turned the long controversy on the nature of force and energy between Descartes, Leibnitz, and their followers.*

The closing section contains some interesting general remarks on the nature of the three laws and the ways of testing their truth.

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ANN ARBOR, MICHIGAN, *January* 1, 1892.

WEIERSTRASS AND DEDEKIND ON GENERAL COMPLEX NUMBERS.

WEIERSTRASS †—*Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen. Göttingen Nachrichten,* 1884.

DEDEKIND—*Zur Theorie der aus n Haupteinheiten gebildeten complexen Grössen. Göttingen Nachrichten,* 1885.

DEDEKIND—*Erläuterungen zur Theorie der sogenannten allgemeinen complexen Grössen. Göttingen Nachrichten,* 1887.

IN closing his second memoir on biquadratic residues ‡ Gauss makes this remark: "Our general arithmetic, which goes so far beyond the limits of the geometry of the ancients, is entirely the creation of recent times. Starting with the notion of whole numbers its field has widened little by little. To whole numbers came fractions, to rational numbers the irrational ones; to the positive came the negative and to the real came the imaginary."

Once convinced that $\sqrt{-1}$ was properly an algebraical quantity and that it had a meaning, mathematicians began to look for other quantities of a similar nature. "Why," they asked themselves, "should algebra yield an imaginary unit which makes it possible to represent two dimensions of space analytically; and fail to yield a second imaginary unit which can be used to represent the third dimension?" The thing needed only to be sought for apparently, and at first they looked amongst the functions of $\sqrt{-1}$. Unfortunately it turned out that even the most promisingly irrational of these could all be broken up into a real part and $\sqrt{-1}$ times a second real quantity; algebra had done her best; if mathematicians wanted more imaginaries they must invent them. From the time of Gauss, then, until the present day the architects and the masterbuilders have turned occasionally

* See for instance E. MACH, *Die Mechanik in ihrer Entwickelung,* Leipzig, Brockhaus, 1889, pp. 254–259.

† Extract from a letter to Schwarz.

‡ *Werke,* II., p. 175.

from their labors upon the theory of functions, that monument which of all that human hands have built will rise the highest and stand the longest, to try their skill in constructing systems of imaginary, or complex, numbers.

Gauss himself was of the opinion that no complex numbers except those of type $x + \sqrt{-1}\,y$ would be found admissible into arithmetic,* but does not state his reason for the opinion. The occasion of the articles cited above was an inquiry into his most probable reason, an inquiry which involved a fundamental investigation into the properties of the hyper-complex [*über-complex*] numbers, as Dedekind calls them. After full and interesting researches, of which this paper aims to give a sketch, these great mathematicians came to opposite conclusions. The fact that in the field of complex numbers the product xy may vanish when neither x nor y is zero, a fact made public by Peirce long before,† seemed to Weierstrass so unlike anything in ordinary mathematics that he concluded this must have been Gauss's reason for excluding hyper-complex quantities from arithmetic. On the other hand Dedekind asserts that it is quite a common thing in ordinary arithmetic for such a product to vanish, and concludes that Gauss's reason for excluding quantities of a nature different from $x + iy$ ‡ was the fact that such quantities, conditioned as they must be, do not exist.

To construct a complex number Weierstrass writes down a system of n units $e_1, e_2, \ldots e_n$ and multiplies each by an ordinary real number ξ_r; then the expression $x = \xi_1 e_1 + \ldots + \xi_n e_n$ is a number of the kind considered. His first undertaking is so to define the fundamental operations of arithmetic for quantities of this kind that $x + y$, $x - y$, xy, x/y may all be linear expressions of the same form as x; and that the commutative, associative and distributive laws of addition and multiplication may hold good for them. It appears that the multiplication table for the units may be constructed in an infinite number of ways so as to satisfy all these requirements. Of course the fundamental condition is the first one, which comes to the same thing as this, that *every rational function of the units shall be expressible in the form*

$$\xi_1 e_1 + \ldots + \xi_n e_n.$$

Division is defined by the equation

$$\frac{a}{b} = \gamma_1 e_1 + \ldots + \gamma_n e_n = \gamma.$$

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* *Werke*, II., p. 178.

† *Am. Journ. Math.*, vol. IV. (1881), p. 97.

