

Weierstrass models of elliptic toric $K3$ hypersurfaces and symplectic cuts

Antonella Grassi¹ and Vittorio Perduca²

¹Department of Mathematics, University of Pennsylvania, David
Rittenhouse Laboratory, 209 S 33rd Street, Philadelphia, PA 19104, USA
grassi@math.upenn.edu

²MAP5 — Laboratory of Applied Mathematics, Paris Descartes University
and CNRS, 45 Rue des Saints-Pères, 75006 Paris, France
vittorio.perduca@parisdescartes.fr

Abstract

We study elliptically fibered $K3$ surfaces, with sections, in toric Fano 3-folds which satisfy certain combinatorial properties relevant to F-theory/heterotic duality. We show that some of these conditions are equivalent to the existence of an appropriate notion of a Weierstrass model adapted to the toric context. Moreover, we show that if in addition other conditions are satisfied, there exists a toric semistable degeneration of the elliptic $K3$ surface which is compatible with the elliptic fibration and F-theory/Heterotic duality.

1 Introduction

This paper begins an investigation motivated by the physics of string theory, in particular the “F-theory and Heterotic duality” which predicts unexpected relations between certain n -dimensional elliptically fibered Calabi–Yau varieties with section (the F-theory models) and certain principal bundles over $n - 1$ dimensional Calabi–Yau varieties (the Heterotic models) [17, 18, 32]. Recently, this duality has also been used to construct realistic models in string theory, see, for example, the survey [42].

The duality is conjectured to be between n -dimensional Calabi Yau varieties which are $K3$ fibered and $n - 1$ -dimensional Calabi–Yau variety elliptically fibered over the same base [17, 18]. When $n = 2$, that is the duality is between F-theory on an elliptically fibered $K3$ surface with section and Heterotic theory on an elliptic curve, the elliptic curve and the bundles are obtained by a suitable semistable degeneration of the $K3$ surface; this semistable degeneration is the starting point of the duality in higher dimension [32].

It is neither straightforward nor easy to produce these pairs of dual varieties. In the 1990s, Candelas and collaborators [9–12, 37] and also [6] proposed a clever and quick algorithm to find the Heterotic Calabi–Yau duals (Y, E) for *certain* Calabi–Yau manifolds X , which are hypersurfaces in toric Fano varieties V . Berglund and Mayr [4] proposed an explicit construction that assigns to each given toric variety of dimension n a toric variety of dimension $n - 1$ together with a specific family of sheaves on it. This systematic construction produces many interesting examples of conjectured F-theory/Heterotic dual in the toric context. The essence of these algorithms is a sequence of *suitable* projections from the toric fan of V . Candelas and collaborators, Berglund and Mayr check via other physics dualities that the constructions produces the expected pair of dual varieties and the group, but the mathematical reasoning behind this, that is the connection to the semistable degeneration, remains elusive.

This paper represents a first step towards the ultimate goal, mentioned in the paper of Hu, Liu and Yau [20], to present the F-theory/Heterotic analog of Batyrev’s constructions of toric mirrors. We consider the case of $K3$ surfaces/elliptic curves, which are the building block for higher-dimensional F-theory/Heterotic duality. When the duality is between F-theory on an elliptically fibered $K3$ surface with section and Heterotic theory on an elliptic curve, Clingher and Morgan proved that certain regions of the moduli space of such Heterotic theories and their F-theory counterparts can be identified as dual [13]. The authors consider a partial compactification of the moduli spaces of the smooth elliptic $K3$ surfaces with section, by adding

two divisors at infinity D_1 and D_2 ; the points of the boundary divisors correspond to semistable degenerations of $K3$ surfaces given by the union of two rational elliptic surfaces glued together along an elliptic curve in two different ways. An elliptic curve E (the double curve of the degeneration) is then shown to be the Heterotic dual of the $K3$; the elliptic curve is endowed with a flat G -bundle. The Lie group G is $(E_8 \times E_8)$ for one boundary divisor and $\text{Spin}(32)/\mathbb{Z}/2\mathbb{Z}$ for the other. We give a mathematical definition for the properties of the examples of Candelas and collaborators, which we refer to as Candelas' conditions in Section 3 and prove several theorems, in particular 6.6 and 6.7, which provide an explanation, in term of semistable degeneration, of Candelas' algorithm, and part of Berglund–Mayr's construction. The main idea is to combine techniques from symplectic geometry, namely the symplectic cut of the moment polytope [21, 30], with an appropriate notion of a Weierstrass model, adapted to the toric context. The dual Newton (moment) polytope is the polytope naturally associated to Fano toric varieties, whose lattice points correspond to the sections of the anticanonical bundle.

In Section 2, after some general facts about toric Fano varieties and their Calabi–Yau hypersurfaces we explain how the inclusion of a two-dimensional (2D) reflexive polytope determines the elliptic fibration. This observation, long known in the physics literature, is the starting point of Rohsiepe's analysis for higher dimensional Calabi–Yau elliptic fibrations [39]: we use it to construct the elliptic fibrations in the examples. Here “top and bottom” are used to identify the inverse image(s) of the fixed points of \mathbb{P}^1 under the torus action, and the corresponding singular fibers of the elliptic fibration.

We give a summary of Candelas' conditions in Section 3. The surjection condition (2) appears in all the Candelas' examples: it is a novel idea, somewhat hidden in Candelas' papers and central in Berglund–Mayr. We recast the conditions in a mathematical context in the following Section 4. In Section 4.1, we define the semistable polytopes, which play the role of building blocks in the examples in above mentioned papers. A semistable polytope will satisfy the Candelas' conditions (1) and (3) stated in Section 3. The particular class of polytopes Theorem 4.3 satisfies all the Candelas' conditions; in Section 4.3, we actually show that all the polytopes which satisfy the Candelas' conditions can be constructed from semistable polytopes. In Section 4.2, we show that condition (3) corresponds to the existence of a “section at ∞ ” of the elliptic fibration of the $K3$ surface. We show that the existence of the section at ∞ and condition (1) implies the existence of a *toric* Weierstrass model, which we call the *Candelas–Font Weierstrass* model (Section 4.3). We characterize these models in Theorem 4.12. Theorem 4.13 characterizes Candelas' examples. It was noted by [4, 11] that these Calabi–Yau varieties yield mirror elliptic fibrations.

In Section 5, we prove an easy combinatorial condition for the existence of a section of the elliptic fibration.

The symplectic cut is the focus of Section 6. We show that Candelas' conditions (1) and (2) correspond to the existence of a codimension one slice in the moment polytope, which cuts it into two “nice” parts (Theorem 6.2). We then show that these conditions give a “symplectic cut” which determines a toric semistable degeneration of the original Fano variety into two other semistable toric varieties; this degeneration induces a natural semistable degeneration of the Calabi–Yau hypersurface (Theorems 6.6 and 6.7). Remark 6.9 concludes that Candelas' conditions (1), (2), and (3), or equivalently Theorem 4.13, imply that the elliptically fibered $K3$ degenerates to two rational elliptic surfaces glued along a fiber and the degeneration preserves the elliptic fibration which induces a semistable degeneration of the section at infinity.

The idea of the symplectic cut can be applied also in higher dimensional fibrations; we leave this for further studies: we believe that [20] contains many useful techniques for this purpose.

2 Background and notations

2.1 Toric, Fano varieties

We follow the notation of [3, 15, 16]:

- $N, M \subset \mathbb{Z}^n$ are dual lattices with real extensions $N_{\mathbb{R}}, M_{\mathbb{R}}$; we denote by $\langle *, * \rangle : M \times N \rightarrow \mathbb{Z}$ the natural pairing; $T_N = N \otimes \mathbb{C}^*$ is the algebraic torus.
- $\Delta \subset M_{\mathbb{R}}$ is an integral polytope, that is, each vertex is in M ; the codimension 1 faces of Δ are called facets.
- We assume Δ is *reflexive*, that is, the equation of any facet F of Δ can be written as $\langle m, v \rangle = -1$, where $v \in N$ is a fixed integer point and $m \in F$; then the origin is the only integral interior point in Δ . The dual of $\Delta \subset N_{\mathbb{R}}$, defined as the set $\nabla \stackrel{\text{def}}{=} \{v \in N_{\mathbb{R}} \mid \langle m, v \rangle \geq -1 \text{ for all } m \in \Delta\}$, is also an integral reflexive polytope in $N_{\mathbb{R}}$.
- The normal fan of $\Delta \subset M_{\mathbb{R}}$ in N is the fan over the proper facets of $\nabla \subset N_{\mathbb{R}}$; since Δ is reflexive the rays of its normal fan are simply the vertices of ∇ ; let \mathbb{P}_{Δ} be the associated projective toric variety.
- Given a fan Σ in N , we denote as X_{Σ} the corresponding toric variety; when the meaning is clear we simply write X .

- $\Sigma^{(1)}$ is the set of all rays of Σ ; each ray $v_i \in \Sigma^{(1)}$ corresponds to an irreducible T_N -invariant Weil divisor $D_i \subset X_\Sigma$, the *toric* divisors.
- Δ is *reflexive* if and only if the projective toric variety \mathbb{P}_Δ is Fano. Recall that the dualizing sheaf on a compact toric variety X of dimension n is $\hat{\Omega}_V^n = \mathcal{O}_X(-\sum_i D_i)$, where the sum ranges over all the toric divisors D_i . The canonical divisor is $K_X = -\sum_i D_i$, and therefore \mathbb{P}_Δ is *Fano*, if and only if $\sum_i D_i$ is ample.
- A *projective subdivision* Σ is a refinement of the normal fan of Δ which is projective and simplicial, that is, the generators of each cone of Σ span $N_{\mathbb{R}}$. The associated toric variety X_Σ has then orbifold singularities. Σ is *maximal* if its cones are generated by all the lattice points of the facets of Σ .

Definition Theorem 2.1 (The Cox ring [14]). For each $v_i \in \Sigma^{(1)}$ introduce a variable x_i and consider the polynomial ring

$$S = \mathbb{C}[x_i : v_i \in \Sigma^{(1)}] = \mathbb{C}[x_1, \dots, x_r],$$

where $r = |\Sigma^{(1)}|$. S is graded by $A_{n-1}(X_\Sigma)$ and is called the *homogeneous (Cox) coordinate ring* of X_Σ . A monomial $\prod_i x_i^{a_i} \in S$ has degree $[D] \in A_{n-1}(X_\Sigma)$, where $D = \sum_i a_i D_i$.

