TOPOLOGY OF ALGEBRAIC CURVES.

In the *Mathematische Annalen*, Vol. 38 (1891), Mr. David Hilbert of Königsberg has a very interesting and suggestive article on the real branches of algebraic curves. The simplicity of the method which Mr. Hilbert employs, and the possibility of its being made to yield further important results seem sufficient reasons for presenting here, in some detail, that portion of the article which treats of plane curves. It has seemed to the present writer advisable to amplify portions of Mr. Hilbert's article, with the view of making his method more intelligible, and also to make some changes in the proof of the principal theorem, in order to avoid some slight inaccuracies that have crept into his demonstration.

The first part of the article in question is devoted to the determination of the maximum number of *nested branches* possible to a plane algebraic curve of order n, and of maximum deficiency. By *nested branches* is meant a group of even branches so arranged that the first lies entirely within the second, the second within the third, and so on, like a series of concentric circles.* It should be observed that some or all of the non-nested branches may, in perfect accord with this definition, lie within the ring-shaped regions formed by the nested branches. A single even branch, which neither encloses another branch nor is enclosed by one, may be looked upon as a nested branch or not, according to the nature of the question under discussion. For reasons that will presently appear, Hilbert does not consider the even branches of the conic and cubic as nested. Hilbert bases some of his investigations upon results previously obtained by A. Harnack,† and his method is entirely analogous to that of the latter.

Harnack had proved, in the article referred to, that a plane algebraic curve, without singularities, of order n and of deficiency p, can not have more than $p + 1$, that is, $\frac{1}{2}(n-1)(n-2) + 1$ *real* branches; and, further, that, for every positive integral value of n, a non-singular curve with $\frac{1}{2}(n-1)(n-2) + 1$ real branches actually exists. Setting out from this result of Harnack's, Hilbert shows first that a non-singular curve can have no more than $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ nested branches, according as n is even or odd; for, if it had more, a right line could be drawn meeting the curve in more

* This definition is not scientific but it serves the present purpose. To make it rigorous Mr. Hilbert needs only to define accurately what is meant by *inside* and *outside* of a closed branch. Such a definition has virtually been given by von Staudt, *Geometrie der Lage*, § 1, 16.

† *Mathematische Annalen*, Bd. 10, *Ueber die Vieltheiligkeit der ebenen algebraischen Curven*

than n points.* He proves further the following theorem: *For every positive integral value of n, a non-singular curve of order n exists, having the maximum number of real branches,* $\frac{1}{2}(n-1)(n-2)+1$, *and* $\frac{1}{2}(n-2)$ *or* $\frac{1}{2}(n-3)$ *nested branches, according as n is even or odd.*

We shall, for sake of brevity, designate an even branch by the term "*oval.*" It is evident that all nested branches are ovals. Moreover, we consider that case only where all the nested ovals are grouped in a single nest. We first assume the theorem true for a curve of n^{th} order, C_n, whose equation may be written $f=0$, and we assume further that an ellipse, E_2, whose equation we write $h=0$, can be constructed enclosing one or more of the nested ovals, and cutting a non-nested oval, b, in $2n$ points, whose order of succession shall be the same upon b as upon E_2. It is evident that E_2 and C_n have no other common point. The ellipse E_2 and the branch b form, by their intersections, $2n$ regions, each completely bounded by a *single* segment of E_2 and a *single* segment of b. Within one of these regions there exists one or more nested ovals. Whether this region, which we call R, contains the nested ovals interior to E_2 or exterior to it,† depends upon the nature of b, and its position with respect to E_2. (When E_2 encloses all the nested ovals, it may occur that none of these $2n$ regions contains a nested oval; in that case one of these regions will be all the plane exterior to E_2 and b, and this we designate by R.) Let s be any segment of E_2 determined by the intersections of E_2 and b, *except that segment which forms a portion of the boundary of R.* Upon s we choose $2(n+2)$ points, none of them coincident with the extremities of s, and join by right lines the first and second, the third and fourth, ... , and the $(2n+3)$th and $(2n+4)$th. Let the product of the equations of these $n+2$ right lines be $l=0$. Then for very small values of δ,

$$F \equiv fh \pm \delta l = 0$$

is the equation of a curve, C_{n+2}, of order $n+2$, lying very near the degenerate curve $fh=0$. This C_{n+2} passes through the points common to C_n and the right lines, and through the points common to E_2 and these lines, but not through the intersections of E_2 with C_n.

* This theorem is not true for curves of order lower than the fourth. Moreover, it must be borne in mind that every non-singular curve with the maximum number of real branches has at least *one non-nested oval,* because $\frac{1}{2}(n-2)$ and $\frac{1}{2}(n-3)$ are each less than $\frac{1}{2}(n-1)(n-2)+1$.