‡ x and y real; $i = \sqrt{-1}$.

Multiplying both members by b and equating the coefficients of $e_1, \ldots e_n$ on both sides, a set of n equations is obtained, linear in $\gamma_1, \ldots \gamma_n$. If their determinant vanishes identically, it is impossible to determine $\gamma_1, \ldots \gamma_n$, and therefore all multiplication tables are excluded which would bring this to pass. But even then there will be certain values of b for which this determinant will vanish. Suppose such a value chosen; we can then find a value of γ such that $b\gamma$ shall vanish, both b and γ being different from zero; for $b\gamma = 0$ leads to a system of n equations linear and homogeneous in $\gamma_1, \ldots \gamma_n$ whose determinant vanishes. The quantities b having this unique and wonderful property are called by Weierstrass "divisors of zero" [*Theiler der Null*].

It turns out that when b is a divisor of zero there are an infinite number of quantities γ such that $b\gamma = 0$, and thence it is an easy inference that the equation

$$ka + kbx + kcx^2 + \ldots + klx^m = 0$$

has an infinite number of roots if k is a divisor of zero. We have, in fact, only to make

$$a + bx + \ldots + lx^m = g$$

where g is any one of the infinite number of quantities satisfying the relation $kg = 0$.

"The existence of these divisors of zero which are not themselves zero, seems," says Weierstrass, "to make a real distinction between ordinary arithmetic and the arithmetic of hyper-complex* numbers"; but ordinary algebraic equations exist which have an infinite number of roots, namely those whose coefficients are all zero. As to this point then there is a good enough correspondence between the numbers of our common arithmetic and hyper-complex numbers.

The author now obtains a multiplication table of beautiful simplicity by the following process. He expresses the first, second, . . . , $(n + 1)$-th powers of x, where

$$x = \xi_1 e_1 + \ldots + \xi_n e_n$$

linearly in terms of $e_1, \ldots e_n$; then, excluding the case when the determinant of the right members of the first n equations vanishes, we can express $e_1, \ldots e_n$ in terms of the first n powers of x; and substituting these values in the last equation, obtain a relation among the powers of x of the form

$$\Delta_0 x^{n+1} + \ldots + \Delta_n x = 0$$

where Δ_0 is the determinant just mentioned.

* Weierstrass does not use this term.

Dividing by $\Delta_0 x$ this becomes, if we replace x by the particular value g,

$$g^n + \varepsilon_1 g^{n-1} + \ . \ . \ . \ + \varepsilon_n e_0 = 0 = f(g).$$

Here e_0 is a quantity which satisfies the conditions

$$e_0 z = z e_0 = z$$

for any number of the system. Its value is in fact g/g; which is determinate so long as g is not a divisor of zero. We are now in a position to put every number a in the form

$$a = a_0 e_0 + a_1 g + a_2 g^2 + \ . \ . \ . \ + a_n g^{n-1} = a(g)\ *$$

and the product of any two numbers takes the same form.

Consider now the algebraic equation $f(\xi) = 0$ formed by replacing g in $f(g)$ by ξ; form also the function $a(\xi)$ by replacing g by ξ in $a(g)$. There is no difficulty in seeing that the product $a(\xi) \,.\, b(\xi)$ will vanish if it contains the factor $f(\xi)$. If $f(\xi) = 0$ has a root of multiplicity λ, it can be indicated by writing

$$f(\xi) = f_1^{\lambda}(\xi) \,.\, F(\xi)$$

and the arbitrary function $\varphi(\xi) = f_1(\xi) \,.\, F(\xi) \,.\, \varphi_1(\xi)$ will be of such a nature that $\varphi^{\lambda}(\xi)$ is divisible by $f(\xi)$ and therefore vanishes; but if ξ be replaced by g in $\varphi(\xi)$ we obtain a hyper-complex quantity x whose λ-th power is obtained by replacing ξ by g in $\varphi^{\lambda}(\xi)$. The λ-th power of x will therefore vanish. Hence, if $f(\xi)$ has a multiple root, the equation

$$x^{\lambda} = 0$$

can be satisfied in as many ways as there are different choices of the function $\varphi(\xi)$; but this number is infinite. It is our intention, however, to allow an algebraic equation an infinite number of roots only when each of its coefficients is a multiple of the same divisor of zero †; matters must consequently be so arranged that $f(\xi) = 0$ shall have no multiple roots. To effect this, the original multiplication table must be so constituted that the discriminant of $f(\xi)$ shall not vanish. This imposes another restriction upon the freedom of choice of the coefficients ε_{ijk} in the equations

$$e_i e_j = \sum_1^n{}^k \varepsilon_{ijk} e_k \,.\, (i, j = 1,\ 2,\ .\ .\ .\ n).$$

The simplified multiplication table is now in sight. Take any function $\varphi(\xi)$ of degree $n-1$ and with real coefficients

* This is a departure from the notation of Weierstrass.