Definition Theorem 2.2. For each cone $\sigma \subset \Sigma$ consider the monomial $x^{\hat{\sigma}} = \prod_{v_i \notin \sigma} x_i \in S$, and define the *exceptional set* associated to Σ as the algebraic set in \mathbb{C}^r defined by the vanishing of all of these monomials:

$$Z(\Sigma) = V(x^{\hat{\sigma}} : \sigma \in \Sigma) \subset \mathbb{C}^r.$$

Finally, define

$$G = \left\{ (\mu_1, \dots, \mu_r) \in (\mathbb{C}^*)^r \mid \prod_{i=1}^r \mu_i^{\langle e_1, v_i \rangle} = \dots = \prod_{i=1}^r \mu_i^{\langle e_n, v_i \rangle} = 1 \right\} \subset (\mathbb{C}^*)^r,$$

where $\{e_1, \dots, e_n\}$ is the standard basis in M . Then:

$$X_\Sigma \simeq (\mathbb{C}^r - Z(\Sigma))/G.$$

2.2 Calabi–Yau varieties and reflexive polytopes

V is a Calabi–Yau variety if $K_V \sim \mathcal{O}(V)$, $h^i(\mathcal{O}_V) = 0$, $0 < i < \dim V$. If V is an hypersurface in a toric variety, then the condition $h^i(\mathcal{O}_V) = 0$ is automatically satisfied.

In fact the lattice polytope Δ_D corresponds to the very ample Cartier divisor D which determines the embedding of X_Σ in some projective space; Σ is the normal fan to Δ_D . On the other hand, let $\mathbb{P}_\Delta := X_\Sigma$ be the toric variety associated to the normal fan Σ of Δ . Δ determines a Cartier divisor D_Δ and an ample line bundle \mathcal{L}_Δ ; its global sections (which provide the equations of the projective embedding) corresponds to the lattice points of $\nu\Delta$ as explained above.

Theorem 2.3 (Ch. 4 [15]). *If $\Delta \subset M_\mathbb{R} \simeq \mathbb{R}^n$ is a reflexive polytope of dimension n , then the general member $\bar{V} \in |-K_{\mathbb{P}_\Delta}|$ is a Calabi–Yau variety of dimension $n - 1$. If Σ is a projective subdivision of the normal fan of Δ , then*

- X_Σ is a Gorenstein orbifold with at worst canonical singularities.
- $-K_{X_\Sigma}$ is semiample and Δ is the polytope associated to $-K_{X_\Sigma}$.
- The general member $V \in |-K_{X_\Sigma}|$ is a Calabi–Yau orbifold with at worst canonical singularities.

In particular, in dimension three or lower the following are equivalent:

- (1) $\Sigma = \Sigma_{\max}$ is maximal.
- (2) Σ is given by a triangulation of the facets of ∇ into elementary triangles $v_{i_1}v_{i_2}v_{i_3}$ such that for all i , the vectors $v_{i_1}, v_{i_2}, v_{i_3}$ span the lattice N (equivalently, the convex hull of $\{v_{i_1}, v_{i_2}, v_{i_3}, \mathbf{0}\}$ is a tetrahedron with no lattice points other than its vertices).
- (3) $X_{\Sigma_{\max}}$ is smooth.

If $n = 3$, V_{\max} is a smooth $K3$ surface, while if $n = 2$, V_{\max} is an elliptic curve.

Remark 2.4. The defining equation of the Calabi–Yau hypersurface $V \in |-K_{X_\Sigma}|$, with X_Σ toric Fano, can be written explicitly. With the above notation, let z_1, \dots, z_k be the homogeneous coordinates of X_Σ . V is defined by the vanishing of the generic polynomial whose monomials are the sections of the line bundle $\mathcal{O}_{X_\Sigma}(-K_{X_\Sigma})$; then V has equation

$$\sum_{m \in \Delta \cap M} a_m \prod_{i=1}^k z_i^{\langle m, v_i \rangle + 1} = 0,$$

where the a_m s are generic complex coefficients. The defining equation of V is invariant modulo the action of $SL(3, \mathbb{Z})$ on M . These equations are easily implemented in the computer algebra system SAGE [35, 41] which has a dedicated package for working with reflexive polytopes.

Batyrev [2], Koelman [24] and then [26] independently classified all the reflexive polytopes of dimension 2 up to $SL(2, \mathbb{Z})$ transformations, see figure 1. In dimension 3, which is the one relevant for $K3$ surfaces, the complete classification was carried out by Kreuzer and Skarke [26, 27] using the software package PALP [28]. There are 4319 reflexive polytopes in dimension 3. Their coordinates are stored in SAGE and can be found on the web page [25]. See also [22, 23]. Rohnsiepe's tables [38] contain a list of all 3D reflexive polytopes, in the same ordering as in SAGE, the only difference being that the first reflexive polytope in Rohnsiepe's tables is indexed by 1 whereas SAGE indexing starts from 0. Throughout this paper, we adopt the same indexing as in SAGE.

2.3 Intersection on toric $K3$ hypersurfaces

The following facts about the intersection on toric $K3$ hypersurfaces are known in physics literature [36].

Let Σ be a projective subdivision of the fan over the proper facets of a 3D reflexive polytope ∇ . For each ray v_i in Σ , let D'_i be the intersection of $D_i \subset X_\Sigma$ with the general $K3$ hypersurface $V \in |K_{X_\Sigma}|$. Three cases can occur: (1) D_i does not intersect V , i.e., $D'_i = 0$; (2) D'_i is irreducible on V : we call it a *toric* divisor; and (3) D'_i is the sum of irreducible divisors on V : we call its irreducible components *non-toric* divisors.

Let Σ be **maximal**. In this case: (1) $D'_i = 0$ if v_i is in the interior of a facet of ∇ ; (2) D'_i is toric if v_i is a vertex of ∇ ; and (3) D'_i is the sum of $l'(\theta^*) + 1$ non-toric divisor if v_i is in the interior of an edge θ of ∇ , where $l'(\theta^*)$ is the *lattice length* of the dual of θ (i.e., the number of lattice points in the interior of $\theta^* \subset \Delta$) [15]. Now let v_1, v_2 be two distinct rays in Σ . The intersection $D_1 \cdot D_2 \cdot V$ can be non-zero iff v_1, v_2 are in the same cone in Σ , that is there are two elementary triangles T, T' in the triangulation of the facets of ∇ that have the segment $v_1 v_2$ in common. Let v_3 be the third vertex of T and v_4 be the third vertex of T' , and denote as $m_{123} \in M$ the dual of the facet of ∇ carrying T . Then

Theorem 2.5 ([36]). $D_1 \cdot D_2 \cdot V = \langle m_{123}, v_4 \rangle + 1$. In particular, $D_1 \cdot D_2 \cdot V = 0$ if v_1 and v_2 are not neighbors along an edge of ∇ . If v_1 and v_2 are neighbors along an edge θ_{12} , then

$$D_1 \cdot D_2 \cdot V = l_{12} = l'(\theta_{12}^*) + 1,$$

where $l'(\theta_{12}^*)$ is the lattice length of θ_{12}^* .

2.4 Elliptic fibrations

The morphism $\pi_V : V \rightarrow B$ denotes an *elliptic fibration*, that is, $\pi_V^{-1}(p)$ is a smooth elliptic curve $\forall p \in B$, general. In addition, $\pi_V : V \rightarrow B$ is an *elliptic fibration with section* if there exists a morphism $\sigma_V : B \rightarrow V$ which composed with π_V is the identity; $\sigma_V(B)$ is a section of π .

$\phi : X_\Sigma \rightarrow B$ is a *toric fibration* if X_Σ (from here on denoted simply as X) and B are toric and if ϕ is induced by a lattice morphism between the corresponding lattices $\varphi : N_X \rightarrow N_B$ which is compatible with the fans. The kernel of φ is a sublattice $N_\varphi \subset N$.

We now assume that X is a Fano variety and that X_ϕ the general fiber of ϕ is a Fano surface; the restriction of the fan of X to N_φ defines the fan of X_ϕ . This fibration determines a 2D reflexive polytope $\nabla^\varphi \subset N_{\varphi, \mathbb{R}}$ corresponding to the toric variety X_ϕ ; let $E \in |-K_{X_\phi}|$ be a general element, a smooth elliptic curve.

Assumption 2.6. *We also assume that there is a section $\sigma : B \rightarrow V$ of $\pi : V \rightarrow B$ induced by a **toric section** $\sigma_X : B \rightarrow X$ of $\phi : X \rightarrow B$ such that $\sigma_X(B) = D$ is a toric divisor. The restriction of D to V can be either reducible or irreducible.*

Nakayama showed that an elliptic fibration $V \rightarrow B$ with section has a Weierstrass model in a precisely defined projective bundle [33]. In particular, when V is a $K3$ surface and $B = \mathbb{P}^1$, the projective bundle is $\mathbf{P} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \mathcal{O}_{\mathbb{P}^1}(-6))$. Every projective bundle is a toric variety whose corresponding fan can be computed by following the construction described by Oda, Section 1.7. However, it turns out that the toric variety \mathbf{P} is not [34] Fano [15]. For the elliptic $K3$ s which are hypersurfaces in Fano toric 3-folds, we would like their Weierstrass models to be hypersurfaces in a Fano toric 3-folds as well.

We focus on 3D toric varieties and elliptic $K3$ hypersurfaces. As noted in [27, 39], if a reflexive 3D polytope ∇ contains a reflexive subpolytope ∇^φ , then a suitable refinement of the normal fan of Δ always gives rise to a natural toric fibration $X \rightarrow \mathbb{P}^1$ with general fiber a toric Fano surface with reflexive polytope ∇^φ . This can be seen explicitly by describing the toric morphisms in homogeneous coordinates [35] and the appendix of the present paper. Other explicit examples of elliptic fibrations of toric $K3$ s in homogeneous coordinates can be found in [1]. Rohnsiepe searched all 4319 reflexive polytopes for toric elliptic fibrations [39], the results can be found in the tables [38].

