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Bosonic part of $4d$ $N = 1$ supersymmetric gauge theory with general couplings: local existence

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In this paper, we prove the local existence of the bosonic part of $N = 1$ supersymmetric gauge theory in four dimensions with general couplings. We start with the Lagrangian of the vector and chiral multiplets with general couplings and the scalar potential is turned on. Then, for the sake of simplicity, we set all fermions to vanish at the level of equations of motion, so we only have the bosonic parts of the theory. We apply Segal's general theory to show the local existence of solutions of equations of motion by taking Kähler potential to be bounded above by $U(n)$ symmetric Kähler potential and the first derivative of gauge couplings to be at most linear growth functions.

1. Introduction

In 1963, I. Segal developed a method to prove the existence and the uniqueness of the solutions of a semi-linear evolution equations using a semigroup [1]. Then, ten years later, he applied his method to prove the existence of local and global solutions for four dimensional Yang-Mills equations in the temporal gauge condition [2]. Such a study has been extended to the case of Yang-Mills theory coupled to scalar fields in three dimensions [3], and in four dimensions [4, 5].

Our interest here is to extend the results in [4] to the case of minimal ($N = 1$) supersymmetric Yang-Mills theory coupled to chiral multiplets. This theory has become a prominent subject over the last four decades since it could provide solutions to major problems in the Standard Model of particle physics such as the unification of gauge couplings and the hierarchy problems.

In this paper, we prove the local existence of solutions of $N = 1$ supersymmetry gauge theory in four dimensions with general couplings. Our starting point is to consider $N = 1$ Lagrangian which consists of chirals and vectors with general couplings so that we have a nonlinear σ-model with Kähler metric, general analytic gauge kinetic functions determined by holomorphic functions, and the scalar potential. Then, we derive field equations of motion, by setting all the fermionic fields to be zero at this level for the sake of simplicity. Thus, we have an effective bosonic theory that describes the interaction between the bosonic field (ϕ, A), where ϕ is the complex scalar fields and A is the gauge fields.

By assuming that the temporal component of gauge fields vanishes (in analogy with the temporal gauge in Yang-Mills theory) and introducing new fields (π, E), one can then transform the equation of motions into a semi-linear form which contains linear and non-linear terms. For our analysis, we take the fields $u = (A, E, \phi, \pi, \bar{\phi}, \bar{\pi})$ lying in $\mathcal{H} = (H_2 \times H_1)^3$ where H_p denotes a Sobolev space. We show that the linear terms are globally defined in $\mathcal{H}$ and generate a one parameter semigroup. Finally, by Segal general theory [1], the local existence of a semi-linear evolution equation is established by showing that the non-linear parts satisfy the local Lipshitz condition.

Since we have a generalized semi-linear evolution equation, we have to consider several assumptions on the general couplings such that Segal's general theory can be used for our problem. First, we assume that Kähler potential is bounded above by $U(n)$ symmetric Kähler potential and we can derive several estimates for the Kähler potential and the Christoffel symbol. These estimates can be used to eliminate the quantity associated with Kähler metric in our analysis. Second, we apply some conditions on the gauge kinetic couplings, namely the derivative of the gauge coupling must be at most a linear growth. Finally, we assume that the scalar potential has to be at least C^3-functions and that its derivative is locally Lipshitz function.

Another problem that arises is the constraint equation which can be solved by a technique developed in [4]. This method can be mentioned in order. We first decompose the E field into unique transverse (E_T) and longitudinal parts (E_L). Then by introducing a new field E_C such that $E_C = E_L$ if the constraints fulfilled, we modify the original equation of motions by replacing E_L with E_C. Using the above mentioned conditions on general couplings, we prove that the non-linear part is locally Lipshitz function. Hence, they admit local solutions. At the end, we show that the solutions of the modified equations with the constraints satisfied are the solution of the original equation of motions.

The organization of the paper is as follows. In Section 2, we shortly review a four dimensional $N = 1$ supersymmetric gauge theory in which the vector multiplets are coupled to arbitrary chiral multiplets. Section 3 is devoted to discussing several aspects of field equations of motion including a modification of the equations of motion to solve the constraint problems. In Section 4, we discuss the internal scalar manifold and derive several estimates. In Section 5, we prove that the non-linear part of the equations of motion satisfies the Lipshitz condition and finally we prove the local existence.

2. General couplings of chiral and vector multiplets

In this section, we shortly review four dimensional $N = 1$ supersymmetric gauge theory in which the vector multiplets are coupled generally to arbitrary chiral multiplets. Here, we only write terms which are useful for our analysis in the paper. For an excellent review, an interested reader can further consult, for example, [6–8].

The theory consists of n_v vector multiplets, (A^a_μ, λ^a) coupled to n_c chiral multiplets, (ϕ^i, χ^i) where the Latin alphabets $a, b = 1, ..., n_v$, and $i, j = 1, ..., n_c$ show the number of multiplets, while the Greek alphabets $\mu, \nu = 0, ..., 3$ show the spacetime indices. In the vector multiplets we have gauge fields A^a_μ together with their fermionic partners λ^a. On the other side, the chiral multiplets contain complex scalars ϕ^i and their fermionic partners χ^i.

Furthermore, $N = 1$ supersymmetry demands the following conditions: First, the scalars ϕ^i span a Kähler manifold endowed with metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ where $K \equiv K(\phi, \bar{\phi})$ is a real function called Kähler potential. Second, there exists a set of holomorphic functions, namely (f_{ab}, X^i_a, W) which are gauge couplings, Killing vectors, and a superpotential, respectively. Finally, the existence of a real function is called the scalar potential which can be written as

$$V = g^{i\bar{j}}\, \partial_i W \partial_{\bar{j}} \bar{W} + \frac{1}{8} h^{ab} P_a P_b \ , \tag{2.1}$$

where P_a are real functions is called the Killing potentials (or momentum maps), determined by X^i_a via

$$X^i_a = \frac{i}{2} g^{i\bar{j}}\, \partial_{\bar{j}} P_a \ , \tag{2.2}$$

with h^{ab} is the inverse of $h_{ab} \equiv \mathrm{Re} f_{ab}$. Then, one can write down the bosonic part of the $N = 1$ Lagrangian as

$$\mathcal{L} = -g_{i\bar{j}} D^\mu \phi^i D_\mu \bar{\phi}^{\bar{j}} - \frac{1}{4} h_{ab} \mathcal{F}^a_{\mu\nu} \mathcal{F}^{b\mu\nu} + \frac{1}{4} k_{ab} \mathcal{F}^a_{\mu\nu} \tilde{\mathcal{F}}^{b\mu\nu} - V \ , \tag{2.3}$$

where $k_{ab} \equiv \mathrm{Im} f_{ab}$, the covariant derivative $D_\mu \phi^i \equiv \partial_\mu \phi^i + X^i_a A^a_\mu$, and the gauge field strength $\mathcal{F}^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu$. The dual field $\tilde{\mathcal{F}}^{a\mu\nu}$ is defined as $\tilde{\mathcal{F}}^{a\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^a_{\rho\sigma}$. It is worth to mention that the Lagrangian (2.3) is invariant under the following supersymmetry transformation of the fields up to three-fermion terms

$$\begin{aligned} \delta\lambda^a_\bullet &= \frac{1}{2}\left(\mathcal{F}^a_{\mu\nu} - \mathrm{i}\widetilde{\mathcal{F}}^a_{\mu\nu}\right)\gamma^{\mu\nu}\epsilon_\bullet + 2\mathrm{i}h^{ab}P_b\,\epsilon_\bullet\ , \\ \delta\chi^i &= \mathrm{i}\partial_\nu\phi^i\,\gamma^\nu\epsilon^\bullet + 2g^{i\bar{j}}\partial_{\bar{j}}\bar{W}\epsilon_\bullet\ , \\ \delta A^a_\mu &= \frac{\mathrm{i}}{2}\bar{\lambda}^a_\bullet\gamma_\mu\epsilon^\bullet + \frac{\mathrm{i}}{2}\bar{\epsilon}_\bullet\gamma_\mu\lambda^{\bullet a}\ , \\ \delta\phi^i &= \bar{\chi}^i\epsilon_\bullet\ . \end{aligned} \tag{2.4}$$

Additionally, in the theory one can replace P_a by $P_a + \xi_a$ where ξ_a are real constants which give rise to Fayet-Iliopoulos term.

