ON THE NON-EUCLIDIAN GEOMETRY.

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The celebrated tracts of Lobatschewsky and Bolyai, in which those writers showed what geometry might become if the parallel-axiom were left out, were long since translated into the chief languages of the continent, but have until the last year remained inaccessible to those whose only tongue is the English. The thanks of this large class are due to Professor Halsted * for supplying the deficiency in good clear style. The critical remarks made by Professor Halsted upon geometrical text books in current use show much acuteness, but they partake of the nature of ephemeral polemics, and will doubtless be omitted in any future edition of these translations, which ought to be republished together in permanent form as a standard work. The practical services which he has rendered to other mathematicians, not only by his valuable text-books, but also in the publication of his elaborate "Bibliography of Hyper-Space and Non-Euclidian Geometry" (*American Journal of Mathematics*, vols. 1 and 2), and now by these translations, are eminently deserving of appreciation and imitation.

Lobatschewsky, acting on suggestions of Gauss, delivered his first lectures on non-euclidian geometry in 1826, his completed work following in 1840. Bolyai's tract was published in 1832 as an appendix to a work of his father, who was also a friend of Gauss. Both of these authors begin their investigations by assuming that through a given point in a plane more than one line can be drawn which shall never meet a given line. Their results, as far as they cover the same ground, are identical in substance, though different in form. The sum of the angles of a triangle is less than two right angles, so that a rectangle is impossible; the angle-sums of two triangles of equal area are equal; no two triangles not equal can have the same angles, so that similar triangles not of the same size are impossible; if two equal perpendiculars are erected to the same line, their distance apart increases with their length; a line every point of which is equally distant from a given straight line is a curved line; any two lines which do not meet, even at infinity, have one common perpen-

* Geometrical Researches on the Theory of Parallels. By Nicolaus Lobatschewsky. Berlin, 1810. Translated by George Bruce Halsted. Austin: published by the University of Texas, 1891.

Scientiæ Baccalaureus, vol. 1, No. 4, June, 1891. The Science Absolute of Space. By John Bolyai. · Translated into English by George Bruce Halsted.

dicular which measures their minimum distance; and lines which meet at infinity are parallel. Each of these authors finds that there is a peculiar curved surface—"boundary surface" of Lobatschewsky, "surface F" of Bolyai—produced by the revolution of "a curve for which all perpendiculars erected at the mid-points of chords are parallel to each other," a surface on which the sum of the angles of any triangle is two right angles and on which, therefore, euclidian geometry is valid.

Three necessary and sufficient theorems of plane trigonometry are given by Bolyai, one of which however may be derived from the other two, which are

$$\sinh\frac{a'}{k} = \sinh\frac{h'}{k}.\ \sin\alpha', \tag{1}$$

$$\cosh\frac{h}{k} = \cosh\frac{a'}{k}.\ \cosh\frac{b'}{k}, \tag{2}$$

where a' and b' represent the sides, and h' the hypothenuse, of a right angled triangle, α' the angle opposite a', and k a constant; sinh and cosh standing as usual for hyperbolic sine and cosine respectively. The constant k is presumed uniform throughout space. If it is infinite in value, all the non-euclidian formulæ deducible from (1) and (2) assume euclidian forms, and our geometry for that value of k is euclidian geometry. For, when $k = \infty$, $k\ \sinh\frac{x}{k} = x$, and since $\cosh x = \sqrt{(1 + \sinh^2 x)}$, the non-euclidian theorems (1) and (2) readily assume the usual euclidian forms, $a' = h' \sin\alpha'$, $h'^2 = a'^2 + b'^2$.

If we were to substitute sin and cos in equations (1) and (2) for sinh and cosh, we should have the two theorems necessary and sufficient for the development of spherical trigonometry. To effect such a substitution we need only write $k\sqrt{-1}$ for k. It follows that when k is finite, we have Lobatschewsky's geometry, when k is infinite that of Euclid, and when k is imaginary that of the surface of the sphere. Observing this, Riemann* suggested the idea of a surface of uniform negative curvature, in all respects converse to the uniform positive curvature of the surface of the sphere. The geometry of such an ideal surface would be Lobatschewsky's. From this point of view the plane geometry of Euclid is on a surface of zero curvature. This was a wide

* Habilitationsschrift, "Ueber die Hypothesen welche der Geometrie zu Grunde liegen," read at Göttingen in 1854, printed posthumously in 1868, and translated by Clifford (*Nature*, vol. 8) in 1873.

departure from the ideas of Gauss, Lobatschewsky, and Bolyai, who dealt only with honest planes and genuine straight lines.

