remember that if the power of abstraction fails their pupils they are often tempted to superficial and insincere study. They must also be willing to teach the higher subjects to small classes only.

Practical applicability in physics and astronomy must be the test by which it is decided what can be demanded of a large majority of their scholars; for it is these sciences which render the abstractions of pure mathematics not only intelligible but interesting to many who have not the utter disregard of the outside world which is characteristic of the pure mathematician.

ON THE TEACHING OF ELEMENTARY GEOMETRY.

Plane Geometry. On the heuristic plan. By G. I. HOPKINS. Boston, D. C. Heath, 1891.

Elementary Synthetic Geometry. By N. F. DUPUIS. New York, Macmillan, 1889.

Introductory Modern Geometry. By W. B. SMITH. New York, Macmillan, 1893.

Elementary Synthetic Geometry. By G. B. HALSTED. New York, Wiley & Sons, 1892.

NOWHERE has the conflict between the forces of conservatism and radicalism waged hotter than in the domain of geometry. The nature of the axioms, the character of the reasoning employed, the method in which the science shall be taught, have each given occasion for many a battle. Peace is not yet, but progress toward it is discernible.

To begin with, it is coming to be generally admitted that geometry is a physical science and that the truth of certain of its axioms, instead of being necessary and self-evident, is dependent upon the nature of space and our means of observation. Space being an hypothesis that the mind makes to explain phenomena, the character of space depends upon the character of the phenomena observed. That the phenomena that give our space-conceptions should be observed, and carefully too, the struggle for existence has inexorably compelled.

Then, too, it is here and there perceived that the reasoning of geometry, of which the characteristic is to spin out as many conclusions from as few data as possible, is not ideal. If observation, it is said, without our scarcely being aware of it, has given us our data, may it not equally have been playing a part in all our reasoning? Do we not reason rightly because we perceive rightly, so that geometry, as Gauss would have it, is "the science of the eye"? Would it not be more logical to consciously and avowedly use our eyes? And is it not safer to observe much and draw few conclusions, rather than little and draw many?

As for the manner of teaching, the number of those who would be content to set a student to memorize the demonstrations of a text grows daily less. That a student should observe and compare, and then draw his own conclusions; that he should have continual opportunity to apply his knowledge; that he should test both his own guesses and the statements of the book by careful constructions, and use these same constructions to suggest new theorems and methods,—all this bids fair to become a matter of course.

Of this general progress and of attempts at improvement in many minor details, the books whose titles head this article furnish instances. All deem it necessary to state that figures can be moved about in space without changing their size or shape. Mr. Dupuis and Mr. Halsted each distinctly calls this an assumption. Mr. Smith goes further and intimates that space may be boundless without being infinite. Moreover, he states some of the properties of space of uniform positive and of uniform negative curvature. All give plenty of problems for the student to work, and Mr. Hopkins's book is indeed mainly a collection of problems. All except Mr. Hopkins give some prominence to modern synthetic geometry, while Mr. Halsted gives the student a taste of even the more recent Lemoine-Brocard geometry. Let us take up these books in some detail.

Though Mr. Hopkins does not go beyond the time-honored bounds of elementary geometry, he claims a substantial improvement by presenting the subject *heuristically*. It surprises one, then, to find the book beginning with ten or a dozen pages of definitions and axioms. Does not heuristic treatment require that technicalities should be brought in by degrees rather than all at once? Again, we find given, for the student to demonstrate, at the very start, such propositions as: "all right angles are equal;" "if two angles are equal, the complements of those angles are equal": propositions whose truth to the student will seem as plain as any demonstration that can be given. Surely it would be better at first to confine the student to reasoning from what seemed self-evident to what did not. He would at least not run so great danger of thinking that in reasoning the chief essential was formality. As a further example of the heuristic method the author expresses his "firm belief that mathematicians have no right to amalgamate the *proportion form* and the *equation form* of expression." Yet we are not told why; on the contrary, the two are said to be equivalent! Quite as remarkable is the author's original demonstration of the Pythagorean proposition, which he makes depend upon this:

If, in any circle, there be drawn a diameter perpendicular to a chord, and if from one end of that diameter a second chord be drawn intersecting the first, then the rectangle on this second chord and that segment of it that meets the diameter is equivalent to the square on the concurring segment of the diameter increased by the square of half the first chord!

What is there heuristic about the book? Well, perhaps that certain of the demonstrations are given by means of leading questions; or that definition and consideration of limits and symmetry are relegated to the appendix; or that, now and then, there is such excellent advice to the student as, "use the most unfavorable figure:" but chiefly, I think, that the book is a collection of problems, that leaving out these it is impossible to find a continuous text for memorizing. If, by this, a few more teachers are driven to requiring problem working of their students, the book will do good.