In our analysis, we assume that the scalar potential to be at least a C^2 function and satisfies the local Lipshitz condition

$$\|\partial_j V(\phi') - \partial_j V(\phi)\| \leq C(\|\phi'\|, \|\phi\|)\|\phi' - \phi\|\ , \tag{2.5}$$

where $C(\|\phi'\|, \|\phi\|)$ is a bounded function depend on $\|\phi\|$. The condition above implies that the holomorphic superpotential has to be at least a C^3 function.

3. Field equations of motion

This section is devoted to discussing several aspect of field equations of motion. In particular, we take all fermions to be trivial at this level. Thus, the gauge fields and the scalars are the main ingredients of our analysis in this paper. To simplify the analysis we take a condition $A_0 = 0$ which is not a gauge condition since in general Lagrangian (2.3) is no longer gauge invariant [1].

First of all, the gauge field equation of motion is given by

$$D_\mu\left(h_{ab}\mathcal{F}^{b\mu\nu} - k_{ab}\tilde{\mathcal{F}}^{b\mu\nu}\right) = g_{i\bar{j}}\left(X^i_a D^\nu \bar{\phi}^{\bar{j}} + \bar{X}^{\bar{j}}_a D^\nu \phi^i\right)\ , \tag{3.1}$$

with

$$D_\mu\left(h_{ab}\mathcal{F}^{b\mu\nu}\right) = \partial_\mu\left(h_{ab}\mathcal{F}^{b\mu\nu}\right) + f^d_{ac}A^c_\mu\left(h_{db}\mathcal{F}^{b\mu\nu}\right)\ . \tag{3.2}$$

[1]For renormalizable Yang-Mills-Higgs theory with constant f_{ab} and $g_{i\bar{j}}$ the condition $A_0 = 0$ is called the temporal gauge, see for example, [9].

The field strength tensor $\mathcal{F}^a_{\mu\nu}$ satisfies the Bianchi identity,

$$D_\mu \tilde{\mathcal{F}}^{b\mu\nu} = 0 \ . \tag{3.3}$$

By defining

$$E^{as} = -\mathcal{F}^{a0s} = -\partial^0 A^{ar} \ , \tag{3.4}$$

and using (3.3), we can rewrite (3.1) as

$$\begin{aligned}\frac{\partial E^{ar}}{\partial t} &= \partial_s \partial^s A^{ar} - \partial^r \partial_s A^{as} + f^a_{bc} \partial_s \left(A^{as} A^{ar}\right) + h^{ab} \mathcal{F}^{csr} \partial_s h_{bc} \\ &\quad + h^{ab} h_{de} f^d_{bc} A^c_s \mathcal{F}^{esr} - h^{ab} E^{cr} \partial_0 h_{bc} - h^{ab} \tilde{\mathcal{F}}^{c0r} \partial_0 k_{bc} - h^{ab} \tilde{\mathcal{F}}^{csr} \partial_s k_{bc} \\ &\quad - h^{ab} \left(k_{de} f^d_{bc} - k_{bd} f^d_{ce}\right) A^c_s \mathcal{F}^{esr} - h^{ab} g_{i\bar{j}} \left(X^i_a D^r \bar{\phi}^{\bar{j}} + \bar{X}^{\bar{j}}_a D^r \phi^i\right) ,\end{aligned} \tag{3.5}$$

together with the constraint equation

$$\mathcal{C}^a(t) = -\partial_s E^{as} + 4\pi\rho^a \ , \tag{3.6}$$

where

$$\begin{aligned}4\pi\rho^a &= h^{ab} g_{i\bar{j}} \left(X^i_b D^0 \bar{\phi}^{\bar{j}} + \bar{X}^{\bar{j}}_b D^0 \phi^i\right) + h^{ab} \tilde{\mathcal{F}}^{cs0} \partial_s k_{bc} - h^{ab} E^{cs} \partial_s h_{bc} \\ &\quad h^{ab} \left(k_{de} f^d_{bc} - k_{bd} f^d_{ce}\right) A^c_s \tilde{\mathcal{F}}^{es0} - h^{ab} h_{de} f^d_{bc} A^c_s E^{es} \ ,\end{aligned} \tag{3.7}$$

with the initial value of (3.6) is $\mathcal{C}^a(0) = 0$. Next, we consider the scalar field equation of motion which has been modified into

$$\begin{aligned}\frac{\partial \pi^i}{\partial t} &= \partial_s \partial^s \phi^i + \Gamma^i_{kl} \left(\partial_r \phi^k \partial^r \phi^l - \pi^k \pi^l\right) + X^i_a \partial_r A^{ar} \\ &\quad + A^{ar} \partial_r \phi^k \nabla_k X^i_a + g^{i\bar{j}} g_{k\bar{l}} A^{ar} D_r \phi^k \nabla_{\bar{j}} \bar{X}^{\bar{l}}_a \\ &\quad - g^{i\bar{j}} \left(\frac{1}{4} h_{ab\bar{j}} \mathcal{F}^a_{\mu\nu} \mathcal{F}^{b\mu\nu} - \frac{1}{4} k_{ab\bar{j}} \mathcal{F}^a_{\mu\nu} \tilde{\mathcal{F}}^{b\mu\nu} + \partial_{\bar{j}} V_S\right) \ ,\end{aligned} \tag{3.8}$$

where

$$\pi^i = -D^0 \phi^i \ , \tag{3.9}$$

$$\nabla_k X^i_a = \partial_k X^i_a + \Gamma^i_{kl} X^l_a \ , \tag{3.10}$$

together with its complex conjugate.

Now, we rewrite the equations of motion (3.1) and (3.8) into the following form

$$\frac{du}{dt} = \mathcal{A}u + J(u) \ , \tag{3.11}$$

where

$$u = \begin{bmatrix} A^{as} \\ E^{as} \\ \phi^{i} \\ \pi^{i} \\ \bar{\phi}^{\bar{j}} \\ \bar{\pi}^{\bar{j}} \end{bmatrix}, \qquad \mathcal{A}u = \begin{bmatrix} E^{as} \\ \partial_r \partial^r A^{as} - \partial_r \partial^s A^{ar} \\ \pi^{i} \\ \partial_r \partial^r \phi^{i} \\ \bar{\pi}^{\bar{j}} \\ \partial_r \partial^r \bar{\phi}^{\bar{j}} \end{bmatrix},$$