The effort of Riemann and others to explain the paradoxes of Lobatschewsky by assuming his planes and lines to have curvature has led to much brilliant work,* much of which however may be ranked as analytical rather than geometrical in its essence. There is no real surface converse to the sphere, on which Lobatschewsky's geometry holds good, in the same sense that spherical geometry holds good on the surface of the sphere. The only surface of uniform negative curvature is that of Beltrami's pseudosphere,† a surface of revolution, as to which I have computed axial coordinates (x) for successive equidistant values of the radius of revolution (r), as follows:

r	x	r	x
1.0	0.000	0.4	0.650
0.9	0.031	0.3	0.920
0.8	0.093	0.2	1.313
0.7	0.181	0.1	1.998
0.6	0.299	0.0	∞
0.5	0.451		

Those who will take the trouble to envisage these numbers will see before them a surface resembling that of a straight flaring trumpet having an infinitely elongated mouthpiece, or, as some one has put it, of a champagne glass having its stem extended to infinity. On it any figure may be changed in place while still fitting the surface, like figures on the surface of the sphere, but with this difference, that on the pseudosphere the figure removed does not retain its rigidity, but submits to bending, somewhat as a plane figure may by bending be fitted to the surface of a cylinder. By treating this surface as a plane and its geodesics (corresponding to the great circles of a sphere) as straight lines, we may develop upon it all the theorems of Lobatschewsky's geometry. But Lobatschewsky wrote of planes and lines and not of surfaces resembling trumpets.

To Cayley, Fiedler, Beltrami, and Klein is due the simpler

* Cf. Killing, *Die Nicht-Euklidischen Raumformen*, Leipzig, 1885; Beez, *Ueber Euklidische und Nicht-Euklidische Geometrie*, Plauen i. V., 1888. Prominent among American writers in this direction are Newcomb and C. S. Peirce.

† *Giornale di Matematiche*, 1868. See also Cayley, *Proceedings of the Royal Society*, vol. 37, 1884.

explanation,* namely, that we need no change in our notions of planes and straight lines, but rather an extension of our ideas concerning measurement. From this point of view, the theory of projective metrics, as developed by Cayley and Klein (some account of which is contained, *inter alia*, in the article "Measurement" of the *Encyclopædia Britannica*), constitutes a system of which Lobatschewsky's geometry is a special case. In Cayley's metric system, the "distance" between two points is defined to be a constant multiplied into the logarithm of the cross-ratio of the real distances between each of such points and each of the two other points in which the line containing them intersects a given quadric curve (for a plane) or surface (for three dimensions) known as the absolute quadric. If the absolute in a plane be a large circle of radius k, and if all distances within the circle be measured upon this metric system, with corresponding measurement of angles, the plane geometry obtainable with such data—straight lines being really straight lines—is that of Lobatschewsky. I cannot do better here than to quote from Cayley's own explanation in his Southport address:

"We measure distance, say, by a yard measure or a foot rule, anything which is short enough to make the fractions of it of no consequence (in mathematical language by an infinitesimal element of length); imagine, then, the length of this rule constantly changing (as it might do by an alteration of temperature), but under the condition that its actual length shall depend only on its situation on the plane and on its direction: viz., if for a given situation and direction it has a certain length, then whenever it comes back to the same situation and direction it must have the same length. The distance along a given straight or curved line between any two points could then be measured in the ordinary manner with this rule, and would have a perfectly determinate value; it could be measured over and over again, and would always be the same; but of course it would be the distance, not in the ordinary acceptation of the term, but in quite a different