3 Candelas' examples

We summarize here the main characteristic of the Candelas' examples of elliptic $K3$ F-theory models: recall that $Y \subset V$ is an anticanonical general surface in the toric Fano 3-fold X ; all the statements are up to a lattice automorphism $SL(3, \mathbb{Z})$.

- (1) The lattice $N_\varphi \subset N$ is a summand of N , with induced morphism of lattices: $N \rightarrow N_\varphi (z_1, z_2, z_3) \mapsto (z_1, z_2)$.
- (2) Under this morphism the lattice points of the reflexive polytope ∇ are sent onto points of the reflexive polytope ∇^φ .
- (3) There exists a vertex $v_z = (a, b, 0)$ of ∇^φ which is not a vertex of ∇ .

Note that conditions (2) and (3) imply that the edge L of ∇^φ through v_z is defined by the equation $z_1 = a, z_2 = b$. Condition (1) induces a split of the dual lattice M , with coordinates (z_1^*, z_2^*, z_3^*) ; $\Delta \cup \{z_3^* = 0\}$, Δ^φ , the dual of the polytope ∇^φ .

In Section 4, we show that condition (3) corresponds to the existence of a “section at ∞ ” of the elliptic fibration; in Section 6, we also assume that X is simplicial, which is also a case in the Candelas' examples, and show that conditions (1) and (2) imply the existence of the semistable degeneration of X and Y . We then discuss the case when all the conditions are satisfied.

Example 3.1 (Polytope 3737). Let ∇ be the polytope with vertices $v_s = (-1, -1, 1)$, $v_t = (-1, -1, -1)$, $v_a = (-2, 1, 1)$, and $v_b = (-2, 1, -1)$, $v_c = (-1, 1, 1)$, $v_d = (-1, 1, -1)$. The dual Δ is the *diamond* in the Example 3.3; ∇^φ is the 2D reflexive polytope number 15 given by points $v_x = (2, -1, 0)$, $v_y = (-1, 1, 0)$, and $v_z = (-1, -1, 0)$. Clearly all the conditions (1), (2) and (3) hold.

Example 3.2 (Polytope 4318 fibered by 9). Let ∇ be the polytope with vertices $v_x = (-1, 1, 0)$, $v_t = (2, -1, 0)$, $v_p = (-1, -1, -6)$, and $v_q = (-1, -1, 6)$, this is, the polytope 4318 in the list [25]. We consider the fiber ∇^φ given by points $v_x, v_y = (1, -1, -2)$, and $v_z = (1, -1, 2)$; ∇^φ is the 2D polytope number 9. Consider the projective subdivision of the normal fan to Δ given by the rays $v_x, v_y, v_z, v_s = (0, -1, 4), v_t, v_p, v_q$. ∇ satisfies the conditions (1) and (3) but not (2). Then this example cannot have a semistable degeneration as described in Section 6 and an Heterotic dual with gauge group $E_8 \times E_8$. In fact a singular fiber of the elliptic fibration is of Kodaira type I_{12}^* , which correspond to a “gauge group” of type $SO(32)$ (or better, $Spin(32/\mathbb{Z}/2\mathbb{Z})$). We will consider again this polytope in the Example 4.5 (with a different fibration).

Example 3.3 (Polytope 113: “Diamond” fibered by 15). Let ∇ be the reflexive polytope with vertices $v_x = (2, -1, 0)$, $v_y = (-1, 1, 0)$, $v_z = (-1, -1, 0)$, $v_s = (0, 0, 1)$, and $v_t = (0, 0, -1)$; ∇^φ has vertices v_x, v_y, v_z , and is the 2D polytope number 15. Conditions (1) and (2) are fulfilled while condition (3) does not hold.

Example 3.4 (Polytope 4: “Diamond” fibered by 1). Let ∇ be the reflexive polytope with vertices $v_x = (1, 0, 0)$, $v_y = (0, 1, 0)$, $v_z = (-1, -1, 0)$, $v_s = (0, 0, 1)$, and $v_t = (0, 0, -1)$; ∇^φ has vertices v_x, v_y, v_z and is the 2-dimensional polytope number 1, the fan of \mathbb{P}^2 . Conditions (1) and (2) are fulfilled and condition (3) does not hold. The general $K3$ (in the anticanonical system) does not have a section, this is in fact the hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ of degree $(3, 2)$.

4 Semistable polytopes, sections at infinity, Weierstass models (*Candelas’ conditions recasted*)

We start by defining the semistable polytopes, which play the role of building blocks in the Candelas’ examples: a semistable polytope will satisfy the Candelas’ conditions (1) and (3) stated in Section 3. The particular class of polytopes in Theorem 4.3 satisfies all the Candelas’ conditions; in Section 4.3, we actually show that all the polytopes, which satisfy the Candelas’ conditions can be constructed from semistable polytopes.

4.1 Semistable polytopes

(On conditions (1) and (3))

2D reflexive polytopes ∇^φ are classified up to $\mathrm{SL}(2, \mathbb{Z})$ (see, for example, [16]), and are listed in Figure 1. We denote by $\nabla^{i;\varphi}$ the i th 2D reflexive polytope in the figure, and by $\Delta^{d(i);\varphi}$ its dual. Then $d(1) = 6$, $d(6) = 1$, $d(2) = 7$, $d(7) = 2$, $d(3) = 8$, $d(8) = 3$, $d(4) = 9$, $d(9) = 4$, $d(5) = 10$, $d(10) = 5$, $d(11) = 16$, $d(16) = 11$, $d(12) = 12$, $d(13) = 13$, $d(14) = 14$, and $d(15) = 15$.

Definition 4.1. Fix a vertex v_z of a 2D reflexive polytope $\nabla^\varphi \subset N_{\varphi, \mathbb{R}} \subset N_{\mathbb{R}}$ such that $N_\varphi \subset N$. Denote by v_z both the point in N and the unit vector on a ray Σ_φ ; let L be a segment of lattice length 2 centered at v_z such that the vertices v_s and v_t of L together with the vertices of ∇^φ generate $N_{\mathbb{R}}$. Assume that the lattice polytope $\nabla^s \subset N_{\mathbb{R}}$, spanned by the vertices of ∇^φ and L is reflexive: ∇^s is a **semistable polytope** with fiber ∇^φ .

Notation 4.2. We denote a semistable polytope with $\nabla^{i, v_z, L; s}$ and its dual with $\Delta^{i, v_z, L; s}$; here s stands for *semistable*. The fan over $\nabla^{i, v_z, L; s}$ has

simplicial cones over the vertices of ∇^φ , v_s and v_t and it determines a simplicial Fano toric variety $X^{i,v_z,L;s}$ together with a morphism $X^{i,v_z,L;s} \rightarrow \mathbb{P}^1$. For ease of notation we drop one or more apices among i, v_z, L when the meaning is clear from the context.

The general hypersurface $V \subset X^s$ in $|-K_{X^s}|$ is an elliptically fibered $K3$ surface.

Theorem 4.3 (Existence and uniqueness). *Assume that the vertices $\{v_s, v_t\}$ together with the vertices of ∇^φ generate the lattice N . Then the convex hull of these vertices is a reflexive polytope and is a semistable polytope ∇^s . Moreover the fiber of the elliptic fibration of a very general³ $K3$ surface $V_{\max} \rightarrow \mathbb{P}^1$ over the fixed points of \mathbb{P}^1 are smooth, while the other singular fibers are semistable. This polytope is unique up to $\mathrm{SL}(3, \mathbb{Z})$.*

Proof. Let v_x, v_y, v_z, v_{w_j} be the vertices of ∇^φ (depending on ∇^φ either there is no v_{w_j} or $j \in \{1, 2, 3\}$); without loss of generality we take $v_z = (z_1, z_2, 0)$ and $v_s = (z_1, z_2, 1)$ and $v_t = (z_1, z_2, -1)$. Let us consider the fan Σ_{\max} of the maximal resolution of ∇^s ; by construction the rays of this fan are either v_s, v_t or are also rays of ∇^φ . Note that the semistable polytope is simplicial and then the very general $K3$ surface $V_{\max} \subset X_{\max}$ has the same Picard number of the ambient Fano 3-fold X_{\max} [8]. Then the singular fibers of the elliptic fibration are either nodes or restrictions of toric divisors corresponding to points of ∇^φ which are reducible when restricted to the $K3$. From Section 2.3, we see that for each edge e in ∇^φ which is also an edge of ∇ there are r semistable fibers I_{q+1} in V_S , where q and r are the lattice lengths of e and its dual in ∇ , respectively. This implies the first statement.

For each edge e of ∇^φ let $\langle m_e, v \rangle + 1 = 0$ be its equation; $m \in \mathbb{Z}^2$ because ∇^φ is reflexive. If e is one of the two edges originating from v_z , then the equation of the vertical facet defined by e and L is $\langle m, v \rangle + 1 = 0$, where $m = (m_e, 0) \in \mathbb{Z}^3$. If e does not pass through v_z , the equation of the facet through e and v_s is $\langle m, v \rangle + 1 = 0$ where $m = (m_e, -(az_1 + bz_2 + 1)) \in \mathbb{Z}^3$ (and similarly for the facets through e and v_t). It follows that ∇ is reflexive. \square

The condition that $\{L, \nabla^\varphi\}$ generates N does not hold for all the semistable polytopes, see Example 4.6. However, this condition is fulfilled when L is centered in a vertex of ∇^φ which satisfies a nice combinatorial property:

³A property is said to be *very general* if it holds in the complement of a countable union of subschemes of positive codimension [29].

Proposition 4.4. *Let $\nabla^s \subset N_{\mathbb{R}}$ be a semistable polytope and L its edge of lattice length 2 centered in a vertex v_z of $\nabla^\varphi \subset N_{\varphi, \mathbb{R}}$. Let v_1, v_2 be the two lattice neighbors of v_z along the two edges of ∇^φ through v_z . If $v_z = v_1 + v_2$ then $\{L, \nabla^\varphi\}$ generates N .*

The condition $v_z = v_1 + v_2$ is not always satisfied: the vertices fulfilling this condition are marked with a square in figure 1; see also Section 5. Compare the above Proposition 4.4 with Theorem 5.1.

