LACHLAN'S MODERN PURE GEOMETRY.

An Elementary Treatise on Modern Pure Geometry. By R. LACHLAN, M.A. Macmillan & Co., 1893. 8vo, pp. x. + 288.

IN this work, with slight exception, the field of operations is the plane. In the plane there are considered points, lines, and circles; and concerning these there is an excellent store of geometric facts, illustrating various methods. Of these methods so many are to be found in English works on plane geometry, whether they are called pure or analytic, that the delimitation is not very evident. It appears to be accepted as the fundamental distinction that in the books on pure geometry the treatment of curves by means of their equations is excluded. Whether co-ordinates are excluded is mainly a question of words; the algebra employed does not look like algebra, but this is of no serious importance; what is really out of bounds is the equation between variable co-ordinates, or the theory of the ternary form. With respect to the use of algebra, it can hardly be maintained, in the present case, that every equation ought to be associated in one's mind with a direct geometric concept; this would, in many cases, be a direct waste of energy, as, for instance, where trigonometry is employed. Since some algebra is permitted, the question arises whether a formal use of the algebra of binary forms is not proper in a work of this kind, and I shall briefly return to this point later.

In Dr. Lachlan's book, after three preliminary chapters, we have the explanation of harmonic ranges and pencils, and then a chapter on involution. The statement (page 40) that the double points of an involution can be imaginary, is misleading; it is better to say that they can be off the line in question. Chapter VI. gives an account of the triangle and of some of its more remarkable circles. Then follows (chapter VII.) the theory of four points or lines, or, as they are called after Townsend, tetrastigms and tetragrams. One hopes that more pleasing pet names will yet be adopted. This chapter contains some extensions which deal mostly with six points or lines. Among these may be noticed (page 99) the case of two triangles triply in perspective, or rather of three triangles, any two of which are in perspective as seen from any point of the third. A simple instance is when two equilateral triangles have the same centre; the centres of perspective form a third equilateral triangle concentric with the others.

The theory of perspective is contained in chapter VIII. The symmetry of the diagram on page 100 should be remarked, as is done by Reye in the preface to his "Geometrie der Lage." It is a configuration of 10 points and 10 lines; there are 3 points on each line and 3 lines through each point. Taking 5 points in space, we have 10 lines through 2 points and 10 planes through 3 points, and the configuration is cut out of any plane by these lines and planes. The chapter includes an account of Pascal's theorem and of some of its developments.*

In chapter IX. is the theory of similar figures, which appears to be one of the most important outcomes of the work recently bestowed on the triangle. We have then (chapter X.) the theory of pole and polar with regard to a circle, of conjugate triangles, of four points of a circle, and of four tangents. Chapter XI. gives the theory of reciprocation, the next chapter that of two circles, and chapter XIII. that of coaxial circles, with a valuable discussion of Poncelet's theorem. The idea of the porism, incidentally mentioned on page 215, might have been introduced in other places; for example, in the diagrams of pages 100 and 164 poristic properties lie unconcealed.

Chapter XIV. deals with inversion. One important distinction between the theories of inversion and projection ought to be brought out. To suit the theory of projection we regard ∞ as a *line;* but for similar reasons ∞ is a *point* in the theory of inversion.

The next chapter on systems of circles contains much new matter. Among other things the triangles formed by three circles not meeting in a point receive here a fuller treatment than they are used to. The author makes use of the work of Mr. A. Larmor, and much also is no doubt his own property. Why does he not refer us to his own papers, in particular to the memoir in the *Philosophical Transactions,* vol. 177?

The last chapter contains the theory of cross-ratios. These are defined trigonometrically for the rays of a pencil, as was done in the harmonic case. There seems no sufficient reason for putting so fundamental an invariant as the cross-ratio so late.

The book contains many examples for solution. We welcome the author's statement that in all cases the solution rests on propositions immediately preceding, for exactly what the beginner needs, when he thinks he understands some theory, is a special application—an example and not a problem. We could wish that the names of the responsible parties were added to the examples in more cases; the practice of attaching a name to a result, even if the result be trifling, has

---

* It may not be out of place to refer to Mr. Richmond's paper on Pascal's hexagram, *Cambridge Phil. Trans.* vol. XV., pp. 267–302.

become common of late, and is apt to stimulate those who fight *pro areis et focis*.