and

$$J(u) = \tag{3.12}$$

$$\begin{bmatrix} 0 \\ f^{a}_{bc}\partial_r \left(A^{bs}A^{cr}\right) + h^{ab}\left(\mathcal{F}^{esr}\partial_r h_{bc} - \tilde{\mathcal{F}}^{esr}\partial_r k_{bc} + h_{de} f^{d}_{bc} A^{c}_{r} \mathcal{F}^{esr}\right) + \mathcal{D}_1 \\ 0 \\ \Gamma^{i}_{kl}\left(\partial_s \phi^{k} \partial^{s} \phi^{l} - \pi^{k}\pi^{l}\right) + X^{i}_{a}\partial_s A^{as} + A^{as}\nabla_k X^{i}_{a}\partial_s \phi^{k} + g^{i\bar{j}} g_{l\bar{k}} A^{a}_{s} \nabla_{\bar{j}} \bar{X}^{\bar{k}}_{a} D^{s}\phi^{l} + \mathcal{D}_2 \\ 0 \\ \Gamma^{\bar{j}}_{\bar{k}\bar{l}}\left(\partial_s \bar{\phi}^{\bar{k}} \partial^{s} \bar{\phi}^{\bar{l}} - \bar{\pi}^{\bar{k}}\bar{\pi}^{\bar{l}}\right) + \bar{X}^{\bar{j}}_{a}\partial_s A^{as} + A^{as}\nabla_{\bar{k}} \bar{X}^{\bar{j}}_{a}\partial_s \bar{\phi}^{\bar{k}} + g^{i\bar{j}} g_{k\bar{l}} A^{a}_{s} \nabla_{i} X^{k}_{a} D^{s}\bar{\phi}^{\bar{l}} + \bar{\mathcal{D}}_2 \end{bmatrix}.$$

where,

$$\begin{aligned} \mathcal{D}_1 = & -h^{ab}\left(h_{bci}\pi^{i} + h_{bc\bar{j}}\bar{\pi}^{\bar{j}}\right)E^{cs} - h^{ab}\left(k_{bci}\pi^{i} + k_{bc\bar{j}}\bar{\pi}^{\bar{j}}\right)\tilde{\mathcal{F}}^{c0s} \\ & - h^{ab}\left(k_{de} f^{d}_{bc} - k_{bd} f^{d}_{ce}\right) A^{c}_{r} \tilde{\mathcal{F}}^{esr} \\ & - h^{ab} g_{i\bar{j}}\left(X^{i}_{b} D^{s} \bar{\phi}^{\bar{j}} + X^{\bar{j}}_{b} D^{s}\phi^{i}\right), \end{aligned} \tag{3.13}$$

$$\mathcal{D}_2 = g^{i\bar{j}}\frac{h_{ab\bar{j}}}{4}\left(2E^{a}_{s}E^{bs} - \mathcal{F}^{a}_{rs}\mathcal{F}^{brs}\right) + g^{i\bar{j}}\frac{k_{ab\bar{j}}}{2}\epsilon^{srl}E^{a}_{s}\mathcal{F}^{b}_{rl} - g^{i\bar{j}}\partial_{\bar{j}}V \ , \tag{3.14}$$

$$\bar{\mathcal{D}}_2 = g^{i\bar{j}}\frac{h_{abi}}{4}\left(2E^{a}_{s}E^{bs} - \mathcal{F}^{a}_{rs}\mathcal{F}^{brs}\right) + g^{i\bar{j}}\frac{k_{abi}}{2}\epsilon^{srl}E^{a}_{s}\mathcal{F}^{b}_{rl} - g^{i\bar{j}}\partial_{i}V \ , \tag{3.15}$$

together with the constraint equation (3.6).

To solve the constraint problem, we use the method in [4] which can be structured as follows. First, we modify (3.1) by decomposing E field into unique transverse parts (divergence free) E_T and longitudinal parts (curl free) E_L

$$E = E_T + E_L \ , \tag{3.16}$$

with

$$\partial_s E_T^s = 0, \qquad \varepsilon_{qrs}\partial^r E_L^s = 0 \ . \tag{3.17}$$

Then, replacing E_L with the new fields, E_C which equals E_L when the constraint is satisfied (see Lemma 3),

$$E_L^s \to E_C^s = \partial_s \left(-\frac{1}{4\pi R} * \rho \right) \ , \tag{3.18}$$

where $-\frac{1}{4\pi r} * \rho$ represents the convolution of ρ with the fundamental solution of Poisson equation,

$$-\frac{1}{4\pi R} * \rho = -\frac{1}{4\pi} \int_{\mathbb{R}^3} dx' \left(\frac{\rho(x')}{|x - x'|} \right) . \tag{3.19}$$

Let $\mathcal{H} = (H_2 \times H_1)^3$, where H_p represent a Sobolev space of square integrable functions over $\mathbb{R}^3$ with their derivative up to order p are also being square integrable and let $\| \, . \, \|_{H_p}$ represent a Sobolev norm defined as

$$\|u\|_{H_p} = \left[\sum_{|\alpha|=0}^{p} \|D^\alpha u\|_{L_2}^2 \right]^{\frac{1}{2}} . \tag{3.20}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a multi-index of non-negative integers and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$. The H_p space is also a Hilbert space. In our analysis, we take the fields $u = (A, E, \phi, \pi, \bar{\phi}, \bar{\pi})$ to lie in $\mathcal{H}$.

Now we prove the following lemmas, which are an extension of lemmas in [4] to incorporate the gauge kinetic functions and show that for fields u lying in $\mathcal{H}$, the condition $E_L = E_C$ implies that E satisfies the constraint equation.

Lemma 1. *Let ρ is defined as in (3.7). If the gauge kinetic function is also lies at least in H_1, then $E_C^s = \partial^s \left(-\frac{1}{4\pi r} * \rho \right) = \frac{1}{4\pi} \left(\frac{\hat{r}}{r^2} * \rho \right)^s$.*

Proof. Since $\frac{1}{r}$ is a weak $L_{3,w}(\mathbb{R}^3)$ function, then using the generalized Young inequality for convolution product [10], we have

$$\left\|\frac{1}{r} * \rho\right\|_{L_p} \leq C\|\rho\|_{L_q}\left\|\frac{1}{r}\right\|_{3,w} \leq C'\|\rho\|_{L_q}, \tag{3.21}$$

for $\frac{1}{q} = \frac{1}{p} + \frac{2}{3}$. It means that $\frac{1}{r} * \rho \in L_p(\mathbb{R}^3)$ provided that $\rho \in L_q(\mathbb{R}^3)$. Since all fields $(A, E, \phi, \pi, \bar{\phi}, \bar{\pi})$ at least lie in H_1 and using the assumption that the gauge kinetic function also lies at least in H_1, then by Sobolev embedding theorem, they all lie in L_s for $2 \leq s \leq 6$. Then it follow that $\rho \in L_q$ for $1 \leq q \leq 3$, thus $\frac{1}{r} * \rho \in L_p(\mathbb{R}^3)$ for $p > 3$.

Let V be an arbitrary C^∞ rapidly decreasing vector field in $\mathbb{R}^3$, we have

$$\begin{aligned}\langle V_s, E_C^s\rangle &= \frac{1}{4\pi}\left\langle V_s, -\partial^s\left(\frac{1}{r} * \rho\right)\right\rangle \\ &= \frac{1}{4\pi}\left\langle \partial^s V_s, \left(\frac{1}{r} * \rho\right)\right\rangle \\ &= \frac{1}{4\pi}\left\langle \frac{1}{r} * \partial^s V_s, \rho\right\rangle. \end{aligned}\tag{3.22}$$

Since $\left|\frac{\hat{r}}{r}\right|$ is a weak $L_{3/2,w}(\mathbb{R}^3)$ function and $\rho \in L_q(\mathbb{R}^3)$ for $1 \leq q \leq 3$, then

$$\left|\frac{\hat{r}}{r}\right| * |\rho| \in L_r, \tag{3.23}$$

for all $\frac{3}{2} < r < 3$. Hence, we have,

$$\begin{aligned}\frac{1}{4\pi}\left\langle \frac{1}{r} * \partial^s V_s, \rho\right\rangle &= \frac{1}{4\pi}\int V_s\left[\int \frac{x^s - x'^s}{|x - x'|^3}\rho(x')dx'\right]dx \\ &= \frac{1}{4\pi}\left\langle V_s, \left(\frac{\hat{r}}{r^2} * \rho\right)^s\right\rangle. \end{aligned}\tag{3.24}$$

And finally we have $E_C^s = \partial^s\left(-\frac{1}{4\pi r} * \rho\right) = \frac{1}{4\pi}\left(\frac{\hat{r}}{r^2} * \rho\right)^s$ which completes the proof. □

Lemma 2. *$E_C \in L_p$ for all $\frac{3}{2} < p < \infty$. In particular $E_C \in L_2$.*

Proof. From the proof of the previous lemma, we already have $\rho \in L_q(\mathbb{R}^3)$ for $1 \leq q \leq 3$. Then using the representation of E_C in previous lemma,

$$\|E_C\|_{L_p} = \left\|\frac{\hat{r}}{r^2} * \rho\right\|_{L_p} \leq C\|\rho\|_{L_q}\left\|\frac{\hat{r}}{r^2}\right\|_{3/2,w} \leq C'\|\rho\|_{L_q}, \tag{3.25}$$

for $1 < p, q < \infty$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{3}$, then $E_C \in L_s$ for all $\frac{3}{2} < s < \infty$.