Proof. Suppose $v_1 = (a_1, a_2, 0), v_2 = (b_1, b_2, 0)$ and let $v_s = (\alpha, \beta, \gamma)$ be the vertex of L with $\gamma > 0$. We show $\gamma = 1$ by proving that if $\gamma > 1$ then there is a lattice point in the interior of the segment $v_z v_s$.

It can be easily checked that for the vertices v_z s.t. $v_z = v_1 + v_2$ (denoted by a square in figure 1) we always have $d := a_1 b_2 - a_2 b_1 = \pm 1$. The facet of ∇^s through v_z, v_s, v_1 has equation $\langle m_{zs1}, v \rangle + 1 = 0$, where

$$m_{zs1} = \left(-b_2 d^{-1}, b_1 d^{-1}, \frac{-b_1 \beta d^{-1} + b_2 \alpha d^{-1} - 1}{\gamma} \right) \in M.$$

Similarly, the facet through v_z, v_s, v_2 has equation $\langle m_{zs2}, v \rangle + 1 = 0$, where

$$m_{zs2} = \left(a_2 d^{-1}, -a_1 d^{-1}, \frac{a_1 \beta d^{-1} - a_2 \alpha d^{-1} - 1}{\gamma} \right) \in M.$$

It follows that

$$\begin{cases} -b_1 \beta + b_2 \alpha - d & \equiv 0 \pmod{\gamma}, \\ a_1 \beta - a_2 \alpha - d & \equiv 0 \pmod{\gamma} \end{cases}$$

because ∇^s is reflexive. By solving the system and because $d = \pm 1$, we obtain:

$$\begin{cases} \alpha & \equiv a_1 + b_1 + 0 \pmod{\gamma}, \\ \beta & \equiv a_2 + b_2 + 0 \pmod{\gamma} \end{cases} \tag{4.1}$$

Let $\langle m_e, v \rangle + 1 = 0$, with $m_e = (A, B) \in M_\varphi$, be the equation in the plane N_φ of an edge e of ∇^φ not passing trough v_z . The facet of ∇^s through v_s

and e has equation $\langle m_{se}, v \rangle + 1 = 0$, where

$$m_{se} = \left(A, B, -\frac{A\alpha + B\beta + 1}{\gamma} \right) \in M.$$

Because ∇^s is reflexive, we have $A\alpha + B\beta + 1 \equiv 0 \pmod{\gamma}$. From equations (4.1), it follows that $A(a_1 + b_1) + B(a_2 + b_2) + 1 \equiv 0 \pmod{\gamma}$, where $A(a_1 + b_1) + B(a_2 + b_2) + 1 \geq 2$ because e does not pass through v_z . In particular, γ is a divisor of an integer > 1 ; suppose $\gamma \geq 2$. Given an integer p such that $0 < p < \gamma$, we have $\lambda_p := p\gamma^{-1} \in (0, 1)$ and $\gamma\lambda_p \in \mathbb{Z}$. We obtain a contradiction by observing that the point $\lambda_p v_s + (1 - \lambda_p)v_z$ in the segment $v_s v_z$ is a lattice point because of Eqs. (4.1). \square

Example 4.5 ([40] (a semistable polytope)). Let ∇ be the reflexive polytope with vertices $v_x = (2, -1, 0)$, $v_y = (-1, 1, 0)$, $v_s = (-1, -1, 1)$, and $v_t = (-1, -1, -1)$ (polytope 88 in the list by Kreuzer and Skarke [25]). In this case $\nabla^\varphi = \nabla^{15;\varphi}$ with vertices v_x, v_y and $v_z = (-1, -1, 0)$; the corresponding toric variety is the weighted projective space $\mathbb{P}^{(2,3,1)}$ with homogeneous coordinates (x, y, z) . We take L to be the edge v_s, v_t of lattice length 2. Clearly $\{\nabla^\varphi, L\}$ generates N , and therefore ∇ is a semistable polytope; we will see in Proposition 4.4 that this is the only semistable polytope $\nabla^{15,v_z;s}$ with marked section corresponding to $v_z = (-1, -1, 0)$. The monomials of the equation of the general K3 surface V (in the anticanonical system) are given by

$$x^3, y^2, a_{12}z^6, a_8xz^4, a_4x^2z^2, a_6yz^3, a_2xyz,$$

where a_i is a general polynomial in s, t of degree i , for $i = 2, 4, 6, 8, 12$. Applying toric automorphisms we obtain the following equation of V .

$$y^2 = x^3 + a(s, t)xz^4 + b(s, t)z^6,$$

where a, b are the generic polynomials of degree 8 and 12, respectively. The discriminant $\delta = 4a^3 + 27b^2$ has degree 24 and thus it vanishes in 24 points in \mathbb{P}^1 . In each of those, the orders of vanishing are $(o(a), o(b), o(\delta)) = (0, 0, 1)$ and thus there are 24 semistable fibers I_1 . It is easy to verify that $z = 0$ is an irreducible section of the fibration.

Example 4.6 (a semistable polytope such that $\{L, \nabla^\varphi\}$ does not generate N). Let ∇ be the reflexive polytope with vertices $v_x = (-1, 1, 0)$, $v_y = (-1, -1, 0)$, $v_s = (1, -1, 2)$, $v_t = (3, -1, -2)$ (polytope 1943) and $v_z = (2, -1, 0)$. We have $\nabla^\varphi = \nabla^{15;\varphi}$ with vertices v_x, v_y and $v_z = (2, -1, 0)$ (note that, with respect to the polytope in the previous example, we changed the names of the coordinates associated to the vertices of $\nabla^{15;\varphi}$). The edge

$L = v_s v_t$ has lattice length 2. The condition is not satisfied: for each point $v \in \nabla^{15;\varphi}$ the matrix (v, v_z, v_s) is not in $SL(3, \mathbb{Z})$. The monomials in the equation of V are

$$y^6, x^2, s^6 z^3, s^4 t^2 z^3, s^2 t^4 z^3, t^6 z^3, s^4 y^2 z^2, s^2 t^2 y^2 z^2, t^4 y^2 z^2, s^2 y^4 z, t^2 y^4 z, xyz.$$

Note that the semistable polytope $\nabla^{15, v_z, L'; s}$ with L' the segment of vertices $v_t = (2, -1, -1)$ and $v_s = (2, -1, 1)$ is also reflexive.

4.2 Infinity sections, toric flexes

(On condition (3))

In both Examples 4.5 and 4.6, D_z , namely $z = 0$, defines the equation of a section of the elliptic fibration $V \rightarrow \mathbb{P}^1$, which is the restriction of the toric section determined by the divisor D_z in X ; this section is the same for all the $K3$ hypersurfaces in the same anticanonical system. From now on we assume that the elliptic fibration has a toric section represented by the divisor D_z .

In analogy with the classical Weierstrass model, and following the above notation, let $\{x, y, z, w_j\}$ be the Cox coordinates corresponding to the vertices of ∇^φ (depending on ∇^φ either there is no v_{w_j} or $j \in \{1, 2, 3\}$), with $z = 0$ be the defining equation of the toric section D_z ; let s, t, r_k be the Cox coordinates corresponding to the remaining vertices of ∇ , with v_s and v_t be the unit lattice points on the edges through v_z , where $\phi(v_s)$ and $\phi(v_t)$ span two different cones of the fan of \mathbb{P}^1 . This will assure the fibration is easily described in terms of the homogeneous coordinates; see Remark 2.4. In many cases, this gives a projective resolution of the normal fan. Let $g(x, y, z, w_j) = 0$ be the equation of E in X_ϕ and $f(x, y, z, w_j, s, t, r_k) = 0$ be the equation of V in X . We often write $f(x, y, z, w_j, s, t, r_k) = G(x, y, z, w_j)$ with $G \in \mathbb{C}[s, t, r_k]$, in the form of the equation of the general elliptic curve in X_ϕ .

Proposition 4.7 (Condition 3). $f_{|z=0}$ does not depend on the coordinates (s, t) if and only if v_z is an interior point of an edge of ∇ with vertices v_s and v_t .

Proof. The non-zero monomials in the polynomial $f_{|z=0}$ are of the form

$$\{s^{\langle m, v_s \rangle + 1} t^{\langle m, v_t \rangle + 1} \prod_k r_k^{\langle m, v_{r_k} \rangle + 1}\} \cdot x^{\langle m, v_x \rangle + 1} y^{\langle m, v_y \rangle + 1} \prod_j w_j^{\langle m, v_{w_j} \rangle + 1}, \quad (4.2)$$

where $m \in M$ satisfies the equation $\langle m, v_z \rangle + 1 = 0$. v_z and the vertices v_s and v_t are collinear if and only if $\langle m, v_s \rangle + 1 = \langle m, v_t \rangle + 1 = 0$, that is if and only if $f|_{z=0}$. □

Definition 4.8. D_z is a *section at infinity* if and only if $f|_{z=0}$ is independent of the particular point in \mathbb{P}^1 ; explicitly there is no dependence in (s, t) .

$f|_{z=0}$ is a *toric flex* if the equation $z = 0$ determines one unique point of the general elliptic curve E of the fibration.

If ∇^s is a semistable polytope, then the toric flex corresponding to the point v_z as in Proposition 4.7 is the analog of a section at infinity. We can see explicitly the morphisms and the defining equations of the $K3$ surfaces in Cox coordinates following [7, 31].

Example 4.9. In Example 3.3, the equation of the general $K3$ hypersurface (in the anticanonical system) is

$$\phi_0 x^3 + \phi_1 x y z + \phi_2 z^6 + \phi_3 y^2 + \phi_4 x z^4 + \phi_5 x^2 z^2 + \phi_6 y z^3 = 0$$

with each $\phi_j(s, t)$ a generic polynomial of degree 2 in (s, t) . It is easy to see that $z = 0$ is a section, but $f|_{z=0}$ is not a section at infinity. The same holds for the other sections coming from toric divisors corresponding to $v_x = 0$ and $v_y = 0$.

Remark 4.10. Under these hypothesis we denote by s, t the two corresponding coordinates.

We further investigate the type of sections of the elliptic fibrations in Section 5.