* Cayley, Sixth Memoir on Quantics, *Philosophical Transactions*, 1859; On the Non-Euclidian Geometry, *Mathematische Annalen*, vol. 2, 1872; Address as President of British Association at Southport, 1883, reprinted from *Nature* in *Littell's Living Age*, vol. 159, p. 177.—Fiedler, *Die neuere Geometrie*, Leipzig, 1862, at end.—Beltrami, *loc. cit.* This is the only citation I am unable to attest directly. For my scant knowledge of Beltrami's alternative (Cayleyan) view I am indebted to notices by Beez.—Klein, Ueber die sogenannte nicht-Euklidische Geometrie, *Mathematische Annalen*, vols. 4 & 6. Klein expresses elsewhere (*Jahrbuch der Mathematik*, vol. 5, p. 273) in so many words his judgment that Lobatschewsky's and Cayley's geometries are identical: ". . . die sogenannte nicht-Euklidische Geometrie, oder, was dasselbe ist, die von Cayley begründete projectivische Massgeometrie."

acceptation. . . . And corresponding to the new notion of distance, we should have a new, non-euclidian system of plane geometry. . . . We may proceed further. Suppose that as the rule moves away from a fixed central point of the plane it becomes shorter and shorter; if this shortening takes place with sufficient rapidity, it may very well be that a distance which in the ordinary sense of the word is finite will in the new sense be infinite; no number of repetitions of the length of the ever shortening rule will be sufficient to cover it. There will be surrounding the central point a certain finite area such that (in the new acceptation of the word distance) each point of the boundary thereof will be at an infinite distance from the central point; the points outside this area you cannot by any means arrive at with your rule; they will form a *terra incognita,* or rather unknowable land: in mathematical language, an imaginary or impossible space: and the plane space of the theory will be that within the finite area—that is, it will be finite instead of infinite. We thus with a proper law of shortening arrive at a system of non-euclidian geometry which is essentially that of Lobatschewsky."

With this explanation it is easy to understand all of Lobatschewsky's paradoxes. Thus, two lines which cross the circle and meet on its circumference are, for dwellers within it, lines which meet at infinity, or what he calls parallel lines; while two lines crossing the circle and meeting outside of it are for them extra-parallel lines.

Given analytically three geometries, for which respectively the k of equations (1) and (2) is real, infinite, or imaginary, a certain change in terminology, so slight that it can scarcely be novel, will enable us to bring more distinctly into view their relation as special cases of one general geometry. It consists in employing a symbol, say c, in lieu of $-k^2$, and in writing Sx' for $c^{-\frac{1}{2}}\sin(c^{\frac{1}{2}}x')$ and Cx' for $\cos(c^{\frac{1}{2}}x')$, so that when $c=1$, the generalized expressions S and C represent sin and cos respectively, and when $c=-1$, sinh and cosh respectively, as functional symbols. That is to say,

$$c\,(Sx')^2 + (Cx')^2 = 1, \tag{3}$$

where

$$Sx' = x - \frac{1}{3!}cx'^3 + \frac{1}{5!}c^2x'^5 - \ldots . \tag{4}$$

$$Cx' = 1 - \frac{1}{2!}cx'^2 + \frac{1}{4!}c^2x'^4 - \ldots . \tag{5}$$

Throughout all the formulæ of general geometry the constant c will enter, as in these definitions, only in integral powers. From (1) and (2) we have these fundamental theorems,

$$\mathrm{S}a' = \mathrm{S}h'.\ \sin \alpha', \tag{6}$$

$$\mathrm{C}h' = \mathrm{C}a'.\ \mathrm{C}b'. \tag{7}$$

Here, as will be remembered, a' and b' are the sides, and h' the hypothenuse, of a right angled triangle, and α' the angle opposite a'. We may, analytically, distinguish our three geometries as those of c positive, c zero, and c negative, respectively. (These correspond to the cases named by Klein as elliptic, parabolic, and hyperbolic.)

How are we to interpret geometrically these formulæ, and those which may be deduced from them? According to Riemann, by means of curved surfaces, modifying our definitions of planes and lines correspondingly; so that for c positive we shall have spherical geometry as usually developed, and for c zero euclidian geometry, regarding the euclidian plane as a spherical surface of infinite radius. But there are difficulties. Merely for two dimensions we have to postulate a sphere of imaginary radius, while for three-dimensional geometry the Riemannian theory, which requires a fourth dimension in which to operate, consists likewise (allowing for a point or two in which Riemann's analysis has been corrected) of analytic verities devoid of geometric meaning.*

The Cayley-Klein interpretation, on the other hand, is intelligible. It is true that the "absolute" circle of Cayley becomes imaginary when c is positive. The objection is not important, as it relates only to the initial step, the definition of distance. Such as it is, we shall see that it may be obviated. The statement which follows contains nothing not deducible analytically from the theory of projective metrics developed by the writers named, which indeed includes much within its scope besides the non-euclidian geometry.