Mr. Dupuis' geometry has been prepared with extreme care and covers, with admirable thoroughness, much ground. Some will doubtless object to his treating distance and direction as simple conceptions; but simpler they certainly are than the reasoning that proves them mysterious, reasoning which after all winds up by adopting for Euclid's space precisely the ordinary common-sense conceptions. The state of the matter is this. Certain notions are derived from our race-experience, among them distance and direction. These notions profound investigations have shown to be compatible with only "the dreary infinitudes of homoloidal space." But what have we to do with any other space in elementary geometry? Ought not a student, must he not, in fact, really begin with his own race- and experience-given notions? When he has learned to reason from these as a basis it will be time to think of how to soar above such petty restrictions into the heaven of the nth dimension. Once break the bounds and where shall we stop? Is there, after all, any more warrant for assuming that space is alike throughout, *homœoidal,* than for saying that it is homoloidal? Can we even maintain that it possesses the property of elementary flatness? These are matters for the professional mathematician and it is not necessary to confuse the learner with them.

The usual tedious calculation of π is omitted, the student being referred elsewhere for this. Algebra is skilfully worked in with the geometry and there is a good chapter on the interpretation of algebraic forms. The allusions to mathematical instruments are valuable, if only the instruments themselves can be put into the student's hands for actual practice. The early introduction of *sine* and *cosine* might well be followed up by a short course in plane trigonometry before going on with the rest of the book. The concluding section on "geometric extensions" (modern geometry) is perhaps as good as can be, if with the circle the other conics are not to be treated. Taken all together the book very well serves the author's stated purpose of an introduction to the modern works on analytical geometry.

Mr. Smith's book has already been reviewed in the BULLETIN.

In reading Mr. Halsted's book it is difficult to rid one's self of the impression that the author somewhat scorns conservatism.

Even the language is original. An indefinite straight line is a *straight,* a limited portion of it is a *sect,* and points upon it are *costraight.* Central symmetry is *symcentry* and a symcentral spherical quadrilateral is a *cenquad.* One is reminded of a certain colloquy beginning: "Do you abbrev.?" "Cert."

The demonstration of all the usual cases of the congruence of triangles is given in eight lines of text with no reference to a diagram. There are, however, on the same page several figures evidently intended for illustration. Demonstration in general terms is, indeed, a marked feature, and a good one, of the book. Of course the student should be exercised in applying these to diagrams, while he should also have practice in changing back from forms with diagrams to forms in general terms.

The arrangement of the subject-matter illustrates the fact that in geometry, as in other sciences, one can begin almost anywhere and go in almost any direction, if he will but proceed circumspectly. And if putting pure spherics near the beginning only leads some to realize that solid geometry can and ought, to some extent, to be taught along with plane, the way will be paved for a distinct advance over the usual presentation.

The chief advantage in bringing in chapters on modern and recent geometry is to teach the progressiveness of the science. The end would have been further secured by historical notes, and the student would have the further advantage of learning how modern and how recent the theorems were.

In books having so much that is unusual in matter and arrangement as those of Dupuis, Smith, and Halsted, an index is especially helpful, and it is gratifying to find that each contains one. All would be improved by having, in addition, synopses and syllabi; while more abundant references to first sources would be valuable to teachers and advanced students.

The book that shall present geometry heuristically is, I fear, an unattainable ideal. Each student would have to be specially written for, and only his particular teacher, daily carefully watching his mental growth, could do it. Even this teacher would run a continual risk of interfering with the student's development. Suppose the student goes wrong at times. An occasional mistake is not necessarily harmful; further investigation is sure, at last, to correct it. Only let the students be kept interested and a year's work will teach them much.

But the teacher must study his class. No amount of routine will quite kill their originality. Howsoever careful and complete the memorizing, the precedence of ideas in the students' minds will not be just that in the book, and each student will actually, in so far as he does learn anything, be learning in his own way. When even teachers and bookmakers differ so much, what must we look for in the yet untrained boy?

Take a case in point.

During the past year I had a small class in geometry. The repugnance of many of them to demonstrating what seemed quite plain without it interested me, and I asked what propositions in their geometry (Byerly's Chauvenet) seemed to them to need no demonstration. Here are the answers on the first book. They had studied three books.

Out of 13 students, 13 thought I self-evident; 13, II; 12, III; 7, IV; 9, V; 7, VI; 7, VII; 10, VIII; 11, IX; 6, X; 6, XI; 8, XII; 6, XIII; 4, XIV; 5, XV; 9, XVI; 12, XVII; 9, XVIII; 6, XIX; 4, XX; 8, XXI; 12, XXII; 11, XXIII; 3, XXIV; 4, XXV; 3, XXVI; 1, XXVII; 5, XXVIII; 9, XXIX; 8, XXX; 6, XXXI; 5, XXXII.