4.3 Candelas–Fonts Weierstrass models

(On conditions (1), (2), and (3))

Candelas and Font [11] fix a 2D reflexive polytope $(\nabla^{i;\varphi})$, a toric Fano B (in particular $B = \mathbb{P}^1$) and proceed, by what it is called “un-Higgsing” in the physics literature, to give examples of elliptic fibrations with interesting singular fibers (“gauge groups”) of Calabi–Yau varieties in toric Fano, fibered over B where the general elliptic curve of the fibration is the general elliptic curve in the anticanonical system of the fan over the fixed polytope $\nabla^{i;\varphi}$. We recast their construction in terms of elliptic fibrations with section at infinity and the semistable polytope.

Definition 4.11. A *Candelas–Font Weierstrass model* $W \rightarrow \mathbb{P}^1$ is an elliptically fibered $K3$ with orbifold Gorenstein singularities, not necessarily general in the anticanonical system, in a variety $X^{i,v_z,L;s}$ with general fiber $E \subset X_{i;\phi}$ and a section at infinity in D'_z (the toric divisor corresponding to v_z).

Note that these singularities are canonical; general anticanonical hypersurfaces in the projective resolution of Fano varieties have orbifold Gorenstein singularities.

Next, we prove a sufficient condition for an elliptically fibered general $K3$ with general fiber $E \subset X_\phi$ to have a Candelas–Font Weierstrass model and we express the condition in term of the combinatorics of the polytope as well as the geometry.

Theorem 4.12 (Candelas–Font Weierstrass models (conditions (1) and (3))). *Let $\nabla^{i;\varphi} \subset N_{\varphi,\mathbb{R}}$ be a 2D polytope and v_z a vertex of $\nabla^{i;\varphi}$ such that there exists a semistable polytope $\nabla^{i,v_z,L;s} \subset N_{\mathbb{R}}$. A general elliptically fibered $K3$ hypersurface V in the anti-canonical of a toric Fano 3-fold X_Σ with section at infinity in the edge L and general fiber $E \subset X_{i;\phi}$ is birationally equivalent to a Candelas–Font Weierstrass model $W \subset X^{i,v_z,L;s}$.*

Proof. By hypothesis the polytope ∇ over the fan of X_Σ contains the polytope $\nabla^{i;s} := \nabla^{i,v_z,L;s}$, hence we have the dual inclusion $\Delta \subset \Delta^{i;s}$. Δ defines a linear subsystem $\mathcal{L} \subset |-K_{X^{i;s}}|$; the resolution of the interminancy locus provides a birational morphism $\mathbb{P}_\Delta \rightarrow X^{i;s}$. \bar{V} , the general hypersurface in \mathcal{L} , is the strict transform of the general hypersurface W in $|-K_{X^{i;s}}|$. This induces a birational morphism between the pullback projective resolution, that is $V \rightarrow W$. □

Theorem 4.13 (conditions (1), (2), and (3)). *Assume that a Newton (moment) polytope is a reflexive subpolytope $\Delta \subset \Delta^{i;s}$ which contains $\Delta^{i;\varphi}$ the dual of $\nabla^{i;\varphi} \subset \nabla^{i;s}$. Then also the vice versa of Theorem 4.12 holds, that is the dual of Δ , ∇ is contained in $\nabla^{X^{i;s}}$.*

Furthermore, the projective resolution of the corresponding $K3$ has a section at infinity.

The condition that ∇ projects onto $\nabla^{i;\varphi}$ together with the existence of a section at infinity characterizes the examples in Candelas’ algorithm.

Proof. It is enough to observe that $\nabla^{i;s} \subset \nabla$ and that ∇ projects onto $\nabla^{i;\varphi}$. □

The transformations can be written explicitly in Cox coordinates, as in [7, 31]:

Example 4.14. Example 3.1: we obtain the projective subdivision of the normal fan to Δ obtained adding the rays $v_x, v_y,$ and v_z . The monomials defining the generic $K3$ in the corresponding toric variety X are

$$x^3a^3b^3, xyzb^2d^2t^2, xyza^2c^2s^2, xyzabcdst, z^6s^6t^6, y^2c^2d^2, xz^4abs^4t^4, \\ x^2z^2a^2b^2s^2t^2, yz^3cds^3t^3.$$

Let L be the edge between the vertices v_a and v_b and consider the semistable polytope $\nabla^{15, v_x, L; s}$. The polytopes satisfy the hypothesis of Theorems 4.12 and 4.13; the equation of the Candelas Weierstrass model W has monomials

$$x^3, y^2, a_{12}z^6, a_8xz^4, a_4x^2z^2, a_6yz^3, a_2xyz,$$

where a_i is a general polynomial in s, t of degree i , for $i = 2, 4, 6, 8, 12$.

Examples 3.2 and 4.6 satisfy the hypothesis of Theorem 4.12 and it is likewise possible to write the transformation to their respective Candelas–Font Weierstrass models.

It is then easy to write the equations of the discriminant of the elliptic fibrations.

Theorem 4.12 provides a criterion for constructing examples $E_8 \times E_8$ F-theory/Heterotic duality in the toric context. In can be verified that all the “gauge groups”, that is all the singular fibers of the elliptic fibrations, are subgroups of $E_8 \times E_8$.

Braun in [5] builds toric Weierstrass models by contracting some toric divisors associated to ∇^φ , and thus changing the elliptic fiber; in view of Section 6, we instead keep the basis fixed. Also [5] considers elliptic Calabi–Yau 3-folds in certain points of the moduli, while we consider an appropriate embedding of the elliptic fibration which makes it general as hypersurface in anticanonical system of the toric ambient space: this makes the resolution process straightforward as it is induced by the resolution of the toric ambient space. It would be interesting to combine the methods.

5 Toric and non-toric sections

We discuss combinatorial conditions for the existence of sections of the elliptic fibration. Let $\nabla \subset N_{\mathbb{R}}$ be a 3D reflexive polytope containing a 2D reflexive polytope ∇^φ and v_z be a vertex of ∇^φ . The equation of $N_{\varphi, \mathbb{R}}$ in $N_{\mathbb{R}}$ is $\langle m_\varphi, v \rangle = 0$, without loss of generality we take $m_\varphi = (0, 0, 1)$. Consider the maximal projective subdivision Σ_{\max} of the fan over the proper faces of ∇ and let V_{\max} be the smooth $K3$ hypersurface in the corresponding smooth Fano toric variety X_{\max} . Let v_s be a lattice point in ∇ at lattice distance one from v_z along an edge of ∇ through v_z not in $N_{\varphi, \mathbb{R}}$. At last, let D'_z, D'_s be the intersection of $D_z, D_s \subset X_{\max}$ with V_{\max} . It can be shown that the fiber of the elliptic fibration $V_{\max} \rightarrow \mathbb{P}^1$ is linearly equivalent to $\sum_{v_i \in \nabla_{\text{top}}} \langle m_\varphi, v_i \rangle D'_i$, where $\nabla_{\text{top}} = \{v \in \nabla \mid \langle m_\varphi, v \rangle > 0\}$ [36].

Theorem 5.1. *Let v_1, v_2 be the two lattice neighbors of v_z along the two edges of ∇^φ through v_z . Suppose $v_z v_1 v_s$ and $v_z v_2 v_s$ are elementary triangles in the maximal triangulation corresponding to Σ_{\max} , D'_z, D'_s are irreducible and ∇ is simple. If $v_z = v_1 + v_2$ then D'_z is a (toric) section of the elliptic fibration; moreover the converse is also true.*

The hypothesis on D'_s is what happens in the cases considered in the previous sections. In fact:

Remark 5.2. If v_s is a vertex of the polytope ∇ , then D'_s is irreducible. This is the case of the basic semistable models. If v_s is not a vertex of ∇ and v_z is on the interior of the same edge, then D'_z is irreducible if and only if D'_s is.

See also Proposition 4.4.

Remark 5.3. The theorem has the hypothesis that ∇ is simple, in particular it is sufficient to ask ∇ simple at v_z . All the reflexive polytopes (which induce elliptic fibrations) we examined in order to prepare this paper satisfy this condition.

Proof. We take $v_1 = (a_1, a_2, 0), v_2 = (b_1, b_2, 0), v_z = (z_1, z_2, 0), v_s = (\alpha, \beta, \gamma) \in \nabla_{\text{top}}$ (i.e., $\gamma > 0$). Moreover $v_z = \lambda v_1 + \mu v_2$, for $\lambda, \mu \neq 0$.

D'_z is section iff $D'_z \cdot \sum_{v_i \in \nabla_{\text{top}}} \langle m_\varphi, v_i \rangle D'_i = 1$. Because ∇ is simple at v_z , the only lattice point in ∇ at lattice distance one along an edge of ∇_{top} through v_z is v_s . We can distribute the intersection over the sum and observe that by the discussion in Section 2.3 all the summands but $\langle m_\varphi, v_s \rangle D'_z \cdot D'_s = \gamma D'_z \cdot D'_s$ are null. Therefore D'_z is a section iff $\gamma D'_z \cdot D'_s = 1$.

Because $v_z v_1 v_s, v_z v_2 v_s$ are elementary triangles, it is straightforward to verify that $\gamma\lambda d = \pm 1$ and $\gamma\mu d = \pm 1$, where $d = a_1 b_2 - a_2 b_1$. Moreover, by Theorem 2.5:

$$D'_z \cdot D'_s = \langle m_{zs1}, v_2 \rangle + 1 = \langle m_{zs2}, v_1 \rangle + 1,$$

where $m_{zs1}, m_{zs2} \in M$ are the dual points of the facets of ∇ span by v_z, v_1, v_s and v_z, v_2, v_s , respectively.

$$\langle m_{zs1}, v_2 \rangle \propto \begin{vmatrix} b_1 & \alpha - z_1 & a_1 - z_1 \\ b_2 & \beta - z_2 & a_2 - z_2 \\ 0 & \gamma & 0 \end{vmatrix} = \begin{vmatrix} b_1 & \alpha - \lambda a_1 - \mu b_1 & (1 - \lambda)a_1 - \mu b_1 \\ b_2 & \beta - \lambda a_2 - \mu b_2 & (1 - \lambda)a_2 - \mu b_2 \\ 0 & \gamma & 0 \end{vmatrix}.$$

Therefore, $\langle m_{zs1}, v_2 \rangle = 0$ iff

$$\begin{vmatrix} b_1 & (1 - \lambda)a_1 \\ b_2 & (1 - \lambda)a_2 \end{vmatrix} = 0,$$

and since v_1 and v_2 are linearly independent this is the case if and only if $\lambda = 1$. A similar argument shows that $\langle m_{zs2}, v_1 \rangle = 0$ iff $\mu = 1$.