In particular, $E_C \in L_2$ and the following inequality holds,

$$\|E_C\|_{L_2} \leq C \|\rho\|_{L_{6/5}}. \tag{3.26}$$

□

Lemma 3. *The condition $E_C = E_L$ is equivalent to $\partial_s E^s = 4\pi\rho$.*

Proof. Let V be an arbitrary C^∞ rapidly decreasing function in $\mathbb{R}^3$, we have

$$\begin{aligned}
\langle V, \partial_s E^s_C \rangle &= \frac{1}{4\pi} \int V(x) \partial_s \left[\int \frac{x^s - x'^s}{|x - x'|^3} \rho(x') dx' \right] dx \\
&= -\frac{1}{4\pi} \int \rho(x') \left[\int \frac{x^s - x'^s}{|x - x'|^3} \partial_s V \, dx \right] dx' \\
&= \frac{1}{4\pi} \int \rho(x') \left[\int V(x) \partial_s \left(\frac{x^s - x'^s}{|x - x'|^3} \right) dx \right] dx' \\
&= \int \rho(x') V(x') dx' \\
&= \langle V, \rho \rangle ,
\end{aligned} \tag{3.27}$$

where we use integration by parts and exchange the order of integration. Thus, E_C satisfies $\partial_s E^s_C = 4\pi\rho$ as distribution.

Since $E_C \in L^2$, we can define it's Fourier transform and decomposing it as

$$\hat{E}_C = \hat{E}^T_C + \hat{E}^L_C, \tag{3.28}$$

with

$$\left(\hat{E}^T_C\right)^s = \left(\delta^{sr} - \frac{k^s k^r}{|k|^2}\right) \left(\hat{E}_C\right)_r \tag{3.29}$$

$$\left(\hat{E}^L_C\right)^s = \frac{k^s k^r}{|k|^2} \left(\hat{E}_C\right)_r . \tag{3.30}$$

Furthermore, since E_C is a gradient, then E_C has a vanishing curl, and $\hat{k} \times \hat{E_C} = 0$, which implies that E_C has a longitudinal component only. Thus

$$\left(E^L_C\right)^s = \frac{k^s k^r}{|k|^2} \left(\hat{E}_C\right)_r = \left(\hat{E}_C\right)^s , \tag{3.31}$$

and because E_C satisfies $\partial_s E_C^s = 4\pi\rho$, taking a Fourier transform, we have

$$\hat{E}_C^s = -i\frac{k^s}{|k|^2}\left(4\pi\hat{\rho}\right). \tag{3.32}$$

Now suppose that the field u satisfies the constraint equation,

$$\partial_s E^s = \partial_s E_L^s = 4\pi\rho, \tag{3.33}$$

where we used the decomposition in (3.16). Taking a Fourier transform of the constraint equation, we get

$$\begin{aligned} \hat{E}_L^s &= -i\frac{k^s}{|k|^2}\left(4\pi\hat{\rho}\right) \\ &= \hat{E}_C^s. \end{aligned} \tag{3.34}$$

Thus any solution of the constraint has $E_L = E_C$. Conversely, if $E_L = E_C$, the constraint is satisfied. □

With the modification, we can rewrite the equation of motions as follows

$$\frac{du}{dt} = \mathcal{A}u + J(u), \tag{3.35}$$

where

$$u = \begin{bmatrix} A^{as} \\ E^{as} \\ \phi^i \\ \pi^i \\ \bar{\phi}^{\bar{j}} \\ \bar{\pi}^{\bar{j}} \end{bmatrix}, \qquad \mathcal{A}u = \begin{bmatrix} E_T^{as} \\ \partial_r\partial^r A^{as} - \partial_r\partial^s A^{ar} \\ \pi^i \\ \partial_r\partial^r\phi^i \\ \bar{\pi}^{\bar{j}} \\ \partial_r\partial^r\bar{\phi}^{\bar{j}} \end{bmatrix},$$

and

(3.36) $J(u) =$

$$\begin{bmatrix} \partial^s\{-\frac{1}{4\pi r} * \rho\} \\ f_{bc}^a\partial_r\left(A^{bs}A^{cr}\right) + h^{ab}\left(\mathcal{F}^{esr}\partial_r h_{bc} - \tilde{\mathcal{F}}^{esr}\partial_r k_{bc} + h_{de}f_{bc}^d A_r^c\mathcal{F}^{esr}\right) + \mathcal{D}_1 \\ 0 \\ \Gamma_{kl}^i\left(\partial_s\phi^k\partial^s\phi^l - \pi^k\pi^l\right) + X_a^i\partial_s A^{as} + A^{as}\nabla_k X_a^i\partial_s\phi^k + g^{i\bar{j}}g_{l\bar{k}}A_s^a\nabla_{\bar{j}}\bar{X}_a^{\bar{k}}D^s\phi^l + \mathcal{D}_2 \\ 0 \\ \Gamma_{\bar{k}\bar{l}}^{\bar{j}}\left(\partial_s\bar{\phi}^{\bar{k}}\partial^s\bar{\phi}^{\bar{l}} - \bar{\pi}^{\bar{k}}\bar{\pi}^{\bar{l}}\right) + \bar{X}_a^{\bar{j}}\partial_s A^{as} + A^{as}\nabla_{\bar{k}}\bar{X}_a^{\bar{j}}\partial_s\bar{\phi}^{\bar{k}} + g^{i\bar{j}}g_{k\bar{l}}A_s^a\nabla_i X_a^k D^s\bar{\phi}^{\bar{l}} + \bar{\mathcal{D}}_2 \end{bmatrix}.$$

4. Scalar internal manifold

This section is assigned for the discussion of the internal scalar manifold. In particular, we consider the case of the Kähler potential that is bounded to a function, then derive the estimates for the Kähler potential and the Christoffel symbol. The estimates we derived in this section are important in our analysis for proving the local existence of (3.35).

As mention in Section 2, in four dimensions, the $N=1$ supersymmetry theory demands that the scalar field $(\phi, \bar{\phi})$ span a Kähler manifold with Kähler potential $K \equiv K(\phi, \bar{\phi})$. In this paper, we consider the case where the Kähler potential is bounded above by a $U(n_c)$ symmetric Kähler potential and satisfies several conditions,

$$K \leq \Phi(|\phi|) \,, \tag{4.1}$$

$$|\Gamma| \leq |\tilde{\Gamma}| \,, \tag{4.2}$$

where $|\phi| = \left(\delta_{i\bar{j}} \phi^i \bar{\phi}^{\bar{j}}\right)^{\frac{1}{2}}$ and $\tilde{\Gamma}$ is the Christoffel symbol of $\tilde{g}$.

