1}{2}$; therefore the logarithm of $-\sqrt{e}$ as well as $+\sqrt{e}$ is $\frac{1}{2}$." Homer nods sometimes, but it is hard to imagine D'Alembert believing that by taking merely arithmetical roots of e he could produce a negative number. It recalls some of Euler's queer fancies about infinite series. After recalling various theories of $\sqrt{-1}$ our author pronounces its true explanation to be that "of a geometric ratio or quaternion" with "at least one other geometric meaning."

He treats next spherical trigonometry by quaternions. Recalling his equations

$$\alpha^A = \cos A + \alpha^{\frac{\pi}{2}} \sin A$$

and

$$\alpha^A = e^{A\alpha^{\frac{\pi}{2}}},$$

he easily concludes again that $A\alpha^{\frac{\pi}{2}}$ is the logarithm of α^A. The formal similarity of the powers of α to those of e^i being complete it can be seen at once how expressions of the most general kind can be obtained for the logarithms of negative numbers. By multiplying together α^A and β^B and taking the real and imaginary parts the fundamental equations of spherical trigonometry are reached almost instantaneously as in the former paper.

A section on circular spirals closes with the differentiation of α^A and other quaternion expressions, and with some remarks on hyperbolic trigonometry and hyperbolic spirals the article terminates. The hyperbolic trigonometry is founded on the equation

$$h\alpha^A = \cosh A + \alpha^{\frac{\pi}{2}} \sinh A,$$

where the symbols on the right have their usual meaning, and A is the area of a hyperbolic sector.

C. H. Chapman.

University of Oregon, *May* 14, 1894.

NOTE ON THE SUBSTITUTION GROUPS OF EIGHT AND NINE LETTERS.

BY G. A. MILLER, PH.D.

In calculating the possible groups of a given degree it is very helpful to have an accurate list of the groups of the lower degrees. An error in the lower groups is apt to give rise to numerous errors in the higher groups. On this account I have calculated all the possible groups through degree 9 and compared my results with the published lists. No complete lists of the groups beyond degree 9 have yet been published.

In the April number of this journal I noted several errors and one omission in the lists of the groups of eight letters. The following forms a supplement to this note.

There is a primitive group of degree 8 and order 1344, which is not given in the lists referred to in my note in the April number of this journal. The existence of a transitive group of this order and degree can be proved as follows:

$$A(abcdefgh)_8$$