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On the vector bundles associated to the irreducible representations of cocompact lattices of SL(2, ℂ)

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Abstract

We prove the following: let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a cocompact lattice and let $\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C})$ be an irreducible representation. Then the holomorphic vector bundle $E_\rho \longrightarrow \mathrm{SL}(2, \mathbb{C})/\Gamma$ associated to ρ is polystable. The compact complex manifold $\mathrm{SL}(2, \mathbb{C})/\Gamma$ has natural Hermitian structures; the polystability of E_ρ is with respect to these natural Hermitian structures. We show that the polystable vector bundle E_ρ is not stable in general.

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1 Introduction

We first recall the set-up, and some results, of [1]. Let

$$\Gamma \subset \mathrm{SL}(2, \mathbb{C})$$

be a discrete cocompact subgroup. Fixing a SU(2)-invariant Hermitian form on the Lie algebra $\mathrm{sl}(2, \mathbb{C})$, we get a Hermitian structure h on the compact complex manifold $M := \mathrm{SL}(2, \mathbb{C})/\Gamma$. The $(1\,,1)$-form ω_h on M associated to h satisfies the identity $d\omega_h^2 = 0$. Take any homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{GL}(r, \mathbb{C}).$$

This ρ produces a holomorphic vector bundle E_ρ of rank r on M equipped with a flat holomorphic connection ∇^ρ. The homomorphism ρ is called irreducible if $\rho(\Gamma)$ is not contained in some proper parabolic subgroup of $\mathrm{GL}(r, \mathbb{C})$.

If $\rho(\Gamma) \subset U(r)$, then E_ρ is equipped with a Hermitian structure H^ρ such that the associated Chern connection is ∇^ρ.

If

- $\rho(\Gamma) \subset U(r)$ and
- $\rho(\Gamma)$ is irreducible,

then the vector bundle E_ρ is stable [1, Proposition 4.5].

Now assume that ρ is irreducible, but do *not* assume that $\rho(\Gamma) \subset U(r)$. Our aim here is to prove the following (see Theorem 2.2):

The holomorphic vector bundle E_ρ is polystable with respect to the Hermitian structure h on M.

It is known that under some minor condition, the group Γ admits some free groups of more than one generators as quotients [6, p. 3393, Theorem 2.1]. Therefore, there are many examples of pairs $(\Gamma\,, \rho)$ of the above type satisfying the irreducibility condition.

Since E_ρ is polystable, the holomorphic vector bundle E_ρ has an Hermitian–Yang–Mills structure $\mathcal{H}^\rho$ [7] (see also [3]). It may be worthwhile to investigate this Hermitian structure $\mathcal{H}^\rho$. We should clarify that $\mathcal{H}^\rho$ need not be flat. An Hermitian–Yang–Mills structure on a polystable vector bundle with vanishing Chern classes over a compact Kähler manifold is flat, but M is not Kähler.

It is natural to ask whether the polystable vector bundle E_ρ is stable. If we take ρ to be the inclusion of Γ in $\mathrm{SL}(2,\mathbb{C})$, then ρ is irreducible, but the associated holomorphic vector bundle E_ρ is holomorphically trivial, in particular, E_ρ is not stable (see Lemma 2.3 for the details).

Infinitesimal deformations of the complex structure of M are investigated in [8].

2 Polystability of associated vector bundle

The Lie algebra of $\mathrm{SL}(2,\mathbb{C})$, which will be denoted by $\mathrm{sl}(2,\mathbb{C})$, is the space of complex 2×2 matrices of trace zero. Consider the adjoint action of $\mathrm{SU}(2)$ on $\mathrm{sl}(2,\mathbb{C})$. Fix an inner product h_0 on $\mathrm{sl}(2,\mathbb{C})$ preserved by this action; for example, we may take the Hermitian form $(A\,,B)\longmapsto \mathrm{trace}(AB^*)$ on $\mathrm{sl}(2,\mathbb{C})$. Let h_1 be the Hermitian structure on $\mathrm{SL}(2,\mathbb{C})$ obtained by right-translating the Hermitian form h_0 on $T_{\mathrm{Id}}\,\mathrm{SL}(2,\mathbb{C}) = \mathrm{sl}(2,\mathbb{C})$.

Let Γ be a cocompact lattice in $\mathrm{SL}(2,\mathbb{C})$. So Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$ such that the quotient

$$M := \mathrm{SL}(2,\mathbb{C})/\Gamma \tag{2.1}$$

is compact. This M is a compact complex manifold of complex dimension three. The left-translation action of $\mathrm{SL}(2,\mathbb{C})$ on itself descends to an action of $\mathrm{SL}(2,\mathbb{C})$ on M. We will call this action of $\mathrm{SL}(2,\mathbb{C})$ on M the *left-translation action*. The Hermitian structure h_1 on $\mathrm{SL}(2,\mathbb{C})$ descends to an Hermitian structure on M. This descended Hermitian structure on M will be denoted by h. Let ω_h be the C^∞ $(1\,,1)$-form on M associated to h. Then

$$d\omega_h^2 = 0$$

[1, Corollary 4.1].