Referring to the quotation from Cayley's presidential address, we must understand as implied in it that the inhabitants of the charmed circle, as well as their foot-rules, grow smaller as they recede from the central point, without themselves being aware of any change. The contrary supposition, representing c positive, that inhabitants and measuring instruments alike grow larger as they recede from the central point, is naturally correlative to that made by Cayley, which represents c negative. The question arises: How can we deter-

* "*Geometrische Gespenster.*" Unintelligible phrases, admissible as means, are commonly distrusted as ends. *Cf.* Poincaré, *Revue Generale des Sciences*, No. 23, translated in *Nature*, vol. 45, Feb. 25, 1892, who says, apropos of the Riemannian theory: "The minds which space of four dimensions does not repel will see here no difficulty; but these are few. . . . Any one who should dedicate his life to it could, perhaps, eventually imagine the fourth dimension."

mine the laws governing such changes in either case, how can we even be sure that a triangle, for instance, will retain the same measurements of sides and angles after removal to another part of the plane, without recourse to Cayley's logarithmic definitions of linear and angular measurements?

Let a plane be touched by a sphere, both being intersected by another plane passing through the centre of the sphere, in the one case a straight line, in the other a great circle, being the line of intersection. The great circle, or let us say that half of it nearest to the plane, is the central projection of the line upon the surface of the sphere. Similarly, any figure upon the plane will have its projection upon the surface of the sphere, every point of it having its corresponding point on the spherical surface, such two points, of course, lying in one straight line with the centre. Let us call the distance between any two points in the plane their *real distance*, and let us call the length of the arc of a great circle between their projections their *projective distance*. Similarly, let us call the angle made by two lines in the plane their *real angle*, and that made by their projections their *projective angle*. If, now, in our geometrical researches upon the plane, we deal with the real distances and the real angles, we shall employ euclidian geometry. If, on the other hand, we deal with the projective distances and projective angles, we shall develop a non-euclidian geometry, of which the theorems will be those of spherical geometry, since every figure upon the surface of the hemisphere will have its counterpart upon the plane, equivalent to it in all its parts respectively, and *vice versa*.

Any other plane parallel to the tangent plane and not passing through the centre of the sphere will obviously be susceptible of the same projective figures as the tangent plane, on a scale proportional to the distances of the two planes from the centre of the sphere. Upon any such parallel plane, therefore, any real or projective angle or any projective distance will be the same in measure as the corresponding angle or distance upon the tangent plane, while all real distances will be p/r times as great as the corresponding real distances upon the tangent plane; p and r representing respectively the real distances of the parallel and tangent planes from the centre, the latter being also the radius of the sphere.

Having once obtained and noted certain measurement-theorems by means of the spherical projection, we may dismiss the sphere from further consideration, and regard the tangent plane merely as one upon which those theorems hold good, having reference to a certain middle point, once the point of tangency. The theorems themselves retain a trace of the sphere in the constant r^2, for which let us substitute the letter c. Let us call any plane such as our former tangent plane a prime plane, and any plane such as our former parallel plane a secondary plane. Let us retain the word "projective" as a mere name for a mode of measurement, without reference to its meaning, because we must have a name. Let us take all planes passing through a given point as prime planes, having each as its middle point the given point, which let us call the centre of space, or merely the centre. If we take as the middle point of any other plane in space the foot of the perpendicular let fall upon it from the centre, the projective distance between any two points in such other plane is given immediately by known theorems, since the two points are likewise points of a prime plane passing through the centre. Our theorems tell us further that if we compare any and all projective distances in the new plane with the corresponding projective distances in any prime plane, that is, those which have the same projective situation with regard to the middle points, the real distances respectively in the new plane are constant multiples of those in the prime plane, the constant factor being a simple function of the perpendicular. The real angles in the two planes are therefore the same, each to each. The projective angles, as we know from (6) and otherwise, are fixed when the projective distances are fixed, so that they also are identical in the two planes, each to each. The new plane is thus a secondary plane. All planes in space are therefore either prime or secondary, and all projective distances and angles in space are parts of one harmonious system of general geometry of three dimensions.

