POINCARÉ'S MÉCANIQUE CÉLESTE.

Les Méthodes nouvelles de la Mécanique Céleste. Par H. POINCARÉ. Tome I. Paris, Gauthier-Villars, 1892. 8vo.

THE publication of this new work on Celestial Mechanics, embodying some of the results of the labors of mathematicians in that direction during the last fifteen years, comes as a welcome addition to our knowledge of this subject. Until lately, nearly all treatises have been written with a special object, that of obtaining expressions which can be used by the practical astronomer; the mathematical aspects of the problems solved have been almost entirely neglected. These latter have an interest of their own apart from any use which can be made of them, and it is to the study of such questions that M. Poincaré largely devotes himself. At the same time he points out where they can be applied usefully in the case of the problem of three bodies. But this is not all. Most of the results obtained can be applied equally to the general problems of dynamics where there is a force function, and by the use of a dissipation function could doubtless be applied to any natural problem whatever.

The applications are, however, more particularly made to a satellite system, in the special case when the three bodies move in one plane, as well as in the general case. The limitation generally imposed consists in making the ratios of the masses of two of the bodies to that of the third a small quantity, an assumption which, nevertheless, does not limit greatly the usefulness of the results. M. Poincaré says, "Le but final de la Mécanique céleste est de résoudre cette grande question de savoir si la loi de Newton explique à elle seule tous les phénomènes astronomiques," and for this end to be attained it is absolutely necessary to know whether the developments of the expressions for the position of any heavenly body do mathematically represent that position. In general, the series obtained must be convergent, and it is to the questions on the convergence of such series that M. Poincaré has been able to give some definite answers.

In his introduction, the author points out that the starting point of the present developments of the lunar theory, was the publication in Vol. I. of the *American Journal of Mathematics* of a paper by Dr. Hill entitled, "Researches in the lunar theory." It is true that in this memoir, Dr. Hill has largely occupied himself in obtaining exact numerical and algebraical values for certain inequalities in the motion of the moon; but the general considerations involved at the beginning and end of it are of a far-reaching nature. In par-

ticular, a superior limit to the radius vector of the moon is found, and a general study of the surfaces of equal velocity is made. His consideration in a particular case of the moons of different lunations with respect to the primary, will be mentioned below.

M. Poincaré's book is principally based on his own memoir, "*Sur le problème des trois corps et les équations de la dynamique.*"* The arrangement is not quite the same. In the treatise, many of the demonstrations are more completely explained, the applications are more numerous, and much matter that is entirely new has been added. In what follows, I have not in any sense attempted to give a complete account of the book. Much that is given there is outside the scope of an article such as this; the results that are mentioned are chiefly noticed either because they can be given in a few words, or because from their peculiar interest they merit a somewhat longer treatment.

The first chapter deals with some general well-known theorems with respect to differential equations. Two types are selected. The general form which it is necessary to consider is shown by the system

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \ldots n).$$

The X_i are analytic and single-valued functions of the x_i and may or may not contain the time explicitly. This type includes the system of canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i},$$

which possess a set of properties special to themselves. Some space is devoted to the consideration of these properties, and special attention is directed to changes of variables for which the system still remains canonical. The proofs for these theorems are sketched very briefly in cases where they are well-known.

In all the particular cases of the applications of canonical equations to the problem of three bodies, M. Poincaré works out the results with some detail. The masses are taken to be m_1, m_2, m_3; m_1 is the mass of the primary while m_2, m_3, satisfy

$$\beta\mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad \beta'\mu = \frac{(m_1 + m_2)\, m_3}{m_1 + m_2 + m_3},$$

* *Acta Mathematica*, Vol. XIII.

such that μ is small while β, β' remain finite quantities. It is then possible to expand F in a series arranged in ascending powers of μ:

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots$$

In general F_0 will be independent of one system of canonical elements, say, the y_i.

The canonical equations given above correspond to n degrees of liberty. If we know an integral of the system, this number can be lowered by one unit. In general, if we know q integrals, Poisson's conditions must be fulfilled between these integrals taken two and two, in order that the number of degrees of liberty may be lowered by q units. The application of this to the general problem of three bodies is immediate. The three integrals for the motion of the centre of mass of the system being known and fulfilling the conditions, we can reduce the number of degrees of liberty from *nine* to *six*. The three known integrals of areas are also integrals of the system thus reduced, and by using two combinations of these latter, it is possible to reduce the system to *four* degrees of liberty; also in the case when the bodies move in one plane, the system can be reduced to *three* degrees of liberty. The usual transformations are then effected so as to leave the equations still in the canonical form and to carry only the smallest number of degrees of liberty.