For a torsionfree nonzero coherent analytic sheaf F on M, define

$$\mathrm{degree}(F) := \int_M c_1(F)\wedge\omega_h^2 \in \mathbb{R} \quad \text{and} \quad \mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \in \mathbb{R}.$$

A torsionfree nonzero coherent analytic sheaf F on M is called *stable* (respectively, *semistable*) if for every coherent analytic subsheaf

$$V \subset F$$

such the $\text{rank}(V) \in [1\,, \text{rank}(F)-1]$ and the quotient F/V is torsionfree, the inequality

$$\mu(V) \,<\, \mu(F) \quad (\text{respectively, } \mu(V) \,\leq\, \mu(F))$$

holds (see [5, Chapter V, Section 7]). A torsionfree nonzero coherent analytic sheaf F on M is called *polystable* if it is semistable and is isomorphic to a direct sum of stable sheaves.

Remark 2.1. Since a polystable coherent analytic sheaf F is semistable, if $F \,=\, \bigoplus_{i=1}^{\ell} F_i$, then $\mu(F_i) \,=\, \mu(F)$ for all i.

Take any homomorphism

$$\rho \,:\, \Gamma \,\longrightarrow\, \text{GL}(r, \mathbb{C})\,. \tag{2.2}$$

Let $(E_\rho\,, \nabla^\rho)$ be the flat holomorphic vector bundle of rank r over M associated to the homomorphism ρ. We recall that the total space of E_ρ is the quotient of $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ where two points

$$(z_1, v_1), (z_2, v_2) \,\in\, \text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$$

are identified if there is an element $\gamma \in \Gamma$ such that $z_2 \,=\, z_1\gamma$ and $v_2 \,=\, \rho(\gamma^{-1})(v_1)$. The trivial connection on the trivial vector bundle $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r \,\longrightarrow\, \text{SL}(2, \mathbb{C})$ of rank r descends to the connection ∇^ρ. The left-translation action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C})$ and the trivial action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^r$ together define an action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$. This action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ descends to an action

$$\tau \,:\, \text{SL}(2, \mathbb{C}) \times E_\rho \,\longrightarrow\, E_\rho \tag{2.3}$$

of $\text{SL}(2, \mathbb{C})$ on the vector bundle E_ρ. The action τ in (2.3) is clearly a lift of the left-translation action of $\text{SL}(2, \mathbb{C})$ on M.

The homomorphism ρ in (2.2) is called *reducible* if there is a nonzero linear subspace $S \subsetneq \mathbb{C}^r$ such that $\rho(\Gamma)(S) \,=\, S$. The homomorphism ρ is called *irreducible* if it is not reducible.

Theorem 2.2. *Assume that the homomorphism ρ in* (2.2) *is irreducible. Then the corresponding holomorphic vector bundle E_ρ is polystable.*

Proof. Since E_ρ has a flat connection, the Chern class $c_1(\det E_\rho) \,=\, c_1(E_\rho) \,\in\, H^2(M, \mathbb{R})$ vanishes. Hence we have $\text{degree}(E_\rho) \,=\, 0$ (see [1, Lemma 4.2]).

We will first show that E_ρ is semistable. Assume that E_ρ is not semistable. Let

$$0 \subset W_1 \subset \cdots \subset W_{\ell-1} \subset W_\ell = E_\rho \tag{2.4}$$

be the Harder–Narasimhan filtration E_ρ; see [2] for the construction of the Harder–Narasimhan filtration of vector bundles on compact complex manifolds. Since E_ρ is not semistable, we have $\ell \geq 2$ and $W_1 \neq 0$.

Consider the action τ of $\mathrm{SL}(2,\mathbb{C})$ on E_ρ constructed in (2.3). From the uniqueness of the Harder–Narasimhan filtration it follows immediately that $\tau(\{g\} \times W_1) = W_1$ for every $g \in \mathrm{SL}(2,\mathbb{C})$. Therefore, we have

$$\tau(\mathrm{SL}(2,\mathbb{C}) \times W_1) = W_1 . \tag{2.5}$$

Let $C(W_1) \subsetneq M$ be the closed subset over which W_1 fails to be locally free. Since τ is a lift of the left-translation action of $\mathrm{SL}(2,\mathbb{C})$ on M, from (2.5) we conclude that $C(W_1)$ is preserved by the left-translation action of $\mathrm{SL}(2,\mathbb{C})$ on M. As the left-translation action of $\mathrm{SL}(2,\mathbb{C})$ on M is transitive, it follows that $C(W_1)$ is the empty set. Therefore, W_1 is a holomorphic vector bundle on M. Similarly, the closed proper subset of M over which W_1 fails to be a subbundle of E_ρ is preserved the left-translation action of $\mathrm{SL}(2,\mathbb{C})$ on M. Hence this subset is empty, and W_1 is a holomorphic subbundle of E_ρ.