For expressing relations between projective distances and angles, we have all that we need in (6) and (7), and in the other theorems of spherical geometry which spring from those fundamental equations. For relations between real distances and angles we have the usual theorems of euclidian geometry. To connect the two sets of theorems, a single equation between real and projective distances is necessary and sufficient. Reverting to our tangent plane (afterwards "prime plane"), let h be the real distance between any point in it and the middle point of tangency (afterwards "centre"), h' being the corresponding arc of great circle which we call the projective distance between the two points. Then h'/r is the angle at the centre of the sphere between the lines connecting it with the two points, so that $r \tan (h'/r) = h$. But, since $r^2 = c$, $r \tan (h'/r) = \mathrm{S}h'/\mathrm{C}h'$, according to our definitions of S and C. Hence, if we write $\mathrm{T}h'$ for $\mathrm{S}h'/\mathrm{C}h'$, we obtain at once the desired equation,

$$\mathrm{T}h' = h, \tag{8}$$

whence $$h' = h - ch^3/3 + c^2h^5/5 - \ldots . \tag{9}$$

Since $(\mathrm{T}h')^2 = (\mathrm{S}h')^2/(\mathrm{C}h')^2 = h^2$, and, by (3), $c(\mathrm{S}h')^2 = 1 - (\mathrm{C}h')^2$, it follows that

$$\mathrm{C}h' = (1 + ch^2)^{-\frac{1}{2}}, \tag{10}$$

$$\mathrm{S}h' = h\,(1 + ch^2)^{-\frac{1}{2}}. \tag{11}$$

When c is positive, a value of h' in (8) will always be found between the two values of $\pm\pi/2c^{\frac{1}{2}}$, and this must be taken. For distances not measured from the centre, let us first suppose that the straight line or sect (to use a convenient expression suggested by Professor Halsted to denote a limited straight line) measured from the centre by h and h' is the hypothenuse of a right angled triangle, of which the base, measured really and projectively by a and a', proceeds likewise from the centre, while the third side, measured by b and b', extends from the right angle to the original non-central point; and let the angle opposite a be denoted by $\alpha = \alpha'$, since the real angle is the measure of the projective angle between two lines passing through the centre. Then $b = h \sin\alpha$, and, by (6), $\mathrm{S}b' = \mathrm{S}h' \,.\, \sin\alpha$, whence $\mathrm{S}b'/b = \mathrm{S}h'/h$. Combining this with (11) and (10),

$$\mathrm{S}b'/b = \mathrm{C}h'. \tag{12}$$

This concise expression gives the relation between the real and projective measurements of the distance between two non-central points, taking for us the place of Cayley's definition of distance as the logarithm of a cross-ratio. The sect bb' (meaning distance whose real measure is b and projective measure b') begins, it is true, at the middle point (foot of perpendicular from centre) of the line of which it forms a part; but any other sect of the same line is merely the sum or difference of two such sects, say b_1b_1' and b_2b_2'. Distance upon any line is thus measured most simply, for our purposes, from its middle point. Equation (12) may be illustrated by saying that if a real sphere be assumed to exist having h as its diameter, one end of it at our centre, and if distances be measured from the non-central extremity to other points of the surface of the sphere there will in all cases be the same ratio between the real distance and the function S of the projective distance. Such lines will all be represented by bb', while for all of them $\mathrm{C}h'$ will have the same value.

Given (8) and (6) as original definitions, without reference to the spherical projection, but assuming, of course, symmetry as regards the centre, the other formulæ of general geometry can be deduced from them in due order. (I have, in fact, so deduced (7) and many other formulæ, but consider this BUL-

LETIN not intended for the publication of detailed investigations.) Upon this system c may have any value, positive, zero, or negative. With negative values of c we have the geometry of Lobatschewsky and Bolyai, confined, however, to points not really further from the centre than $(-c)^{-\frac{1}{2}}$, the radius of Cayley's circle; because (10) and (11) become imaginary when $ch^2 < -1$. Interesting explanations of what to them were serious paradoxes are readily obtained. For instance, the remarkable curved surface upon which, with projective measurements, euclidian geometry obtains, I find to be a surface of revolution described by a real ellipse; a projective circle is also a real ellipse; and the "line of equal distance" from a given straight line, proved by Lobatschewsky and Bolyai to be a curve, is again a real ellipse. When c is positive, on the other hand, a projective circle is really, according to its position and radius, either an ellipse, a straight line, a parabola, or a hyperbola, and the line of equal distance is a real hyperbola.