The form of the disturbing function is also discussed, and it is considered under what circumstances we can develop it in ascending powers.

The second chapter deals with the general conditions for integration in series, and in particular with the conditions that these series may be convergent. It is here that M. Poincaré's penetrative genius especially shows itself. The complicated forms which appear in the lunar problem render it an almost impossible task to attack directly the question of convergence of the series obtained. But by going back to the differential equations themselves, and considering the disturbing function, he is able to obtain definite results, with respect to the problem of three bodies, for the convergence of those series which may be taken to represent certain particular solutions.

The notation introduced by M. Poincaré a short time back for dealing with questions of convergence is an especially happy one. It is as follows:—If we have two functions φ, ψ, expanded in ascending powers of x, y,

$$\varphi \ll \psi \qquad (\text{arg. } x, y)$$

denotes that the coefficient of every term in ψ is greater in absolute value than the corresponding term in φ, the "argu-

ments" in terms of which the expansion is made being written as above. This can of course be used for any number of arguments. An extension of this notation is given at the end of the chapter. The coefficients, instead of being constants, are supposed to be periodic functions of the time; then, if every coefficient of ψ in its expansion according to powers of $x, y, e^{\pm it}$ is real, positive, and greater in absolute value than the corresponding coefficient in φ,

$$\varphi \ll \psi \qquad (\text{arg. } x, y, e^{\pm it}).$$

Cauchy's general theorems on convergence are quoted and extended to the case in which the function is expanded in terms of several variables. If we have a system of differential equations

$$\frac{dx}{dt} = \theta(x, y, z, \mu), \frac{dy}{dt} = \varphi(x, y, z, \mu), \frac{dz}{dt} = \psi(x, y, z, \mu)$$

where θ, φ, ψ are expanded in powers of x_0, y_0, z_0, and μ, t, there will exist three series expanded in powers of x_0, y_0, z_0 and μ, t which will satisfy these equations and reduce respectively to x_0, y_0, z_0 when $t = 0$. For these to be convergent it is necessary that $|x_0|, |y_0|, |z_0|, |\mu|, |t|$ should be sufficiently small. The restriction $|t|$ sufficiently small is evidently inconvenient, and Poincaré is able to get rid of it and to say that the series are convergent if t lies between given limits provided that $|\mu|$ be sufficiently small.

In most cases, however, expansion is not made in powers of the time, but in trigonometrical functions of it, and it therefore becomes necessary in the first instance to examine a system of differential equations,

$$\frac{dx_i}{dt} = \varphi_{i,1}x_1 + \varphi_{i,2}x_2 + \ldots + \varphi_{i,n}x_n \quad (i = 1, 2, \ldots n)$$

where the φ are all periodic functions of the time. The general solution found is,

$$x_i = c_1 e^{a_1 t} \lambda_{1,i} + c_2 e^{a_2 t} \lambda_{2,i} + \ldots + c_n e^{a_n t} \lambda_{n,i}$$

the λ being periodic functions of the time only, the a_i dependent on the roots of a determinantal equation, and the c_i arbitrary constants.

These a_i are called the characteristic exponents (*exposants caractéristiques*) of the solution. On them depends the whole nature of the various solutions. Thus if two of the exponents are equal, the time appears as a factor; if they are all

pure imaginaries, the general solution contains periodic terms only, and so on. Also, on them depend the "asymptotic solutions." Chapter IV. is devoted to the consideration of these exponents.

Chapter III., which deals with periodic solutions, is perhaps the most interesting from the point of view of its immediate application to some of the problems in the lunar theory. In this connection, a periodic solution is defined as being such that the system at the end of a finite time T comes into the same *relative* position as at the beginning of that time. The period is then T. Thus if $\varphi(t)$ represent a periodic solution of period T

$$\varphi(t + T) = \varphi(t);$$

also if

$$\varphi(t + T) = \varphi(t) + 2k\pi \qquad (k = \text{whole number})$$

$\varphi(t)$ is still said to be a periodic solution. These two types are analogous to linear and angular coördinates, respectively. In the canonical system of coördinates as applied to dynamical problems, one set of elements belongs to the first type, and the conjugate set in general to the second type. It is to be noted that by defining a periodic solution in this way, the system can, so to speak, be separated from its external relations. The motions both of rotation and translation of the system as a whole can be detached, and those of its various parts amongst themselves considered.