Mr. Matthew Arnold in his "Science and Literature" gives a canon for judging a work of art: "Fit details strictly combined, in view of a large general result nobly conceived." The present book would avoid condemnation under this canon if some remarks were made at the outset on the principle of correspondence. We should then see that in projection we are studying a one-to-one correspondence of point to point, or of line to line; in inversion a different one-to-one correspondence of point to point, in which a circle answers to a circle, and the line appears as a special circle; in reciprocation a one-to-one correspondence of point and line; and we should be prepared for the direct correspondence of point and circle. which is discussed in chapter XV.—this apparently novel correspondence being, however, two-to-one. Lastly, we should see that involution belongs both to the projective theory and to that of inversion.

Throughout the book metrical methods are freely employed; in fact, there is little of the ideas of the *reine Geometrie der Lage*, and the works of the constructors of that edifice are not so much as referred to. Herein is a manifest injustice to the student, seeing that a mere note would have sufficed to put him on the track.

On the other hand, it seems a pity to make no use of the idea of the stroke or vector, when this would involve merely a modification of those proofs which are based on geometric constructions. Instances of important geometric ideas thus passed over, but attainable without any greater use of algebra, are those of involution as applied to points of a plane, and of the cross ratios of four points.

It seems not unlikely that a work of this kind is the proper place for the systematic development of the geometric meaning of the elements of the theory of binary forms. Beginning with the range of points, let us measure from a selected point, replace AB by $b-a$, and so forth. We then handle the quadratic equation, and deal with involutions and harmonic pairs as in Salmon's "Conic Sections"; using, however, geometric constructions freely. Let us then bear in mind that a and b can be complex, and make the necessary modifications in our diagrams. Sooner or later the single point of reference is replaced by two fixed points, and the determination of any point is effected by assigning the ratio of the strokes to it from the two points. We have then homogeneous co-ordinates. Proceeding to the cubic, or triangle of points, we should naturally put in the foreground the covariants—the polars of a point, the Hessian pair, the Jacobian triangle; and we should have at hand a scientific theory which covers many of the facts recently added to our knowledge of the triangle, and which indicates many others. A similar treatment of the quartic would follow. It is hardly an objection to say that the reader is not supposed to know about covariants, for here is as good a way as any to introduce them to him. The plan thus briefly sketched has, of course, no pretension to novelty; its claim to consideration here lies in its ability to classify and connect much of that matter of the book which is not already connected and classified by the methods of projection and reciprocation.

I have dwelt enough on defects in the author's programme. But in its handling so much power is displayed that one hopes he will set his hand to the plough again, and deal largely with some part of the present outlook as stated in Klein's "Vergleichende Betrachtungen" (BULLETIN, July, 1893); throwing examination schedules, if they interfere, to the four winds of heaven. FRANK MORLEY.

WEST FALMOUTH, MASS., *August*, 1893.

PAPERS OF MR. CHARLES CHREE ON VORTICES IN A COMPRESSIBLE AND ROTATING FLUID.

THE recent appointment of Mr. Charles Chree, M.A., of King's College, Cambridge, Eng., to be Superintendent of Kew Observatory as the successor of the late Mr. Whipple, leads us to hope that the study of the mathematical and physical problems whose solution is so important to the progress of meteorology will now receive a great stimulus in England. Mr. Chree has hitherto been known to us mostly through his excellent works in pure mathematics as applied to the subjects of elasticity and of vortex motion: his papers on vortex motion are probably but little known to the mathematical physicists of America, and with the author's help we are able to present the following account of his memoirs, which will be of interest to meteorologists in so far as the assumptions which underlie his mathematical solutions harmonize with the conditions that prevail in the atmosphere. On this latter point Mr. Chree very properly and modestly says:

"I should be sorry if any one supposed I profess to have actually solved the exact problems presented by nature. The lay reader is so much at the mercy of the mathematician that I think the latter is taking a most unfair advantage of his position if he avoid taking a reasonable care to prevent the