On the one hand, if $v_z = v_1 + v_2$ then we have $d := a_1 b_2 - a_2 b_1 = \pm 1$ (as we already observed in the proof of Proposition 4.4) and $\lambda = \mu = 1$. Therefore $\langle m_{zs1}, v_2 \rangle = \langle m_{zs2}, v_1 \rangle = 0$, $\gamma = 1$ and therefore D'_z is a section. On the other hand, if D'_z is a section, then $\lambda = \mu = \gamma = 1$ and in particular $v_z = v_1 + v_2$. □

6 Symplectic cut, degenerations, physics duality

In this section, we assume that the polytope Δ is simple, that is, the normal toric variety \mathbb{P}_Δ is simplicial [16]; this is the case in all the Candelas' examples.

Lemma 6.1. *If $\Delta \subset M_{\mathbb{R}}$ is a polytope associated to a toric Fano 3-fold \mathbb{P}_Δ satisfying the condition (1) of Definition 3, then the following are equivalent:*

- i. (2) holds.*
- ii. If D is a facet, a codimension 1 face in ∇ , with inner normal vector $\nu_D = (w_1, \dots, w_n)$, then: $w_n > 0$ (resp. $w_n < 0$) if and only if D lies entirely in the half-space $N_{\leq 0} = \{(z_1, \dots, z_n) \in N \text{ such that } z_n \leq 0\}$ (resp. $N_{\geq 0}$).*

Proof. $i. \iff ii.$: If $n = 2$, then the statement is immediate, as ∇ is convex. Otherwise, let us consider the plane passing through the z_n axis and parallel to ν_D , and let D_2 be the intersection of D with such a plane. We can then reduce to the case $n = 2$. \square

In [19], Hu shows that suitable partitions of *simple* polytopes $\Delta \subset M_{\mathbb{R}}$ induces a semistable (or weakly semistable) degeneration of the toric variety associated to the polytope.

Theorem 6.2. *Let $\Delta \subset M_{\mathbb{R}}$ be the polytope associated to a toric Fano 3-fold X satisfying conditions (1) and (2) of Definition 3. We also assume that Δ is simple, that is, X is simplicial. Then the polytope $\Delta^\varphi \subset \Delta$, dual of ∇^φ , determines a symplectic cut, a simple, semistable partition of Δ [19].*

Proof. Δ^φ divides the polytope Δ in two polytopes Δ_1 and Δ_2 . Lemma 6.1 shows that each Δ_j , $j = 1, 2$, is simple, that is, the partition is simple. We need to verify that the conditions stated in [19] are satisfied, namely that any ℓ -face of Δ_j , $\ell = 1, 2$, is contained in exactly $k - \ell + 1$ polytopes Δ_j if there is a k -face of Δ containing it. This follows from a straightforward verification. \square

Lemma 6.3. *Let Δ be a polytope as in 6.2. Then the semistable partition determined by Δ^φ is also balanced, in the sense of [19].*

Proof. By construction all the vertices of Δ_j which are not vertices of Δ lie on an edge of Δ , which makes the subdivision balanced. Note that these vertices are the vertices of Δ^φ which are not vertices of Δ . \square

Definition 6.4 ([19]). The semistable, balanced subdivision of Δ determined by Δ^φ is mildly singular if the vertices of Δ_j which are not vertices of Δ are non-singular in each Δ_j , that is the primitive vectors at such vertex span the lattice M (over \mathbb{Z}).

Theorem 6.5 (Theorem 3.5 [19]). *Let $\{\Delta_j\}$, $j = 1, 2$ be a mildly singular semistable partition of Δ : then there exists a weak semistable degeneration of \mathbb{P}_Δ , $f : \tilde{\mathbb{P}}_\Delta \rightarrow \mathbb{C}$ with central fiber $\mathbb{P}_{\Delta,0} = \cup_j \mathbb{P}_{\Delta_j}$. The central fiber is completely described by the polytope partition $\{\Delta_j\}$ and $\mathbb{P}_{\Delta_1} \cap \mathbb{P}_{\Delta_2} = \mathbb{P}_{\Delta^\varphi}$.*

The following corollary follows from Lemma 6.1:

Theorem 6.6. *Let $\Delta \subset M_{\mathbb{R}}$ be the polytope associated to a toric Fano 3-fold X satisfying conditions (1) and (2) of Definition 3. We also assume that Δ is simple and that $\Delta^\varphi \subset \Delta$, the dual of ∇^φ , determines a simple, mildly singular semistable partition of Δ [19]. Let $f : \tilde{\mathbb{P}}_\Delta \rightarrow \mathbb{C}$ be the induced weak semistable degeneration. The rays of the toric fan of \mathbb{P}_{Δ_1} are the rays of \mathbb{P}_Δ*

with $z_3 \geq 0$ together with the ray $z_1 = z_2 = 0, z_3 \geq 0$; similarly for the rays of the toric fan of \mathbb{P}_{Δ_2} (with the ray $z_1 = z_2 = 0, z_3 \leq 0$).

The degeneration of Theorem 6.5 induces a degeneration of the general hypersurface $\bar{V} \subset \mathbb{P}_{\Delta}$, Section 4 [19]; let \mathcal{L}_j be the (ample) line bundle on \mathbb{P}_{Δ_j} associated to Δ_j and \bar{S}_1 and \bar{S}_2 be the associated hypersurfaces. If all the vertices of ∇^φ are also vertices of ∇ , then \mathbb{P}_{Δ} , \mathbb{P}_{Δ_1} and \mathbb{P}_{Δ_2} have a fibration with general fiber $\mathbb{P}_{\Delta^\varphi}$ and the degeneration preserves the fibration. We have proved the following:

Theorem 6.7. *Theorem 6.5 induces a weakly semistable degeneration of the general hypersurface \bar{V} to $\bar{S}_1 \cup \bar{S}_2$; in addition $\bar{S}_1 \cap \bar{S}_2$ is the general elliptic curve in Δ^φ . The construction of the degeneration shows that there is a naturally induced semistable degeneration of the maximal resolution V to S_1 and S_2 .*

In addition:

Proposition 6.8. *The surface $\bar{S}_j \in |\mathcal{L}_{Z_j}|$ is a rational elliptic surface, where \mathcal{L}_{Z_j} is the line bundle determined by the polytope Δ_{Z_j} .*

Proof. Note in fact that $-K_{Z_1} = \sum_{v_k \in \Sigma(1)} v_k$ and that $\mathcal{L}_{Z_1} = + \sum_{v_k \in \Sigma(1)} v_k - e_3$. Hence, $K_{S_1} = -e_{3|S_1}$. Note also that $e_3 = -K_{S_1}$ is the divisor of a general fiber of $\pi_1 : Z_1 \rightarrow \mathbb{P}^1$. □

If a section at infinity exists $S_j \rightarrow \bar{S}_j$ is not a crepant resolution; the exceptional curve is a section of the elliptic fibration.

Remark 6.9. Condition (3) assures that all the $K3$ hypersurfaces in the anticanonical of the toric Fano have the same infinity in section: this is consistent with the F-theory/Heterotic duality, which is for families of $K3$.

Example 6.10. Examples 3.1, 3.3, 4.5, and 3.4 satisfy the hypothesis of the above theorems. Note that the general $K3$ hypersurface in the “diamond” 3.4 $X \rightarrow \mathbb{P}^1$ with fiber $E \subset \mathbb{P}^2$ does not have a section; but it has a semistable degeneration induced by the symplectic cut. In this case, all the vertices of ∇^φ are also vertices of ∇ .

Note that the Examples 3.4 and 3.3, which admit a semistable degeneration as hypersurfaces in toric Fano, do not satisfy condition (3) in the Candelas’ conditions. In fact, the symplectic cut construction is more general, and holds for more examples, with no condition on the existence of a section.

Example 6.11. Example 3.2 does not satisfy condition (2) and it does not have a semistable degeneration in the sense of the above theorems.

Appendix A. Equations of elliptic curves in toric Del Pezzo surfaces

Figure 1 depicts all 2D reflexive polytopes $\nabla^{i;\varphi}$ up to $SL(2, \mathbb{Z})$ transformations. In this figure, an arrow denotes a pair of dual polytopes, if there is no arrow, the polytope is autodual. Let $\Sigma_{i;\varphi}$ be the normal fan of $\Delta^{i;\varphi}$ and $X_{i;\varphi}$