The question which is put forward for examination is as follows: If for $\mu = 0$ we have a periodic solution, what are the conditions necessary in order that the solution shall still remain periodic when μ is not zero but a small quantity? It must be remembered that in this and in what follows, the term "periodic" has the meaning which has just been given to it. In order to answer the question, M. Poincaré considers the system

$$\frac{dx_i}{dt} = X_i$$

where the X_i are functions of the time periodic and of period 2π, as well as of the x_i. Space will not permit me to reproduce the argument, which finally reduces the answer to the consideration of the properties of a certain curve in the neighborhood of the origin. This curve is examined in certain particular cases and notably in the case where there are an infinite number of periodic solutions for μ zero, *i.e.* when the period is an arbitrary constant of the general solution. Generally, it is found that in these cases the equations *do* admit periodic solutions. In another partic-

ular case, the equations when $\mu = 0$ admit a solution of period 2π, and when μ is small but not zero, save in an exceptional case, the equations admit a solution of period $2k\pi$ (k being a whole number) which is different from the solution of period 2π, and is only not distinct from this latter when μ becomes zero.

If the X_i are periodic with respect to the time, the solution in general if periodic must have the same period. When however the time does not enter into the X_i explicitly, the period of the solution can be anything whatever. Suppose that the period selected when $\mu = 0$ be T. The question resolves itself into finding under what circumstances a solution of period $T + \tau$ is possible when μ is small. The argument proceeds in a somewhat similar manner as in the first case and similar results follow.

To apply these results to the problem of three bodies, suppose $\mu = 0$. Then two of the bodies describe ellipses about the third. At the end of a certain period measured by the difference of their mean motions, the system is found in the same relative position as at the beginning of the period. The solution for $\mu = 0$ is then periodic. Will periodic solutions be still possible when μ, instead of being zero, has a small positive value? From what has been proved above, we can say that such solutions *are* in general possible. M. Poincaré distinguishes three classes :—(1) when the inclinations and eccentricities are zero, (2) when the inclinations only are zero, (3) when the latter are not zero. He then examines these in detail.

Under (1) comes, as a particular case, Dr. Hill's now classic solution, where the mass of one body is supposed to be infinitely great and at an infinite distance, but to have a finite mean motion, and the mass of the other is infinitely small. The solutions are referred to axes moving with the infinitely distant body which takes a circular orbit. The period is one of the arbitraries and can be anything whatever. When $\mu = 0$ the motion is circular, and when μ is small, the curve does not differ much from a circle, and is somewhat elliptical in shape with its shorter axis directed constantly towards the sun. [If the sun be not infinitely distant, the only change in the curve is a loss of symmetry with regard to the line joining the earth and the sun.] Dr. Hill calculated the various shapes which the curve takes for different values of the arbitrary period, corresponding to gradually decreasing values of the constant of vis viva. As this latter constant diminishes, the ratio of the magnitude of the axes becomes greater, until for one particular value of it a cusp appears at each end of the greater axis. This gives what Dr. Hill calls, "the moon of maximum lunation." At the cusp and therefore in quad-

rature, the moon becomes for a moment stationary with respect to the sun.

He did not pursue the calculations beyond this point. It was stated however that any member of this class of satellites if prolonged beyond the moon of maximum lunation would oscillate to and fro about a mean place in syzygy, never being in quadrature. M. Poincaré points out an inaccuracy in this statement. The satellites which are never in quadrature, are indeed possible but belong to a different class of solution, and are not the analytical continuation of those studied by Dr. Hill. He shows that if we prolonged them beyond the critical orbit, they would cross the line of quadratures six times, cutting their own orbits twice and forming a curve with three closed spaces. The class to which the moons without quadrature belong has, as a limiting case, a moon which is stationary with respect to the sun and which is always either in conjunction or opposition.

M. Poincaré next goes on to consider the canonical system when F_0 is supposed independent of the y_i. This is the general problem of dynamics where the forces depend on the distances only and where we proceed by successive approximations. The first approximation is

$$x_i = const = a_i, \qquad \frac{dy_i}{dt} = const = n_i.$$

If the solution is to be periodic and of period T, all the $n_i T$ must be multiples of 2π. It is then shown that unless the Hessian of F_0 with respect to the x_i vanish, we can have a periodic solution of period T or differing little from T when μ is small. If this Hessian vanish we can sometimes find a function of F_0 whose Hessian does not vanish. If we cannot do this the case must be otherwise examined. Such an examination shows, that when the Hessian of F_0 vanishes, if the mean value R of F_1, with respect to t, admits of a maximum or a minimum, periodic solutions are still possible.