As regards angles, it is enough to say here that, with c positive or negative, the projective measure of an angle whose vertex is not at the middle point is the same as its real measure when, and only when, it is a right angle between two lines, one of which passes through the middle point of the plane in which it lies.

If we attempt to draw a projective square, taking two sides of equal length x', with a projective right angle between them, and at their ends erect projective perpendiculars z' meeting each other in a point, the two lines z' being of course projectively equal to each other, we have this relation,

$$\mathrm{T}z' = \mathrm{S}x'. \tag{13}$$

It is readily to be seen that z' cannot equal x' unless $\mathrm{C}z' = 1$, and this cannot happen unless $c = 0$, that is to say, unless our geometry is euclidian, in which projective and real measurements coincide. When c is positive, $z' < x'$, and *vice versa*. Again, if we attempt to construct a rectangle by erecting two projectively equal projective perpendiculars to a given base line, and connecting their extremities, we shall find that when c is positive the connecting line is projectively shorter, and when c is negative longer, than the base. In other words, projective rectangles are impossible, except in the euclidian case $c = 0$. It is understood, of course, that the erection of a perpendicular is to be accomplished as in euclidian geometry, but using projective measurements, or say projective calipers.

If we suppose ourselves incapable of making any but projective measurements, in consequence of unsuspected changes constantly occurring in our own magnitude and in that of our foot-rules, arising from changes of our location in space, we shall see that we cannot in that event discover the location of the centre of space, though such centre must really exist. If on the same supposition the fourth side of an attempted rectangle is found greater than the base, we shall merely develop the geometry of Lobatschewsky; if less, the theorems of spherical geometry. In the former case we must infer that our visible universe is comprised within the limits of one of Cayley's finite real spheres, of which an unlimited number may exist in space; and we cannot possibly discover whether we and our foot-rules happen to be near the centre of our sphere, and therefore are large in real size, or near the surface, and therefore small or infinitesimal. The existence of such a state of things within any one or within any number of such physically bewitched universes would not prevent the real measurement of the whole system by unchangeable foot-rules in the hands of geometers able to traverse it without variation in their personal magnitude. On the other hand, if the fourth side is found less than the base, we shall develop the theorems of spherical geometry, and shall also, without knowing it, grow larger in size and take larger paces as we recede from the centre, so that while still measuring a finite distance according to our own understanding, we shall easily pass a point really infinite in distance from the centre, ourselves at the same time inconceivably great in real bodily magnitude.

The reader will have observed that I have repeatedly assigned limits to the theory of the case $c > 0$. Attention was confined in the first place to the projections of points in the tangent plane upon that half of the sphere nearest to the plane. By this means the projective distance of any point on the plane from the centre was confined, as subsequently stated, to values limited to $\pm \pi / 2c^{\frac{1}{2}}$, the greatest possible projective length of any sect being therefore $\pi/c^{\frac{1}{2}}$, the measure of a line whose real length extends from $+\infty$ to $-\infty$. If we remove the limitation, we shall find that the end-point of a straight line proceeding from the centre goes regularly forward till the projective distance reaches $\pi / 2c^{\frac{1}{2}}$, at real infinity, when it starts back discontinuously to negative infinity to continue the projective measurement, reaching the same centre point again after traversing a total projective distance of $\pi / c^{\frac{1}{2}}$. Similarly, the total projective length of every straight line, counting from any one point to itself, subject to the same discontinuity, is $\pi / c^{\frac{1}{2}}$; so that if two lines cross, they meet again at the same point, each having traversed the projective length $\pi / c^{\frac{1}{2}}$, without meeting at any other point. For any distance d', therefore, we may write $d' + k\pi / c^{\frac{1}{2}}$, where k is integral.