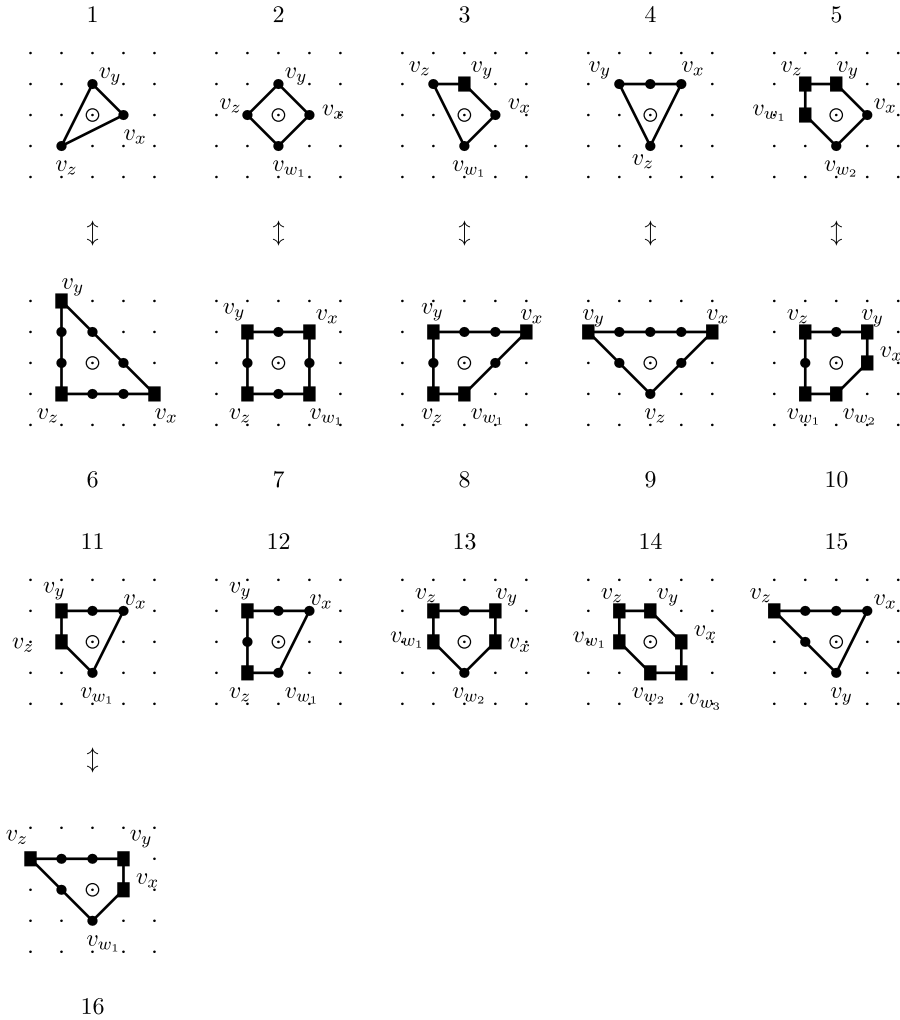


Figure 1: Reflexive polytopes in the plane.

the corresponding (Del Pezzo) toric variety: $X_{i;\phi} = (\mathbb{C}^r - Z(\Sigma_{i;\varphi}))/G_{i;\varphi}$, where r is the number of vertices of $\nabla^{i;\varphi}$. For each i we compute the equation of the generic elliptic curve embedded in $X_{i;\phi}$. We name the vertices as in the figure.

- $i = 1$ (i.e., here we consider $\nabla^{1;\varphi}$ and its dual $\Delta^{d(1),\varphi} = \Delta^{6,\varphi}$): $X_{1;\phi} = \mathbb{P}^2_{(x,y,z)}$; the monomials in the equation of the generic elliptic curve are

$$x^3, y^3, z^3, yz^2, y^2z, xz^2, xyz, xy^2, x^2z, x^2y.$$

- $i = 6$ (i.e., here we consider $\nabla^{6;\varphi}$ and its dual $\Delta^{d(6),\varphi} = \Delta^{1;\varphi}$): $X_{6;\phi} = \mathbb{P}^2_{(x,y,z)}/\mathbb{Z}_3$, where the relations are

$$(x, y, z) = (\mu, \epsilon_k) \cdot (x, y, z) = \left(\mu x, \epsilon_k \mu y, \frac{1}{\epsilon_k} \mu z \right),$$

with $\mu \in \mathbb{C}^*$ and $\epsilon_k = e^{2\pi i \frac{k}{3}}, k = 0, 1, 2$. The monomials in the equation of the generic elliptic curve are

$$x^3, y^3, z^3, xyz.$$

- $i = 2$: $X_{2;\phi} = \mathbb{P}^1 \times \mathbb{P}^1$, the monomials in the equation of the generic elliptic curve are

$$y^2z^2, x^2y^2, z^2w_1^2, x^2w_1^2, yz^2w_1, xzw_1^2, xyzw_1, xy^2z, x^2yw_1.$$

- $i = 7$: $X_{7;\phi} = (\mathbb{P}^1_{(x,z)} \times \mathbb{P}^1_{(y,w_1)})/\mathbb{Z}_2$, where the relations are

$$(x, y, z, w_1) = (\mu, \lambda, \bar{0}) \cdot (x, y, z, w_1) = (\mu x, \lambda y, \mu z, \lambda w_1),$$

and

$$(x, y, z, w_1) = (\mu, \lambda, \bar{1}) \cdot (x, y, z, w_1) = (\mu x, \lambda y, -\mu z, -\lambda w_1),$$

Monomials:

$$x^2w_1^2, x^2y^2, z^2w_1^2, xyzw_1.$$

- $i = 3$: $X_{3;\phi} = \mathbb{C}^4 - Z(\Sigma_{3;\varphi})/G_{3;\varphi}$, where $Z(\Sigma_{3;\varphi}) = V(x, z) \cup V(y, w_1)$ and the relations are

$$(x, y, z, w_1) = (\mu x, \lambda y, \mu z, \mu \lambda w_1), \mu, \lambda \in \mathbb{C}^*.$$

Monomials:

$$y^2z^3, x^3y^2, zw_1^2, xw_1^2, yz^2w_1, xyzw_1, x^2yw_1, xy^2z^2, x^2y^2z.$$

- $i = 8$: Monomials:

$$x^2y^2, x^3w_1, y^3z, z^2w_1^2, xyzw_1.$$

- $i = 4$: $X_{4;\phi} = \mathbb{P}_{(x,y,z)}^{(1,1,2)}$. Monomials:

$$y^4, x^4, z^2, x^3y, x^2y^2, xy^3, x^2z, xyz, y^2z.$$

- $i = 9$: $X_{9;\phi} = \mathbb{P}_{(x,y,z)}^{(1,1,2)}/\mathbb{Z}_2$, where the relations are:

$$(x, y, z) = (\mu, \bar{0}) \cdot (x, y, z) = (\mu x, \mu y, \mu^2 z),$$

and

$$(x, y, z) = (\mu, \bar{1}) \cdot [x, y, z] = (-\mu x, \mu y, -\mu^2 z), \mu \in \mathbb{C}^*.$$

Monomials:

$$y^4, x^4, z^2, x^2y^2, xyz.$$

- $i = 5$:

$$y^2z^3w_1^2, x^2y^2z, zw_1^2w_2^2, xw_1w_2^2, x^2yw_2, yz^2w_1^2w_2, xyzw_1w_2, xy^2z^2w_1.$$

- $i = 10$:

$$x^2y^2w_2, xy^2z^2, z^2w_1^2w_2, xw_1^2w_2^2, yz^3w_1, xyzw_1w_2.$$

- $i = 11$:

$$y^4z^3, x^3y, zw_1^2, x^2w_1, xy^3z^2, x^2y^2z, y^2z^2w_1, xyzw_1.$$

- $i = 16$:

$$yz^4, x^2y^3, z^3w_1, xw_1^2, xy^2z^2, xyzw_1.$$

- $i = 12$:

$$x^2y^2, x^2w_1, y^4z^2, z^2w_1^2, y^2z^2w_1, xy^3z, xyzw_1.$$

- $i = 13$:

$$yz^3w_1^2, x^2y^3z, z^2w_1^2w_2, xw_1w_2^2, x^2y^2w_2, xy^2z^2w_1, xyzw_1w_2.$$

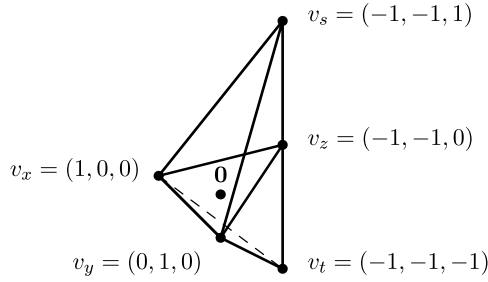


Figure 2: The polytope $\nabla \subset N_{\mathbb{R}}$.

- $i = 14$:

$$xy^2z^2w_1, x^2y^2zw_3, zw_1^2w_2^2w_3, xw_1w_2^2w_3^2, yz^2w_1^2w_2, x^2yw_2w_3^2, xyzw_1w_2w_3.$$

- $i = 15$: $X_{15;\phi} = \mathbb{P}_{(x,y,z)}^{(2,3,1)}$. Monomials:

$$z^6, x^3, y^2, xz^4, x^2z^2, yz^3, xyz.$$

Appendix B. Elliptic fibration in homogeneous coordinates

Example B.1. Consider the reflexive polytope $\nabla \subset N$ is the 3D reflexive polytope with vertices $v_x = (1, 0, 0), v_y = (0, 1, 0), v_s = (-1, -1, 1), v_t = (-1, -1, -1)$ (polytope number 1 in the list [25]). ∇^φ is the 2-dimensional subpolytope of ∇ given by the vertices $v_x, v_y, v_z = (-1, -1, 0), \nabla^\varphi = \nabla^{1,\varphi}$. The dual $\Delta \subset M_{\mathbb{R}}$ to is the reflexive polytope with vertices $(2, -1, 0), (-1, 2, 0), (-1, -1, 3),$ and $(-1, -1, -3)$; see figures 2 and 3.

Consider the fan Σ with rays v_x, v_y, v_z, v_s, v_t . We have $X_\phi = \mathbb{P}_{(x,y,z)}^2$, and $X := X_\Sigma = (\mathbb{C}^3 - Z(\Sigma))/(\mathbb{C}^*)^2$, where

$$(x, y, z, s, t) \sim (\lambda x, \lambda y, \lambda \mu^2 z, \mu^{-1} s, \mu^{-1} t), \lambda \cdot \mu \neq 0. \tag{B.1}$$

Observe that ∇^φ lies on the lattice $N_\varphi = \{v \in N : \langle v, m_\varphi \rangle = 0\}$, where $m_\varphi = (0, 0, 1) \in M$. Moreover,

$$\nabla = \nabla^\varphi \cup \nabla_{\text{top}} \cup \nabla_{\text{bottom}},$$

where $\nabla_{\text{top}} = \{v \in \nabla | \langle v, m_\varphi \rangle > 0\}$ and $\nabla_{\text{bottom}} = \{v \in \nabla | \langle v, m_\varphi \rangle < 0\}$.