In the problem of three bodies, F_1 corresponds to the disturbing function, and we are led to periodic solutions of the second and third kinds. Here R does admit of a maximum or a minimum, and hence such periodic solutions are always possible. The periodic solutions of the first kind only cease to exist when n' is a multiple of $n - n'$. When, however, this ratio $n' : n - n'$ is nearly a whole number, as happens in several cases in the solar system, a large inequality will exist and its principal part can be calculated suitably by the help of these periodic solutions.

In the next chapter M. Poincaré passes on to the consideration of the characteristic exponents. One solution of a

system of differential equations being known, it is required to find a solution differing little from it. The equations of variations are formed in the usual way, and these bring in the equations given above, which involve the characteristic exponents. As an example of the use of these equations, Dr. Hill's work on the motion of the lunar perigee is quoted, where he obtains the principal part of it accurately to a large number of places of decimals.*

It is then considered under what circumstances one or more of the exponents become zero, and their effect on the existence of a periodic solution. The argument and result depend chiefly on two things: first, the presence in, or absence from, the X_i of the time explicitly, and, secondly, the existence or non-existence of single-valued integrals of the system. If canonical equations be used, the exponents are equal and opposite in pairs. With the limitation that F_0 does not depend on the y_i, two exponents will be zero, and unless certain conditions be fulfilled, two exponents only will be zero. In the periodic solutions of the problem of three bodies, whether in one plane or not, two exponents and two only are zero. The solutions corresponding to these exponents are called "*solutions dégénérescentes*," and are of the form

$$\xi_i = S_i'', \quad \eta_i = T_i'',$$

$$\xi_i = S_i^* + t\, S_i'', \quad \eta_i = T_i^* + t\, T_i'',$$

in which the S, T are periodic.

The canonical system given above has an integral which is known, namely the integral of vis viva. The author devotes himself in Chapter V. principally to prove that, save in certain exceptional cases, there does not exist any single-valued algebraic or transcendental integral other than that of vis viva. For this a function Φ is supposed to be analytic and single-valued for all values of x, y, μ within a certain region, and within this region to be developable according to powers of μ, thus:

$$\Phi = \Phi_0 + \mu\, \Phi_1 + \mu^2\, \Phi_2 + \ldots$$

As long as Φ_0 is not a function of F_0, it is proved that $\Phi = const.$ cannot be an integral of the system. If Φ_0 be a function of F_0, it is possible to find another integral which is distinct from F, and which does not reduce to F_0 when μ is zero. In case, however, the Hessian of F_0 be zero, an exceptional case arises, and it is in this exceptional case that the

* *Acta Mathematica*, Vol. VIII. See also a note by the writer in *Mo. Not. R. A. S.*, Vol. XVII. No. 6.

importance of the principle applied to problems in dynamics is seen. A general set of conditions is found, necessary but not sufficient for the existence of another integral of the equations. These conditions take the form of relations between the co-efficients in the development of F.

Applying these to the problem of three bodies, the author arrives at the conclusion, *that there cannot exist any new transcendental or algebraic single-valued integral of the problem of three bodies other than the well-known ones, whether we consider the particular cases of two, three, or the general case of four degrees of liberty mentioned above.* This important result is of course applicable here to the case only when μ is small, a restriction which nevertheless occurs in most problems of celestial mechanics. It is pointed out, however, that M. Bruns has demonstrated that there cannot exist any other *algebraic* single-valued integral for *any* values of the masses. In actual application M. Poincaré's theorem will be found the more useful, since he includes transcendental as well as algebraic forms in his demonstration.

The most interesting example given to illustrate the general theorem is that of the motion of a solid suspended from a fixed point and acted on by gravity only. The distance of the centre of mass of the body from the point of suspension is supposed small. Two integrals are known : is it possible that a third can exist ?* When the conditions are applied it is found that there is nothing to prevent the existence of a third integral, but since the conditions are necessary and not sufficient nothing proves that it does exist ; such an integral however cannot be algebraic.

Chapters VI. and VII. treat of the disturbing function and M. Poincaré's asymptotic solutions, respectively. In the consideration of the latter a series appears which is divergent in a manner analogous to Sterling's series.

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* For an elementary discussion of this problem, see Routh's Rigid Dynamics (4th ed.) Vol. II., Chaps. IV., V.