We will show that the flat connection ∇^ρ on E_ρ preserves the subbundle W_1 in (2.4).

To show that ∇^ρ preserves W_1, first note that the flat sections of the trivial connection on the trivial vector bundle $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^r \longrightarrow \mathrm{SL}(2,\mathbb{C})$ are of the form

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^r, \quad g \longmapsto (g,\, v_0)\,,$$

where $v_0 \in \mathbb{C}^r$ is independent of g. On the other hand, the image of such a section is an orbit for the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^r$; recall that the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^r$ is the diagonal one for the left-translation action of $\mathrm{SL}(2,\mathbb{C})$ on itself and the trivial action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^r$ (see the construction of τ in (2.3)). Also, recall that the connection ∇^ρ on E_ρ is the descent of the trivial connection on the trivial vector bundle $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^r \longrightarrow \mathrm{SL}(2,\mathbb{C})$. Combining these, from (2.5) we conclude that ∇^ρ preserves W_1.

The homomorphism ρ is given to be irreducible. Therefore, the only holomorphic subbundles of E_ρ that are preserved by the associated connection

∇^ρ are 0 and E_ρ itself. But $\ell \geq 2$ and $W_1 \neq 0$ in (2.4). So W_1 neither 0 nor E_ρ.

In view of the above contradiction, we conclude that the holomorphic vector bundle E_ρ is semistable.

We will now prove that E_ρ is polystable.

Consider all nonzero coherent analytic subsheaves V of E_ρ such that

- V is polystable and
- degree$(V) = 0$.

Let

$$\mathcal{F} \subset E_\rho \tag{2.6}$$

be the coherent analytic subsheaf generated by all V satisfying the above two conditions. It is known that $\mathcal{F}$ is polystable with $\mu(\mathcal{F}) = \mu(E_\rho) = 0$ (see [4, p. 23, Lemma 1.5.5]). Therefore, the subsheaf $\mathcal{F}$ is uniquely characterized as follows: the subsheaf $\mathcal{F}$ is the unique maximal coherent analytic subsheaf of E_ρ such that

- $\mathcal{F}$ is polystable and
- degree$(\mathcal{F}) = 0$.

Note that the quotient $E_\rho/\mathcal{F}$ is torsionfree, because if $T \subset E_\rho/\mathcal{F}$ is the torsion part, then $\varphi^{-1}(T) \subset E_\rho$, where

$$\varphi : E_\rho \longrightarrow E_\rho/\mathcal{F}$$

is the quotient map, also satisfies the above two conditions, while $\mathcal{F} \subsetneq \varphi^{-1}(T)$ if $T \neq 0$.

Consider the action τ of $\mathrm{SL}(2, \mathbb{C})$ on E_ρ constructed in (2.3). From the above characterization of the subsheaf $\mathcal{F}$ in (2.6) it follows immediately that

$$\tau(\mathrm{SL}(2, \mathbb{C}) \times \mathcal{F}) = \mathcal{F}. \tag{2.7}$$

As it was done for W_1, from (2.7) we conclude that $\mathcal{F}$ is a holomorphic subbundle of E_ρ.

As it was done for W_1, from (2.7) it follows that the flat connection ∇^ρ on E_ρ preserves the subbundle $\mathcal{F}$ in (2.6). Since ρ is irreducible, either

$\mathcal{F} = 0$ or $\mathcal{F} = E_\rho$. The rank of $\mathcal{F}$ is at least one because the semistable vector bundle E_ρ of degree zero has a nonzero stable subsheaf of degree zero. Therefore, we conclude that $\mathcal{F} = E_\rho$. Consequently, E_ρ is polystable. □

We may now ask whether the polystable vector bundle E_ρ in Theorem 2.2 is stable. The following lemma shows that E_ρ is not stable in general.

Let

$$\delta : \Gamma \hookrightarrow \mathrm{SL}(2, \mathbb{C}) \tag{2.8}$$

be the inclusion map. This homomorphism δ is clearly irreducible. Let $(E_\delta, \nabla^\delta)$ be the corresponding flat holomorphic vector bundle on M.

Lemma 2.3. *The above holomorphic vector bundle E_δ is holomorphically trivial.*

Proof. Recall that the vector bundle E_δ is a quotient of $\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^2$. Consider the holomorphic map

$$\mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^2 \longrightarrow \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^2$$

defined by $(g, v) \longmapsto (g, g(v))$. This map descends to a holomorphic isomorphism of vector bundles

$$E_\delta \longrightarrow M \times \mathbb{C}^2$$

over M. Therefore, this descended homomorphism provides a holomorphic trivialization of E_δ. □

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