† Weierstrass, *loc. cit.*, p. 399.

and break up into partial fractions the quotient of $\varphi(\xi)$ by $f(\xi)$. This yields the equation

$$\frac{\varphi(\xi)}{f(\xi)} = \frac{A_1}{\xi - b_1} + \frac{A_2}{\xi - b_2} + \ . \ . \ . \ + \frac{C_1 + D_1 \xi}{\xi^2 + 2h_1\xi + k_1} + \ . \ . \ . \ ;$$

the quadratic denominators corresponding to pairs of conjugate imaginary roots of $f(\xi) = 0$.* The quantity $\dfrac{A_1 f(\xi)}{\xi - b_1}$ is a polynomial in ξ of degree $n - 1$ and may be changed into a hyper-complex quantity, c_1, by replacing ξ by g as above. In the same way $\dfrac{A_2 f(\xi)}{\xi - b_2}$ leads to another quantity c_2. The product $c_1 c_2$ is obtained by replacing ξ by g in $A_1 A_2 \dfrac{f(\xi) f(\xi)}{(\xi - b_1)(\xi - b_2)}$; but this product vanishes and, in consequence, $c_1 c_2 = 0$. If then $f(\xi) = 0$ has the m real roots $b_1, \ . \ . \ . \ b_m$, we may construct m hyper-complex quantities $c_1, c_2, \ . \ . \ . \ c_m$ such that the product of any two of them vanishes. Moreover we can obtain

$$\left(\frac{A_1 f(\xi)}{\xi - b_1}\right)^2 = \vartheta(\xi) f(\xi) + B_1 \frac{f(\xi)}{\xi - b_1},$$

where B_1 is a constant. This reduces to $B_1 \dfrac{f(\xi)}{\xi - b_1}$ and we infer that c_1^2 is equal to c_1 times a real quantity. Moreover this real multiplier cannot be 0.

Again the product $\dfrac{(C_1 + D_1\xi) f(\xi)}{\xi^2 + 2h_1 \xi + k_1}$ will, when ξ is replaced by g, yield two hyper-complex quantities, c_{m+1}, c'_{m+1} since C_1 and D_1 are both arbitary. These quantities form the doubly extended manifoldness $C_1 c_{m+1} + D_1 c'_{m+1}$; and each pair of conjugate imaginary roots of $f(\xi) = 0$ enables us to form a similar manifoldness. Repeating the reasoning already given we find that the product of any two quantities belonging to different manifoldnesses vanishes; thus

$$(C_r c_{m+r} + D_r c'_{m+r})\ (C_s c_{m+s} + D_s c'_{m+s}) = 0$$

whether D_r and D_s be different from zero or not; and that the product of two quantities belonging to the same manifoldness also belongs to that manifoldness. Suppose the whole number of partial fractions to be r; each fraction yields a simple or complex quantity $a_1, \ . \ . \ . \ a_r$ and any hyper-complex quantity whatever can be expressed in the form

$$x = \xi_1 a_1 + \ . \ . \ . \ + \xi_r a_r.$$

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* The notation of Weierstrass is here altered for simplicity.

If y be any other quantity

$$y = \eta_1 a_1 + \ . \ . \ . \ + \eta_r a_r$$

then the rule for multiplication is

$$xy = \xi_1 \eta_1 a_1^2 + \xi_2 \eta_2 a_2^2 + \ . \ . \ . \ + \xi_r \eta_r a_r^2$$

If now in x the coefficients $\xi_1, \xi_2, \ . \ . \ . \ \xi_k$ all vanish, and in y the coefficients $\eta_k, \eta_{k+1}, \ . \ . \ . \ \eta_r$ all vanish, then xy will vanish while neither x nor y is zero.