We may vary our general view by assuming that all figures upon the tangent plane are re-projected perpendicularly upon a plane parallel to it and passing through the centre of the sphere, the latter now being taken as the typical prime plane. On this assumption, a point on the surface of the hemisphere is projected first upon the tangent plane by a line from the centre, and thence by a perpendicular to the prime plane; the prime plane, with all points and figures upon it, being in fact the same as the tangent plane translated. Let us now assume another tangent plane for the other hemisphere, with a similar re-projection of all points upon the prime plane. Half of every great circle upon the sphere will therefore be represented by a straight line upon the prime plane, and the other half by another, let us say conjugate straight line in the same prime plane, lying opposite and parallel to the first, and at an equal distance from the centre. From this point of view, every projective straight line upon the prime plane requires for its full representation two real straight lines. For every point of one line there is a conjugate point in the conjugate line, the two lying in a straight line with the centre, and equally distant from it. For every secondary or non-central plane there is a parallel conjugate plane, every point of which is conjugate to a point of the first plane. Two straight lines of a secondary plane which cut each other in a given point are continued projectively in the conjugate plane and cut each other again in the point conjugate to the given point, after measuring the projective distance $\pi/c^{\frac{1}{2}}$; and if the measurement be continued, they will cut each other again in the original plane at the given point, after describing a total projective distance of $2\pi/c^{\frac{1}{2}}$. This view corresponds to what Klein calls the double-elliptic case, while our original view corresponds to the single-elliptic. In the double case we may suppose the projective measurement of $2\pi/c^{\frac{1}{2}}$ to be made by passing along the line to real infinity, say from east to west, then back over the conjugate line from west to east, then over the original line from east to west to the point of beginning; and in the single case by passing along the line to infinity as before, from east to west, then again from east to west along the same line, and once more from east to west to the point of beginning.

The chief lesson to be obtained from all non-euclidian diversions is that the distinguishing mark of euclidian geometry is fixity of distance-measurement, by which alone it is possible to draw the same figure upon different scales. That the same figure may be drawn upon different scales might well be laid down as the axiom necessary and sufficient to distinguish euclidian from non-euclidian geometry.* Or, more directly, we might adopt the really sensible assumption of Clavius, that the "line of equal distance" is straight. Or, again, we might assume that it is possible to construct a rectangle ; remarking that if three sides be constructed, the fourth must be either greater than, or equal to, or less than the base, and that the simplest system, that of equality, is chosen because of its conformity with universal consent. It might also be said that that system is the only one consistent with the received idea, that if a solid be moved along a straight line without revolving, all points of it describe equal distances ; for if this be admitted, it follows that two perpendiculars to the same line are everywhere equally distant.

MORRISTOWN, *October* 18, 1892.

A NEW LOGARITHMIC TABLE.

Tables des Logarithmes a huit décimales des nombres de 1 *à* 125000, *et des fonctions goniométriques sinus, tangente, cosinus et cotangente de centimiligone en centimiligone et de microgone en microgone pour les* 25000 *premiers microgones, et avec sept décimales pour tous les autres microgones.* Par J. de MENDIZÁBAL-TAMBORREL, Ingénieur-Géographe. Paris ; Hermann, 1891. Folio, pp. 320.

IN this folio volume, piously dedicated to the memory of Le Verrier, is contained a very extended table of logarithms, differing materially from any of its predecessors. The unit which the author has adopted for decimal subdivision, in place of the *degree,* is not the *quadrant,* or unit of the recent eight-figure tables issued by the French government, but the *entire circumference.* The author considers this unit more logical, and instances two advantages peculiar to it. These are, first, that the unit of measure for time being the *day,* the corresponding unit for angle should be the whole circle, and secondly, in the case of angles exceeding the circumference, the trigonometric functions can be found by using simply the fractional part of the angle. The author proposes the name *gone* for his unit, and adopts the symbol γ to represent it. The lower units will then be *décigones, centigones . . . microgones,* the last being 1/1000000 of the circumference, and equivalent to $1''.296$. The first table gives eight-figure logarithms for

* Referred to as "the axiom of similars" by Sir Richard Ball in the article "Measurement" of the *Encyclopædia Britannica.* The axiomatic character of geometric proportion is urged strongly by De Morgan, in the article "Proportion" of the *Penny Cyclopædia.*