We prove a lemma about estimates of Kähler potential and Christoffel symbol,

Lemma 4. *Let $\mathcal{M}$ be a Kähler manifold with Kahler potential $K = K(\phi, \bar{\phi})$. If $\mathcal{M}$ satisfies (4.1), (4.2) and*

$$\left| \frac{F'}{2|\phi|} \right| \leq \epsilon \,, \tag{4.3}$$

where $F(|\phi|) = \frac{1}{4|\phi|^2}\left(\Phi'' - \frac{\Phi'}{|\phi|}\right)$ with $\Phi' = \partial\Phi/\partial|\phi|$ and ϵ is a non negative constant, then we have the following estimates

$$|K| \leq \frac{\epsilon}{6}\,|\phi|^6 + \frac{C_1}{2}\,|\phi|^4 + C_2\,|\phi|^2 + C_3 \,, \tag{4.4}$$

$$|\Gamma| \leq 2\epsilon|\phi|^3 + C_1|\phi| \,. \tag{4.5}$$

Proof. Let $\tilde{\mathcal{M}}$ be a Kähler manifold generated by Φ. We can write the metric $\tilde{g}_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi$ as

$$\tilde{g}_{i\bar{j}} = \frac{\Phi'}{2|\phi|}\delta_{i\bar{j}} + \frac{1}{4|\phi|^2}\left(\Phi'' - \frac{\Phi'}{|\phi|}\right)\delta_{k\bar{j}}\delta_{i\bar{k}}\phi^k \bar{\phi}^{\bar{k}} \,, \tag{4.6}$$

where $\Phi' = \partial\Phi/\partial|\phi|$. The inverse of the metric can written as,

$$\tilde{g}^{i\bar{j}} = \frac{2|\phi|}{\Phi'}\delta^{i\bar{j}} - \frac{2}{\Phi'|\phi|}\left(\frac{\Phi'' - \frac{\Phi'}{|\phi|}}{\Phi'' + \frac{\Phi'}{|\phi|}}\right)\phi^i\bar{\phi}^{\bar{j}} \ . \tag{4.7}$$

The norm of the Christoffel symbol is

$$\begin{aligned} |\tilde{\Gamma}| &= \left(\tilde{g}^{j\bar{j}}\tilde{g}^{k\bar{k}}\tilde{g}_{i\bar{i}}\tilde{\Gamma}^i_{jk}\bar{\tilde{\Gamma}}^{\bar{i}}_{\bar{j}\bar{k}}\right)^{\frac{1}{2}} \\ &= \left(\tilde{g}^{j\bar{j}}\tilde{g}^{k\bar{k}}\tilde{g}_{i\bar{i}}\tilde{g}^{i\bar{l}}\partial_j\tilde{g}_{k\bar{l}}\,\tilde{g}^{l\bar{i}}\,\partial_{\bar{j}}\tilde{g}_{l\bar{k}}\right)^{\frac{1}{2}} \\ &= \left(\tilde{g}^{l\bar{l}}\tilde{g}^{j\bar{j}}\tilde{g}^{k\bar{k}}\partial_j\tilde{g}_{k\bar{l}}\,\partial_{\bar{j}}\tilde{g}_{l\bar{k}}\right)^{\frac{1}{2}} \\ &= |\partial\tilde{g}| \ , \end{aligned} \tag{4.8}$$

and the first derivative of the metric is

$$\partial_j\tilde{g}_{k\bar{l}} = F\left(\delta_{k\bar{l}}\delta_{j\bar{i}} + \delta_{k\bar{i}}\delta_{j\bar{l}}\right)\bar{\phi}^{\bar{i}} + \frac{F'}{2|\phi|}\delta_{k\bar{i}}\delta_{i\bar{l}}\delta_{j\bar{m}}\phi^i\bar{\phi}^{\bar{i}}\bar{\phi}^{\bar{m}} \ , \tag{4.9}$$

where

$$\begin{aligned} F(|\phi|) &= \frac{1}{4|\phi|^2}\left(\Phi'' - \frac{\Phi'}{|\phi|}\right) \ , \\ \frac{F'}{2|\phi|} &= \frac{1}{8|\phi|^3}\left(\Phi''' - \frac{3\Phi''}{|\phi|} + \frac{3\Phi'}{|\phi|^2}\right) \ . \end{aligned} \tag{4.10}$$

If condition (4.3) is satisfied and we use the inequality for integral,

$$\left|\int f(x)dx\right| \leq \int |f(x)|\,dx \ , \tag{4.11}$$

we have the following estimates,

$$\begin{aligned} |F| &\leq \epsilon\,|\phi|^2 + C_1, \\ |\Phi| &\leq \frac{\epsilon}{6}\,|\phi|^6 + \frac{C_1}{2}\,|\phi|^4 + C_2\,|\phi|^2 + C_3 \ , \end{aligned} \tag{4.12}$$

where $C_1 = |F(0)|$, $C_2 = \left|\frac{\Phi'}{2|\phi|}(0)\right|$ and $C_3 = |\Phi(0)|$. Then we have that the norm of the Christoffel symbol satisfies

$$\left|\tilde{\Gamma}\right| \leq 2\epsilon|\phi|^3 + C_1|\phi| \ . \tag{4.13}$$

Hence, by our assumption in 4.1 and 4.2, we have

$$|\Phi| \leq \frac{\epsilon}{6}\,|\phi|^6 + \frac{C_1}{2}\,|\phi|^4 + C_2\,|\phi|^2 + C_3\,, \tag{4.14}$$

$$|\Gamma| \leq 2\epsilon|\phi|^3 + C_1|\phi|\,. \tag{4.15}$$

This completes the proof. □

Our assumption in (4.3) is satisfied for several examples of Kähler manifold, for examples are $\mathbb{C}^n$ and $\mathbb{C}P^n$ which are widely used in the theory. For $\mathbb{C}^n$, the Kähler potential is given by $|\phi|^2$, then clearly F vanishes, hence $\frac{F'}{2|\phi|}$ is bounded by 0. In case of $\mathbb{C}P^n$, the Kähler potential (using standard Fubini-Study metric) is given by

$$\Phi_{\mathbb{C}P^n}(|\phi|) = \ln(1 + |\phi|^2)\,. \tag{4.16}$$

Then we have,

$$F = -\frac{1}{(1 + |\phi|^2)^2}\,, \tag{4.17}$$

and

$$\left|\frac{F'}{2|\phi|}\right| = \frac{2}{(1 + |\phi|^2)^3}\,, \tag{4.18}$$

which is bounded above by 2.

5. Local existence

In this section, we will prove the local existence of the evolution equation (3.35) using Segal's theorem. Furthermore, we shall show that solutions of (3.35) are the solutions of the original equations, namely (3.1) and (3.8).

In Section 3, we have derived the equations of motion for u and have taken the field u lies in $\mathcal{H} = (H_2 \times H_1)^3$. Let us consider the linear part of the evolution equation (3.35),

$$\frac{du}{dt} = \mathcal{A}u\,. \tag{5.1}$$

By decomposing the A and E fields into their transverse and longitudinal components, we can write (5.1) as follows,

$$\frac{d}{dt}\begin{bmatrix} A_T^{as} \\ E_T^{as} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \triangle & 0 \end{bmatrix}\begin{bmatrix} A_T^{as} \\ E_T^{as} \end{bmatrix}, \qquad \frac{d}{dt}\begin{bmatrix} A_L^{as} \\ E_L^{as} \end{bmatrix} = 0\,, \tag{5.2}$$

$$\frac{d}{dt}\begin{bmatrix} \phi^i \\ \pi^i \end{bmatrix} = \begin{bmatrix} 0 & I \\ \triangle & 0 \end{bmatrix}\begin{bmatrix} \phi^i \\ \pi^i \end{bmatrix}, \qquad \frac{d}{dt}\begin{bmatrix} \bar{\phi}^{\bar{j}} \\ \bar{\pi}^{\bar{j}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \triangle & 0 \end{bmatrix}\begin{bmatrix} \bar{\phi}^{\bar{j}} \\ \bar{\pi}^{\bar{j}} \end{bmatrix}. \tag{5.3}$$

Each pair of fields (A_T, E_T), (ϕ, π),and $(\bar{\phi}, \bar{\pi})$ satisfies the linear wave equation and (A_L, E_L) is a constant of the linearized equation. Thus, the linear operator $\mathcal{A}$ generates a one-parameter semigroup on $\mathcal{H}$ and for any initial value $u_0 = u(t_0) \in \mathcal{H}$, the linearized equation admits a classical solution which can be written as [2],

$$u(t) = e^{\mathcal{A}(t-t_0)} u_0\,. \tag{5.4}$$

Then, it follows that the solution of the linearized equation is globally defined on $\mathcal{H}$.