† A nested oval exterior to E_2, since it encloses those interior to E_2, must also enclose E_2 itself. Therefore, when, among a number of isolated ovals, we have to consider *a single one* as nested, we choose as such, one that lies in the interior of E_2.

We proceed now to prove:

(1) that C_{n+2} has $p' + 1$ real branches, p' being the deficiency of C_{n+2};

(2) that C_{n+2} has the maximum number of nested branches; and

(3) that the ellipse, E_2, encloses one or more of the nested ovals of C_{n+2}, and cuts one of its non-nested ovals in $2(n+2)$ points, whose order of succession upon C_{n+2} is the same as upon E_2.

1. Ignoring the branch b for the moment, it appears, from the form of the equation $F=0$, that in the immediate vicinity of every other branch of C_n, there exists a *similar* branch of C_{n+2}. The C_n has by hypothesis $\frac{1}{2}(n-1)(n-2)$ real branches, exclusive of b. These give rise, therefore, to $\frac{1}{2}(n-1)(n-2)$ real branches of C_{n+2}. Furthermore, under proper choice of the sign of δ, there exists, in the vicinity of the complete boundary of each of the $2n$ regions formed by E_2 and b, an oval of C_{n+2}. The latter curve has no real branch save those already enumerated. Therefore *C_{n+2} has $\frac{1}{2}(n-1)(n-2)+2n=\frac{1}{2}(n+1)n+1=p'+1$ real branches.*

2. Each of the nested ovals of C_n gives rise to a nested oval of C_{n+2}. Moreover, the oval of C_{n+2} engendered by the boundary of R is itself a nested oval of C_{n+2}. The latter has, therefore, *one more* nested oval than does C_n. Since increasing n by 2, increases the functions $\frac{1}{2}(n-2)$ and $\frac{1}{2}(n-3)$ by 1, it follows that *C_{n+2} has the maximum number of nested branches.*

3. In the vicinity of that region, a portion of whose boundary is S, there exists an oval of *C_{n+2} which cuts the ellipse in the $2(n+2)$ points already determined upon s, and the order of succession of these $2(n+2)$ points is the same upon C_{n+2} as upon s.*

Hence, if our assumptions concerning C_n and E_2 are valid, the curve C_{n+2} has the maximum number of real branches, and also the maximum number of nested branches. And furthermore—and this is a very important point—the ellipse E_2 has the same position with respect to C_{n+2} that it was assumed to have with respect to C_n. It follows, then, that we may in like manner derive from the C_{n+2} a C_{n+4} having the same properties, and so on. If, then, we can prove our assumption valid for one even value, and for one odd value of n, we may conclude that our theorem is true for all values of n.

That these assumptions are valid when $n=4$ can be demonstrated as follows: Let $f=0$ be the equation of a given ellipse C_2, and $h=0$ that of the auxiliary ellipse E_2. Let E_2 intersect C_2 in 4 real points; and upon any segment, s, of E_2 determined by two successive points of intersection, choose the 8 successive points, 1, 2, 3, . . . 8. Join by right lines, 1 with 2, 3 with 4, . . . , and 7 with 8. Let the product of the equations of these 4 right lines be $l = 0$. Then, for very small values of δ,

$$fh \pm \delta\, l = 0$$

represents a non-singular quartic, C_4, and, by proper choice of the sign of δ, this quartic has four ovals, one of which intersects E_2 in the eight points upon s. Moreover, within E_2 there lie one or two ovals of C_4, one if s is exterior to C_2, and two if s is within C_2. Now a quartic can have no more than $\frac{1}{2}(4-2) = 1$ nested oval. We choose as such, an oval in the interior of E_2. We have then a C_4 with the maximum number of real branches, viz., $\frac{1}{2}(4-1)(4-2)+1 = 4$; with the maximum number of nested ovals, 1; and the ellipse E_2 encloses this nested branch, and cuts a non-nested oval in $2(2+2) = 8$ real points, whose order of succession upon C_4 is the some as upon E_2. Hence *our assumption is valid when* $n = 4$.

That this is true also when $n = 5$ is similarly proved. Let $f = 0$ represent a straight line. Draw the ellipse, E_2, not cutting $f = 0$ in any real point. Upon E_2 choose six points and, as before, join alternate pairs by right lines. Let the product of the equations of these three right lines be $l = 0$. Then, when δ is very small,

$$fh \pm \delta\, l = 0$$

represents a non-singular cubic, C_3, the oval of which intersects E_2 in the six points whose order of succession upon E_2 and the oval is the same. Proceeding one step further, let the equation of C_3 be $f = 0$. Upon any segment of E_2 choose $2(3+2) = 10$ points, and join alternate pairs by right lines, the product of whose five equations is $l = 0$. Then, for sufficiently small values of δ,

$$fh \pm \delta\, l = 0$$

represents a non-singular quintic, C_5, and, upon proper choice of the sign of δ, this C_5 has six ovals, one of which intersects E_2 in ten points. Within E_2 lie two ovals of C_5, one of which we consider a nested oval. Moreover, C_5 has an odd branch in the vicinity of the odd branch of C_3. We have then a quintic with the maximum number of real branches, $\frac{1}{2}(5-1)(5-2)+1 = 7$; with the maximum number of nested branches, $\frac{1}{2}(5-3) = 1$; and with a non-nested oval cut by E_2 in $2(3+2) = 10$ real points; E_2 also encloses the nested branch. Hence, *our assumptions are valid when* $n = 5$. *The theorem is therefore true in general.*