An equation of the form

$$\alpha + \beta x + \gamma x^2 + \ . \ . \ . \ + \omega x^\lambda = 0$$

breaks up into r equations of the form

$$(B) \qquad \alpha_\mu + \beta_\mu x_\mu + \ . \ . \ . \ + \omega_\mu x_\mu{}^\lambda = 0$$

where $\alpha_\mu, \beta_\mu, \ . \ . \ . \ , x_\mu$ are ordinary quantities. Equation (B) can have an infinite number of roots—only in case $\alpha_\mu, \beta_\mu, \ . \ . \ . \ \omega_\mu$ all vanish. Suppose they do vanish : then

$$\alpha = \alpha_1 a_1 + \ . \ . \ . \ + \alpha_{\mu-1} a_{\mu-1} + \alpha_{\mu+1} a_{\mu+1} + \ . \ . \ . \ + \alpha_r \mu_r.$$

Taking any quantity

$$k = k_1 a_1 + \ . \ . \ . \ + k_{\mu-1} a_{\mu-1} + k_{\mu+1} a_{\mu+1} + \ . \ . \ . \ + k_r \mu_r,$$

we can put α in the form

$$\alpha = k\alpha' \text{ where } \alpha'_1 e_0 = \frac{\alpha_1 a_1}{k_1 a_1} = \frac{\alpha_1}{k_1} e_0 \text{ or } \alpha_1' = \frac{\alpha_1}{k_1};$$

similarly for α'_2, and so on. But $\alpha'_\mu = 0/0$; that is it may be anything we please. Proceeding in this way, the equation can be put in the form

$$k\alpha' + k\beta' x + \ . \ . \ . \ + k\omega' x^\lambda = 0$$

where k, having one coefficient zero, is a divisor of zero. Equation (B) having an infinite number of roots, of course x, of which each root of (B) forms a part, has an infinite number of values. We thus see why it is that in this system an equation must have an infinite number of roots when each coefficient is a multiple of the same divisor of zero.

Closing this section of his letter the distinguished author remarks that very likely Gauss's only reason for excluding from arithmetic these hyper-complex quantities was that he regarded the vanishing of xy when neither x nor y is zero as an insurmountable difficulty ; otherwise "it could hardly have escaped him that an arithmetic of these quantities can be constructed in which all the theorems are identical with those concerning ordinary complex quantities, or at least analogous to them." "In fact," he continues, "the arithmetic of hyper-complex quantities can lead to no result which could not be reached by processes known in the theory of ordinary complex quantities."

The views of Dedekind upon this last point quite coincide with those of Weierstrass; but for an account of his beautiful method of generating systems of complex quantities, the reader is for the present referred to the memoirs cited above.

C. H. CHAPMAN.

JOHNS HOPKINS UNIVERSITY, *February* 3, 1892.

EMILE MATHIEU, HIS LIFE AND WORKS.*

IF it were asked what tyranny in this world has least foundation in reason and is at the same time most overbearing and capricious, none could be found to answer better to this description than *fashion;* that fashion which makes us admire to-day what but yesterday would have excited astonishment, and which may provoke ridicule to-morrow. We all know that this sovereign whose iron rule is so much more keenly felt on account of its injustice governs the thousand and one details of every-day life; that it is supreme in literature and in the arts. But those who have not watched closely the life of the scientific world may perhaps be surprised to hear that even there if you would please you must bend the knee to fashion. What? might exclaim the stranger to the world of science, can it be true that the mathematician knows other laws than the inflexible rules of logic? Does he care to obey other orders than the invariable commands of reason?—Well, yes. Of course, to have a mathematical production accepted as *correct,* it is sufficient that it conform to the precepts of logic; but to have it admired as beautiful, as interesting, as of importance, to gain honor and success by it, more is required: it must then satisfy the manifold and varying exactions imposed by the prevailing taste of the day, by the preferences of prominent men, by the preoccupations of the public.

Thus it comes to pass that, in mathematics as elsewhere, fashion will sometimes award the laurels to those who have not deserved the triumph and make victims of men whose lack of success is an injustice. In every country there are such victors and such victims; but nowhere perhaps are they

* Translated from the MS. of the author by Professor ALEXANDER ZIWET.