Following the result above, by writing the evolution equation (3.35) as an integral equation,

$$u(t) = e^{\mathcal{A}(t-t_0)} u_0 + \int_{t_0}^{t} ds\, e^{\mathcal{A}(s-t_0)} J(u(s))\,, \tag{5.5}$$

the local existence of solution of the equation is established by showing that the nonlinear operator J satisfies Lipshitz condition,

$$\|J(u') - J(u)\| \leq C\left(\|u'\|, \|u\|\right) \|u' - u\|\,, \tag{5.6}$$

for all $u', u \in \mathcal{H}$. The norm $\|\,.\,\|$ is designed for $\mathcal{H}$ norm and $C(,)$ is some monotonically increasing, finite function of the norm indicated. Then, for any initial data $u_0 = u(t_0) \in D_{\mathcal{A}}$, where $D_{\mathcal{A}}$ is a domain of linear operator $\mathcal{A}$, the evolution equation (3.35) admits a unique classical solution on some interval (T_1, T_2) containing t_0 either $(T_1, T_2) = (-\infty, \infty)$ or $\|u(t)\| \to \infty$ as $t \to T_1$ or T_2.

Let us write the components of the non-linear operator J as $J = (J_1, J_2, J_3, J_4, J_5, J_6)$. The proof that J satisfies a Lipshitz condition is facilitated by a Schauder ring property for Sobolev space over $\mathbb{R}^3$ and the Sobolev

[2]for details review, see [1], [11].

inequality over $\mathbb{R}^3$,

$$\|\partial^r u\|_{L_p} \leq C\|\partial^s u\|^{\theta}_{L_m} \|u\|^{1-\theta}_{L_q} , \tag{5.7}$$

for real numbers q, m with $1 \leq q, m \leq \infty$ and r, s are integers where $0 \leq r < s$ which satisfy

$$\frac{1}{p} = \frac{r}{3} + \theta\left(\frac{1}{m} - \frac{s}{3}\right) + (1-\theta)\frac{1}{q} , \tag{5.8}$$

with $r/s \leq \theta \leq 1$ and p is non negative, and a constant C depends only on m, j, q, r and θ. For further discussion on Sobolev inequality, see [12].

Now, since the theory has scalar field dependent gauge couplings, we have to make an assumption for the gauge kinetic function in order to prove that E_C is both a mapping from $\mathcal{H}$ to H_2 and locally Lipshitz continuous.

Lemma 5. *Let the fields $(A, E, \phi, \pi, \bar{\phi}, \bar{\pi}) \in \mathcal{H}$. If the first derivative of the real part of the gauge kinetic function is at most a linear growth,*

$$\|\partial_i h_{ab}\|_{H_1} = \|\partial_{\bar{j}} h_{ab}\|_{H_1} \leq C\|\phi\|_{H_2} , \tag{5.9}$$

then E_C lies in H_2. Furthermore, if $\partial_s h_{ab}$ is also a locally Lipshitz function, then E_C is a locally Lipshitz function.

Proof. Define a norm for a gauge indexed field as

$$|A|^2 = h_{ab} A^{as} A^b_s . \tag{5.10}$$

Using (3.26), we have

$$\|E_C\|_{L_2} \leq C\|\rho\|_{L_{6/5}} , \tag{5.11}$$

and by the definition of ρ and using the Holder inequality, we have

$$\begin{aligned} \|\rho\|^2_{L_{6/5}} &\leq C\big\{\|EA\|^2_{L_{6/5}} + \|X\pi\|^2_{L_{6/5}} + \|E\partial_s h\|^2_{L_{6/5}} \\ &\quad + \|\mathcal{F}\partial_s k\|^2_{L_{6/5}} + \|kA\mathcal{F}\|^2_{L_{6/5}}\big\} \\ &\leq C'\big\{\|E\|^2_{L_2}\|A\|^2_{L_3} + \|X\|^2_{L_3}\|\pi\|^2_{L_2} + \|E\|^2_{L_2}\|\partial_s h\|^2_{L_3} \\ &\quad + \|\partial_s k\|^2_{L_3}\|\mathcal{F}\|^2_{L_2} + \|kA\|^2_{L_3}\|\mathcal{F}\|^2_{L_2}\big\} \\ &\leq C''\big\{\|E\|^2_{L_2}\|A\|^2_{H_1} + \|X\|^2_{H_1}\|\pi\|^2_{L_3} + \|E\|^2_{L_2}\|\partial_s h\|^2_{H_1} \\ &\quad + \|\mathcal{F}\|^2_{L_2}\left(\|\partial_s k\|^2_{H_1} + \|kA\|^2_{H_1}\right)\big\} , \end{aligned} \tag{5.12}$$

where we used the Sobolev inequality to show that $\|u\|_{L_3} \leq C\|u\|_{H_1}$.

Since $\partial_s E_C^s = 4\pi\rho$, by taking the Fourier transform on both sides, we have

$$\hat{E}_C^s = -i4\pi \frac{k^s}{|k|^2}\hat{\rho}\,, \tag{5.13}$$

hence, we have

$$\begin{aligned}
\|D^2 E_C\|_{L_2}^2 &= \int_{\mathbb{R}^3} d^3x\, (\partial_q \partial_r E_{Cs})^2 \\
&= \int_{\mathbb{R}^3} d^3k \left| k^2 \hat{E}_{Cs} \right|^2 \\
&= (4\pi)^2 \int_{\mathbb{R}^3} d^3k\, |k_s \hat{\rho}|^2 \\
&= (4\pi)^2 \int_{\mathbb{R}^3} d^3x\, (\partial_s \rho)^2 = (4\pi)^2 \|\partial_s \rho\|_{L_2}^2\,.
\end{aligned} \tag{5.14}$$

Using definition of Sobolev norm and Schauder ring property, we have

$$\begin{aligned}
\|\partial_s\rho\|_{L_2}^2 &\le \|EA\|_{H_1}^2 + \|X\pi\|_{H_1}^2 + \|E\partial_s h\|_{H_1}^2 + \|\mathcal{F}\partial_s k\|_{H_1}^2 + \|kA\mathcal{F}\|_{H_1}^2 \\
&\le \|E\|_{H_1}^2 \|A\|_{H_1}^2 + \|X\|_{H_1}^2 \|\pi\|_{H_1}^2 + \|E\|_{H_1}^2 \|\partial_s h\|_{H_1}^2 \\
&\quad + \|\mathcal{F}\|_{H_1}^2 \left(\|\partial_s k\|_{H_1}^2 + \|kA\|_{H_1}^2 \right)\,.
\end{aligned} \tag{5.15}$$

By definition of field strength, we get the estimate

$$\|\mathcal{F}\|_{H_1} \le C\left(\|A\|_{H_2} + \|A\|_{H_2}^2 \right)\,, \tag{5.16}$$

and using condition (5.9),

$$\begin{aligned}
\|\partial_s h\|_{H_1} &\le \|\partial_i h\|_{H_1} \|\partial_s \phi\|_{H_1}\,, \\
&\le C\|\phi\|_{H_2}^2
\end{aligned} \tag{5.17}$$

and the fact that the gauge kinetic function is a holomorphic function, we have

$$\begin{aligned}
\|E_C\|_{H_2} \le C\big(&\|A\|_{H_2}\|E\|_{H_1} + \|\phi\|_{H_2}\|\pi\|_{H_1} + \|\phi\|_{H_2}^2\|E\|_{H_1} + \|A\|_{H_2}\|\phi\|_{H_2}^2 \\
&+ \|A\|_{H_2}^2\|\phi\|_{H_2}^2 + \|A\|_{H_2}^3\|\phi\|_{H_2} + \|A\|_{H_2}^2\|\phi\|_{H_1}^2 \big)\,,
\end{aligned} \tag{5.18}$$

which shows that E_C lies in H_2 for all $u \in \mathcal{H}$, thus the first part of the lemma is proven.