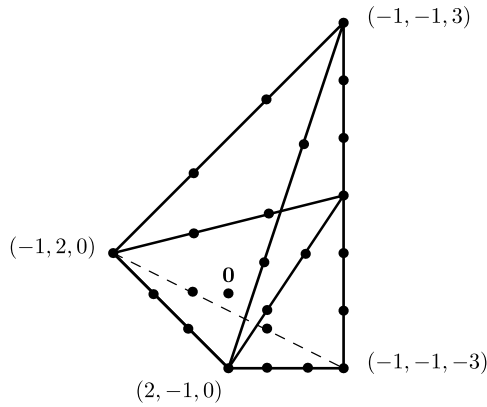


Figure 3: The polytope $\Delta \subset M_{\mathbb{R}}$ dual to $\nabla \subset N_{\mathbb{R}}$.

The homogeneous coordinates for the base of the fibration \mathbb{P}^1 are given by $(z_{\text{top}}, z_{\text{bottom}})$ with

$$z_{\text{top}} = \prod_{v_i \in \nabla_{\text{top}}} z_i^{\langle v_i, m_\varphi \rangle}$$

and

$$z_{\text{bottom}} = \prod_{v_i \in \nabla_{\text{bottom}}} z_i^{-\langle v_i, m_\varphi \rangle}.$$

In this case, we have $z_{\text{top}} = s$ and $z_{\text{bottom}} = t$. It is clear that if we fix a point (s, t) with $s, t \neq 0$, we obtain as a fiber a whole copy of \mathbb{P}^2 . This can also be seen using the equivalence relation (B.1). The generic $K3$ hypersurface V in X has equation

$$\begin{aligned} & a_1 x^3 + x^2(a_2 z + p_1^{(2)} y) + x(p_1^{(4)} z^2 + a_3 y^2 + p_2^{(2)} yz) \\ & + a_4 y^3 + p_3^{(2)} y^2 z + p_2^{(4)} yz^2 + p_1^{(6)} z^3 = 0, \end{aligned}$$

where the a_i are the generic complex numbers, and the $p_i^{(j)}$ are the generic homogeneous polynomials of degree $j = 2, 4, 6$ in s, t . Fixing a point in the base space amounts to fixing the values of the $p_i^{(j)}$. The generic fiber of the fibration restricted to V is a smooth cubic curve (i.e., an elliptic curve) in $\mathbb{P}^2_{(x,y,z)}$.

Acknowledgements

We would like to thank V. Braun, P. Candelas, M. Rossi, U. Whitcher and in particular X. De la Ossa for several useful conversations. This paper is based in part on some sections of V. Perduca's thesis, at University of Turin. We would like to thank the the Mathematics Department of University of Pennsylvania and il Dipartimento di Matematica dell' Università di Torino, especially A. Conte and M. Marchisio. A.G. would like to thank J. Morgan and the staff at Simons Center for Geometry and Physics for their gracious hospitality.

This research project was partially supported by National Science Foundation grant DMS-0636606 and by the Simons Center for Geometry and Physics. V.P. benefited from INDAM - GNSAGA travel fellowships and is presently supported by the *Fondation Sciences Mathématiques de Paris* postdoctoral fellowship program, 2011–2013.

References

- [1] A.C. Avram, M. Kreuzer, M. Mandelberg and H. Skarke, *Searching for K3 fibrations*. Nucl. Phys. B **494**(3) (1997), 567–589; [arXiv:hep-th/9610154v1](#).
- [2] V. Batyrev, *Higher-dimensional toric varieties with ample anticanonical class*, PhD thesis (in Russian). Moscow State University, 1984.
- [3] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Algebr. Geom. **3**(3) (1994), 493–535; [arXiv:alg-geom/9310003v1](#).
- [4] P. Berglund and P. Mayr, *Heterotic string/F-theory duality from mirror symmetry*, Adv. Theor. Math. Phys. **2**(6) (1999) 1307–1372 (1998).
- [5] V. Braun, *Toric elliptic fibrations and F-theory compactifications*, 2011.
- [6] V. Braun, P. Candelas, X. De la Ossa and A. Grassi, *Toric Calabi–Yau fourfolds duality between $n = 1$ theories and divisors that contribute to the superpotential*; [[arXiv:hep-th/0001208](#)].
- [7] G. Brown and J. Buczyński, *Maps of toric varieties in Cox coordinates*, 2010; [arXiv:1004.4924](#), .
- [8] U. Bruzzo and A. Grassi, *Picard group of hypersurfaces in toric 3-folds*, Internat. J. Math. **23**(2) (2012), pp. 1–14; [[arXiv:1011.1003](#) [[math.AG](#)]].

- [9] P. Candelas, *Toric geometry and dualities of string theory*, Fortschr. Phys. **47**(1–3) (1999), 29–37; Quantum aspects of gauge theories, supersymmetry and unification (Neuchâtel, 1997).
- [10] P. Candelas, Toric geometry and dualities of string theory, in *XIIth Int. Congress of Mathematical Physics (ICMP '97) (Brisbane)*, Int. Press, Cambridge, MA, 1999, pp 197–205.
- [11] P. Candelas and A. Font, *Duality between the webs of heterotic and type II vacua*, Nucl. Phys. B **511**(1–2) (1998), 295–325.
- [12] P. Candelas, E. Perevalov and G. Rajesh, *Matter from toric geometry*, Nucl. Phys. B **519**(1–2) (1998), 225–238.
- [13] A. Clingher and J.W. Morgan, *Mathematics underlying the F-theory/heterotic string duality in eight dimensions*, Comm. Math. Phys. **254**(3) (2005), 513–563.
- [14] D.A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebr. Geom. **4**(1) (1995), 17–50; [arXiv:alg-geom/9210008v2](#).
- [15] D.A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, vol 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [16] D.A. Cox, J.B. Little and H.K. Schenck, *Toric varieties*, vol 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
- [17] R. Friedman, J.W. Morgan and E. Witten, *Principal G-bundles over elliptic curves*, Math. Res. Lett. **5**(1–2) (1998), 97–118.
- [18] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, J. Algebr. Geom. **8**(2) (1999), 279–401.
- [19] S. Hu, *Semistable degeneration of toric varieties and their hypersurfaces*, Comm. Anal. Geom. **14**(1) (2006), 59–89; [arXiv:math.AG/0110091](#).
- [20] Y. Hu, C.-H. Liu and S.-T. Yau, *Toric morphisms and fibrations of toric Calabi–Yau hypersurfaces*, Adv. Theor. Math. Phys., **6**(3) (2002) 457–506.
- [21] E.-N. Ionel, *Symplectic sums and Gromov–Witten invariants*. in *Proceedings of the International Congress of Mathematicians*, Vol. II (Beijing, 2002), pp. 427–436, Beijing, 2002, Higher Ed. Press.
- [22] J. Knapp and M. Kreuzer, *Toric methods in f-theory model building*, Adv. High Energy Phys. 2011:513436, 2011.
- [23] J. Knapp, M. Kreuzer, C. Mayrhofer and N.-O. Walliser, *Toric construction of global f-theory guts*, J. High Energy Phys. **03** (2011) 138.
- [24] R. Koelman, *The number of moduli of families of curves on toric varieties*, 1991.

- [25] M. Kreuzer and H. Skarke, Calabi–Yau data. <http://hep.itp.tuwien.ac.at/kreuzer/CY/>.
- [26] M. Kreuzer and H. Skarke, *On the classification of reflexive polyhedra*, Commun. Math. Phys. **185**(2) (1997), 495–508; [[arXiv:hep-th/9512204v2](https://arxiv.org/abs/hep-th/9512204v2)].
- [27] M. Kreuzer and H. Skarke, Classification of reflexive polyhedra in three dimensions, Adv. Theor. Math. Phys., **2**(4) (1998), 853–871; [[arXiv:hep-th/9805190v1](https://arxiv.org/abs/hep-th/9805190v1)].
- [28] M. Kreuzer and H. Skarke, *PALP: a package for analysing lattice polytopes with applications to toric geometry*, Comput. Phys. Commun. **157**(1) (2004), 87–106; [[arXiv:math/0204356v1](https://arxiv.org/abs/math/0204356v1)].
- [29] R. Lazarsfeld, *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 2004., Classical Setting: Line Bundles and Linear Series.
- [30] E. Lerman, *Symplectic cuts*, Math. Res. Lett., **2**(3) (1995), 247–258.
- [31] A. Mavlyutov, Degenerations and mirror contractions of Calabi–Yau complete intersections via Batyrev–Borisov mirror symmetry. Preprint to appear.
- [32] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi–Yau threefolds. I*, Nucl. Phys. B **473**(1–2) (1996), 74–92.
- [33] N. Nakayama, On Weierstrass models, In *Algebraic geometry and commutative algebra*, Vol. II, pp. 405–431, Kinokuniya, Tokyo, 1988.
- [34] T. Oda, *Convex bodies and algebraic geometry*, Vol 15 *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer–Verlag, Berlin, 1988. An introduction to the theory of toric varieties, Translated from the Japanese.
- [35] V. Perduca, Background, 2010, to appear.
- [36] E. Perevalov and H. Skarke, *Enhanced gauge symmetry in type II and F-theory compactifications: Dynkin diagrams from polyhedra*, Nucl. Phys. B **505**(3) (1997), 679–700; [arXiv:hep-th/9704129v2](https://arxiv.org/abs/hep-th/9704129v2).
- [37] G. Rajesh, *Toric geometry and F-theory/heterotic duality in four dimensions*, J. High Energy Phys. (12):Paper 18, 25 pp. (electronic), 1998.
- [38] F. Rohsiepe, *Lists of elliptic toric K3 surfaces, sections and gauge algebras*, <http://www.th.physik.uni-bonn.de/People/rohsiepe/>.
- [39] F. Rohsiepe, *Fibration structures in toric Calabi–Yau fourfolds*, (2005); [arXiv:hep-th/0502138v1](https://arxiv.org/abs/hep-th/0502138v1).

- [40] H. Skarke, *String dualities and toric geometry: an introduction*, *Chaos Solitons Fractals* **10**(2–3) (1999), 543–554; [arXiv:hep-th/9806059v1](#).
- [41] W. Stein, *et al.* *Sage Mathematics Software (Version 2.7)*. The Sage Development Team, (2011). <http://www.sagemath.org>.
- [42] W. Taylor, *Tasi lectures on supergravity and string vacua in various dimension*, [arXiv:11042051v2](#).