There was not a single proposition that some one did not think self-evident, and in the entire five books of plane geometry there were only six, viz.: X of book III; VIII, IX, and X of book IV; IV and IX of book V.

Evidently the list of axioms could be considerably enlarged with perfect satisfaction to the average student. The advisability is shown, it seems to me, of pointing out the reasons for undertaking certain demonstrations.

To still further test the class I drew up the following questions:

1. One of two iron balls is twice as far through as the other. How many times as much does it weigh?

2. How many times as much leather does it take to cover it?

3. If one angle of a triangle is twice as large as another, is the side opposite the first angle twice that opposite the second?

4. Two sides of one triangle are respectively equal to two sides of another triangle, but the third side of the one is once

and a half the third side of the other. Is the angle opposite the third side in the first triangle once and a half that opposite the third side in the second triangle?

5. If one chord of a circle is twice as large as another, will it be twice as near the centre?

6. Will the two lines that trisect the vertical angle of an isosceles triangle also trisect the base?

7. Which will stand the most firmly on an uneven floor, a three-legged, a four-legged, or a five-legged table?

8. Which will take the more stone to build a wall 3 feet high, 1 foot thick, and a mile long, on a level or over a hill? Distance is measured along the wall.

9. If you wanted to go the nearest way to a point due east, would you go directly east?

10. Can you cut out of a potato a flat-faced solid with 5 edges? with 6? with 7? with 8?

11. Can you cut one out all of whose faces are six-sided?

12. How many braces does it take to stiffen a three-sided plane figure? a four-sided? a five-sided?

13. Are lines in space perpendicular to the same line parallel?

14. One of two square pyramids is twice as high as the other, but its base is half as far across. Does it take more or less stone to build it?

In answering these questions, students were asked to give their first impressions, their second impressions, any reasons, even vague ones, that they might have for their answers.

Replies were received from twenty. I give a part of the results:

Question.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Number giving right answer....	4	7	6	2	8	3	13	1	6	2	4	0	3	5
Number giving right reason.....	2	2	5	0	3	1	0	1	1	2	0	0	0	3

The largest number correctly answered by any one student was 8, and for all but one of these answers his reasons were correct. On first impressions he would have answered but two rightly and have been doubtful about one of these.

The tendency to accept as true statements resembling true ones is clearly marked. This is an exceedingly useful tendency; by its aid we frame inductions. But there needs to go along with it the habit of taking the resembling statements only provisionally, to be carefully tested at the first opportunity. In geometry the supreme test is, of course, "Can a valid reason be given?" But, while awaiting that, there is readily available the test of careful drawing and observation. Of this there cannot be too much. But with this there should go reasoning from figures that are purposely drawn false, that are obtrusively, what all figures are when innerly viewed, mere diagrams to symbolize the relations under consideration.

I have referred to the importance of connecting geometry with its practical applications. Here it seems to me that all our texts, even these recent ones, are weak. The early introduction of trigonometry and mensuration is only one of many opportunities offered. The triangle of geometry stands for the triangle of forces, the theory of proportional lines leads at once to graphical methods of computation, that of intersecting lines and planes leads to descriptive geometry and perspective, while these, in turn, furnish the best possible introduction to modern projective geometry. Such are among the many vistas into the surrounding universe that are to be gained from the highway of geometry. Why should these be shut out by the close walls of prejudice and custom on either hand? There is not time for these outlooks! Why, the way to save time is to make the teaching interesting. To this end, varied applications and illustrations are the most effective means.

ELLERY W. DAVIS.

THE MATHEMATICAL CONGRESS AT CHICAGO.

THE mathematical section of the Congress on Mathematics and Astronomy held in Chicago from August 21st to 26th was of the highest interest to all present, particularly on account of the active participation of Prof. Klein, of Göttingen. Only a brief outline of the proceedings can be presented here, but it is hoped that a full official report of the proceedings will ultimately be published.

Monday's session was devoted to preliminary addresses and to organization, Prof. Klein referring in his introductory address* to two of the special papers presented:

Gruppentheorie und Krystallographie, by Prof. Schönflies of Göttingen, and

Ueber einige mathematische Resultate neuerer astronomischer Untersuchungen, insbesondere über irreguläre Integrale linearer Differentialgleichungen, by Dr. Burkhardt of Göttingen.

The so-called "structure theory" of crystals deals essentially with a problem of the theory of groups, namely, with the enumeration of all discontinuous sub-groups which may be formed from the group of space-movements, combined also with reflection. Dr. Schönflies has already (1891) made a

* See this number of the BULLETIN, p. 1.