Now we will prove the Lipshitz condition for E_C which is important for proving the local existence of the evolution equation. From the definition of E_C we have

$$
\begin{aligned}
&E_C^{as}(u') - E_C^{as}(u) \\
&\quad = \frac{\hat{r}^s}{4\pi r^2} * \Big(-f_{bf}^c \{ h'^{ab} h'_{cd} A'^f_r E'^r_d - h^{ab} h_{cd} A^f_r E^r_d \} \\
&\qquad - h'^{ab} g'_{i\bar{j}} \{ X'^i_b \bar{\pi}'^{\bar{j}} + \bar{X}'^{\bar{j}}_b \pi'^i \} + h^{ab} g_{i\bar{j}} \{ X^i_b \bar{\pi}^{\bar{j}} + \bar{X}^{\bar{j}}_b \pi^i \} \\
&\qquad + h'^{ab} \big(k'_{de} f^d_{bc} - k'_{bd} f^d_{ce} \big) A'^{cr} \tilde{\mathcal{F}}'^{es0} - h^{ab} \big(k_{de} f^d_{bc} - k_{bd} f^d_{ce} \big) A^{cr} \tilde{\mathcal{F}}^{es0} \\
&\qquad - h'^{ab} E'^{cs} \partial_s h'_{bc} + h^{ab} E^{cs} \partial_s h_{bc} + h'^{ab} \tilde{\mathcal{F}}'^{cs0} \partial_s k'_{bc} - h^{ab} \tilde{\mathcal{F}}^{cs0} \partial_s k_{bc} \Big) . \qquad (5.19)
\end{aligned}
$$

Recalling the estimate from the prove of the first part of the lemma, and using assumption that $\partial_s h_{ab}$ is locally Lipshitz, then we have

$$
\begin{aligned}
&\|E_C^{as}(u') - E_C^{as}(u)\|_{H_2} \\
&\quad \le K \Big\{ \|A' - A\|_{H_2} \big[\|E'\|_{H_1} + \|\phi\|^2_{H_2} (\|A\|_{H_2} + \|A\|_{H_2} \|A' + A\|_{H_2} \\
&\qquad + \|A\|^2_{H_2} + \|A' + A\|_{H_2}) \big] + \|E' - E\|_{H_1} (\|A\|_{H_2} + \|\phi\|^2_{H_2}) \\
&\qquad + \|\pi' - \pi\|_{H_1} \|\phi\|_{H_2} + \|\phi' - \phi\|_{H_2} \big[\|E\|_{H_1} \|\phi' + \phi\|_{H_2} \\
&\qquad + \|A\|_{H_2} (1 + \|A\|_{H_2}) \|\phi' + \phi\|_{H_2} + \|A'\|^2_{H_2} (1 + \|A'\|_{H_2}) \big] \Big\} \\
&\quad = C_1(\|u'\|, \|u\|) \|A' - A\|_{H_2} + C_2(\|u'\|, \|u\|) \|E' - E\|_{H_1} \\
&\qquad + C_3(\|u'\|, \|u\|) \|\phi' - \phi\|_{H_2} + C_4(\|u'\|, \|u\|) \|\pi' - \pi\|_{H_1} . \qquad (5.20)
\end{aligned}
$$

Then for $u \in \mathcal{H}$, E_C is a locally Lipshitz function which proves the second part of the lemma.

□

The final proof of the Lipshitz condition for J is established by showing that the other components of J are also Lipshitz functions. Using Sobolev inequality, we have

$$
\begin{aligned}
&\|J_2(u') - J_2(u)\|_{H_1}\\
&\qquad \le C\Big\{\|A' - A\|_{H_2}\Big[\|A\|_{H_2} + \|A\|^2_{H_2} + \|E\|_{H_1}\|\phi'\|^2_{H_2} + \|\phi\|^2_{H_2}\\
&\qquad\quad + \|E\|_{H_1}\|\phi'\|^2_{H_2} + \|A' + A\|_{H_2}\big(1 + \|A'\|_{H_2} + \|\phi\|^2_{H_2}\\
&\qquad\quad + \|\phi'\|_{H_2}\|\pi'\|_{H_1}\big) + \|\phi'\|_{H_2}\|\pi'\|_{H_1}\Big]\\
&\qquad\quad + \|E' - E\|_{H_1}\Big[\|\phi'\|_{H_2}\|\pi'\|_{H_1} + \|\phi\|^2_{H_2} + \|A'\|_{H_2}\|\phi'\|^2_{H_2}\Big]\\
&\qquad\quad + \|\phi' - \phi\|_{H_2}\Big[\|\phi'\|_{H_2} + \|\phi' + \phi\|_{H_2}\big(\|E\|_{H_1} + \|A'\|_{H_2} + \|A'\|^2_{H_2}\\
&\qquad\quad + \|A\|_{H_2}\big) + \|\pi\|_{H_1}\|E\|_{H_1} + \|\pi\|_{H_2}\big(\|A\|_{H_2} + \|A\|^2_{H_2}\big) + \|\phi\|_{H_2}\Big]\\
&\qquad\quad + \|\pi' - \pi\|_{H_1}\Big[\|E\|_{H_1} + \|\phi'\|_{H_2}\big(\|A\|_{H_2} + \|A\|^2_{H_2}\big)\Big]\Big\}\\
&\qquad = C_5\big(\|u'\|,\|u\|\big)\|A' - A\|_{H_2} + C_6\big(\|u'\|,\|u\|\big)\|E' - E\|_{H_1}\\
&\qquad\quad + C_7\big(\|u'\|,\|u\|\big)\|\phi' - \phi\|_{H_2} + C_8\big(\|u'\|,\|u\|\big)\|\pi' - \pi\|_{H_1}\,, \qquad (5.21)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|J_4(u') - J_4(u)\|_{H_1}\\
&\qquad \le C\Big\{\|\Gamma' - \Gamma\|_{H_2}\Big[\|\phi'\|_{H_2} + \|A'\|_{H_2}\|\phi'\|^2_{H_2} + \|A'\|^2_{H_2}\|\phi'\|^2_{H_2} + \|\pi\|^2_{H_1}\Big]\\
&\qquad \|E' - E\|_{H_1}\Big[\|\phi\|_{H_2}\|E' + E\|_{H_1} + \|\phi'\|_{H_2}\|\mathcal{F}\|_{H_1}\Big] + \|\pi' - \pi\|_{H_1}\|\Gamma\|_{H_2}\\
&\qquad \|A' - A\|_{H_2}\Big[\|\phi'\|_{H_2} + \|A' - A\|_{H_2}\big(\|\phi'\|_{H_2} + \|\Gamma\|_{H_2}\|\phi\|^2_{H_2}\big)\\
&\qquad + \|\Gamma\|_{H_2}\|\phi\|^2_{H_2} + \big(1 + \|A' + A\|_{H_2}\big)\big(\|\phi\|_{H_2}\|\mathcal{F}' - \mathcal{F}\|_{H_1}\\
&\qquad + \|\phi\|_{H_2}\|E'\|_{H_1}\big)\Big] + \|\phi' - \phi\|_{H_2}\Big[\|\Gamma\|_{H_2}\|\phi' + \phi\|_{H_2}\big(1 + \|A\|^2_{H_2}\big)\\
&\qquad + \|A'\|_{H_2} + \|A\|_{H_2} + \|A\|^2_{H_2} + \|E\|_{H_2}\|\mathcal{F}\|_{H_1} + \|E\|^2_{H_2} + \|\mathcal{F}\|^2_{H_2}\Big]\\
&\qquad + \|\partial_{\bar{j}}V'_S - \partial_{\bar{j}}V_S\|_{H_1}\Big\}\,. \qquad (5.22)
\end{aligned}
$$