Readers of Hilbert's article in the *Annalen* will notice some minor errors in his proof. He states, for instance, that the auxiliary ellipse may lie wholly within the innermost nested oval (see *Annalen*, vol. 38, p. 117). This is impossible, for the ellipse could not then be made to intersect a non-nested oval. Again, he allows the ellipse to cut *any* of the non-nested branches. If the ellipse be drawn to enclose all the nested branches and to intersect in $2n$ points an *odd* branch, the derived C_{n+2} will have indeed the maximum number of real branches, but one fewer than the maximum number of nested branches. And, lastly, Hilbert chooses the $2(n+2)$ points of E_2, through which the lines $l=0$ are to pass, upon any segment of E_2. If, however, these be taken upon that segment of E_2 which forms part of the boundary of R, the branch of C_{n+2} which has these points in common with E_2 will be a *nested* oval, and, though the C_{n+2} will then have $p+1$ real branches, and the maximum number of nested ovals as required, it will be impossible to carry the process further.

It will be observed that Hilbert's results apply only to curves of maximum deficiency, and of the maximum number of real branches, n being given. It by no means follows that a curve of order n and of maximum deficiency, but with fewer than the maximum number of real branches, cannot have more than $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ nested branches. For instance, in the case of the cubic discussed above, if δ be given the opposite sign to the one there chosen, the equation

$$fh \pm \delta l = 0$$

will represent a non-singular quintic, having but three real branches, two of which are nested.

And, in general, it is easily seen that a non-singular curve of even order, and possessing but $\frac{1}{2}n$ real branches, may have them all nested. Similarly, a curve of odd order having only $\frac{1}{2}(n+1)$ real branches, may have $\frac{1}{2}(n-1)$ of them nested. Hilbert leaves untouched also the case of singular curves, and thus excludes from his investigations a large class of curves. It would be interesting to know under what conditions, and in what way, the branches of a singular curve can be nested.

Lack of space prevents any discussion of the second part of Hilbert's article, in which the author determines some of the properties of curves in three-fold space. I give only the results of these investigations. By a method entirely analogous to that presented above, Hilbert proves the theorem: *An irreducible twisted curve of order n, with the maximum number of real branches* [$\frac{1}{4}(n-1)^2+1$ *when n is even, and* $\frac{1}{4}(n-1)(n-3)+1$ *when n is odd*] *can have no more than* $2\nu-2$, $2\nu-1$, $2\nu-1$ *odd branches, according as* $n=4\nu$,

$4\nu + 1$, $4\nu + 3$. *When* $n = 4\nu + 2$, *no odd branch can exist. Exceptional are the cases when* $n = 3, 4, 5$, *the maximum number of odd branches being* 1, 2, 3, *respectively.* Then, by applying Abel's theorem for elliptic functions, he proves, for every value of n, the existence of curves with the maximum number of real odd branches.

L. S. HULBURT.

WORCESTER, MASS., *April* 5, 1892.

FINAL FORMULAS FOR THE ALGEBRAIC SOLUTION OF QUARTIC EQUATIONS.*

BY MANSFIELD MERRIMAN, PH.D.

I. FINAL formulas for the algebraic solution of quadratic and cubic equations are well known. Such formulas exhibit the roots in their true typical forms, and lead to ready and exact numerical solutions whenever the given equations do not fall under the irreducible case. But for the quartic, or biquadratic, equation the books on algebra do not give similar final formulas. The solution of the quartic has been known since 1540, and numerous methods have been deduced for its algebraic resolution, yet in no case does this appear to have been completed in final practical shape. It is the object of this paper to state the final solution in the form of definite formulas.

II. The expression of the roots of the quartic is easily made in terms of the roots of a resolvent cubic, and the cubic itself is solved without difficulty. Yet great practical difficulty exists in treating a numerical equation on account of the presence of imaginaries in the roots of the resolvent. Witness the following example which is generally given to illustrate the method in connection with Euler's resolvent:

"Let it be required to determine the roots of the biquadratic equation,

$$x^4 - 25x^2 + 60x - 36 = 0.$$

By comparing this with the general form the cubic equation to be resolved is,

$$y^3 - 50y^2 + 729y - 3600 = 0$$

* Abstract of a paper presented to the Society at the meeting of May 7, 1892.