Using estimate (4.2) and (5.16) and using the assumption in (2.5), we have

$$
\begin{aligned}
\|J_4(u') - J_4(u)\|_{H_1} \le\ & C_9(\|u'\|,\|u\|)\|A' - A\|_{H_2} + C_{10}(\|u'\|,\|u\|)\|E' - E\|_{H_1}\\
& + C_{11}(\|u'\|,\|u\|)\|\phi' - \phi\|_{H_2}\\
& + C_{12}(\|u'\|,\|u\|)\|\pi' - \pi\|_{H_1}\,. \qquad (5.23)
\end{aligned}
$$

Thus, from Lemma 5, (5.21) and (5.23), the nonlinear operator J is a mapping from $\mathcal{H}$ to itself and satisfies locally Lipshitz condition.

Then by Segal's theorem, for any initial data u_0 in $\mathcal{H}$, there exists a positive constant $T > 0$ depending on u_0 such that the Equation (3.35) admits a unique mild solution which is continuous in $\mathcal{H}$ for an interval $[0, T]$, i.e. $u \in C([0, T], \mathcal{H})$ which satisfies (5.5). Furthermore, the solutions can be extended into maximal mild solutions on interval $[0, T_{\max})$ such that either

1) $T_{\max} = +\infty$ and the equation (3.35) admits a global solution, or
2) $\|u(t)\| \to \infty$ as $t \to T_{\max}$ and the solution blows up on a finite time $T_{\max}$.

If the initial value u_0 lies in $D_{\mathcal{A}}$ then Equation (3.35) admits a unique classical solutions $u(t)$ for an interval $[0, T_{\max})$ which remains in $D_{\mathcal{A}}$ and satisfies the differential equations

$$\frac{du(t)}{dt} = \mathcal{A}u(t) + J(u(t)) \,, \tag{5.24}$$

with $\frac{du(t)}{dt}$ is a continuous curve in $\mathcal{H}$. Then, the solutions u belong to,

$$u \in C^1\left([0, T_{\max}), \mathcal{H}\right) \cap C\left([0, T_{\max}), D_{\mathcal{A}}\right) \,, \tag{5.25}$$

such that either $T_{\max} = +\infty$ or $\|u(t)\| \to \infty$ as $t \to T_{\max}$.

Now, we shall show that the solution of the modified equation (3.35), which satisfies the constraint, is the solution of the original equation.

Following the result of [4] for the constraint equation, we have

$$\|\mathcal{C}(t)\|_{L_2}^2 \leq \|\mathcal{C}(0)\|_{L_2}^2 \exp\left(\int_0^t C' \|E(s)\|_{H_1}\right) \,. \tag{5.26}$$

Since the initial value of the constraint equation is $\mathcal{C}(0) = 0$, then $\mathcal{C}(t)$ vanishes in the interval existence of $u\,(t)$. Hence, the solution of the modified equation (3.35) always satisfies the constraint equation so that it is the solution of the original equation.

Therefore, we have proven,

Theorem 1. *Let u_0 be any initial data lying in $\mathcal{H} = (H_2 \times H_1)^3$. If the conditions (2.5), (4.1), (4.2), and (4.3) are satisfied, then there exists a positive constant $T_{\max} > 0$ depending on u_0 such that the integral equation (5.5) admits a unique maximal solution $u(t)$ on interval $[0, T_{\max})$ which belongs to $u \in C\left([0, T_{\max}), \mathcal{H}\right)$ and either*

1) $T_{\max} = +\infty$ *and the equation (3.35) admits a global solution, or*

2) $\|u(t)\| \to \infty$ *as* $t \to T_{\max}$ *and the solution blow up on a finite time* $T_{\max}$.

Furthermore, if u_0 *lies in* $D_{\mathcal{A}}$ *and satisfies the constraint* $\mathcal{C}(u_0) = 0$, *then the equation (3.35) admits a unique classical solution* $u(t)$ *for an interval* $[0, T_{\max})$ *which remains in* $D_{\mathcal{A}}$, *and belongs to* $u \in C^1([0, T_{\max}), \mathcal{H}) \cap C([0, T_{\max}), D_{\mathcal{A}})$, *and satisfies the constraint* $\mathcal{C}(u(t)) = 0$ *such that either* $T_{\max} = +\infty$ *or* $\|u(t)\| \to \infty$ *as* $t \to T_{\max}$.

Appendix A. Convention and notation

The purpose of this appendix is to inform our conventions used in this paper. The spacetime metric is flat with the signature $(-, +, +, +)$.

The following indices are used:

: $\mu, \nu, \rho, \sigma = 0, \ldots, 3,$	label 4-dimensional flat spacetime
: $r, s, p, q = 1, 2, 3,$	label 3-dimensional flat space
: $i, \bar{i}, j, \bar{j}, k, \bar{k} = 1, \ldots, n_c,$	label n_c dimensional Kähler manifold
: $a, b, c, d = 1, \ldots, n_v,$	label the gauge index

Appendix B. List of inequality

In this appendix, we mention some basic inequalities used in this paper. For detail reviews, see [10, 12].

B.1. Young inequality for convolution

Suppose $f \in L_p(\mathbb{R}^d)$ and $g \in L_q(\mathbb{R}^d)$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1\,, \tag{B.1}$$

with $p, q, r \in \mathbb{R}$ and $1 \leq p, q, r \leq \infty$. Then

$$\|f * g\|_{L_r} \leq \|f\|_{L_p} \|g\|_{L_q}\,. \tag{B.2}$$

If $g \in L_{q,w}(\mathbb{R}^d)$ where $L_{q,w}$ is weak L_q space, then

$$\|f * g\|_{L_r} \leq \|f\|_{L_p} \|g\|_{L_{q,w}}\,, \tag{B.3}$$

with p, q, r are defined and satisfy the condition above.

B.2. Holder inequality

Let p, q, r be positive numbers and satisfy $p, q, r \leq 1$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{B.4}$$

Suppose $f \in L_p(\mathbb{R}^d)$ and $g \in L_q(\mathbb{R}^d)$, then $fg \in L_r(\mathbb{R}^d)$ and

$$\|fg\|_{L_r} \leq \|f\|_{L_p} \|g\|_{L_q} . \tag{B.5}$$

B.3. Sobolev inequality

Let q, m be real numbers with $1 \leq q, m \leq \infty$ and r, s are integers where $0 \leq r < s$ which satisfy

$$\frac{1}{p} = \frac{r}{3} + \theta \left(\frac{1}{m} - \frac{s}{3} \right) + (1 - \theta) \frac{1}{q} , \tag{B.6}$$

with $r/s \leq \theta \leq 1$ and p is non negative. For $u \in H_s(\mathbb{R}^3) \cap L_q(\mathbb{R}^3)$, there is a positive constant C which depends only on m, j, q, r and θ such that the following inequality holds

$$\|\partial^r u\|_{L_p} \leq C \|\partial^s u\|^{\theta}_{L_m} \, \|u\|^{1-\theta}_{L_q} . \tag{B.7}$$

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