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Elliptic genera of Landau–Ginzburg models over nontrivial spaces

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Abstract

In this paper, we discuss elliptic genera of (2,2) and (0,2) supersymmetric Landau–Ginzburg models over nontrivial spaces, i.e., nonlinear sigma models on nontrivial noncompact manifolds with superpotential, generalizing old computations in Landau–Ginzburg models over (orbifolds of) vector spaces. For Landau–Ginzburg models in the same universality class as nonlinear sigma models, we explicitly check that the elliptic genera of the Landau–Ginzburg models match that of the nonlinear sigma models, via a Thom class computation of a form analogous to that appearing in recent studies of other properties of Landau–Ginzburg models on nontrivial spaces.

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1 Introduction

Historically, elliptic genera have provided an important example of mathematics/physics interactions. For mathematicians, elliptic genera (and elliptic cohomology) provided the promise of new mathematical invariants associated to spaces. For physicists, elliptic genera are not only one-loop string partition functions, but have also proven their worth through, e.g., their application to black hole entropy computations [1].

Elliptic genera of (2,2) supersymmetric Landau–Ginzburg models over vector spaces were first computed in [2]. As many Landau–Ginzburg models are on the same Kähler moduli space as ordinary nonlinear sigma models, and elliptic genera are invariant under smooth deformations, computations of elliptic genera of Landau–Ginzburg models often provide efficient ways to compute elliptic genera of corresponding nonlinear sigma models. Unfortunately, most of those Landau–Ginzburg models do not live over vector spaces, or orbifolds thereof, but over more complicated spaces, forming what are sometimes called "hybrid Landau–Ginzburg models."

In this paper, we shall generalize the methods of [2] to Landau–Ginzburg models over nontrivial spaces, typically, total spaces of vector bundles. We study both (2,2) supersymmetric examples, as well as more general (0,2) supersymmetric examples pertinent to heterotic strings. We check our methods by using the renormalization group: if a Landau–Ginzburg model is in the same universality class as an ordinary nonlinear sigma model, then their elliptic genera must match, and we verify this in our examples. We will find that in such cases, the two expressions for the elliptic genus are related by a Thom class, a mathematical gadget that encodes how genera on one space can be calculated in a larger space in which the first space is embedded.

Much of this paper can also be seen as a step in a larger program of generalizing computations for Landau–Ginzburg models on vector spaces, to Landau–Ginzburg models on nontrivial spaces. Other steps in this direction were in, for example, [3, 4], where it was described how to compute correlation functions in A- and B-twisted Landau–Ginzburg models on nontrivial spaces. There, results were checked by comparing Landau–Ginzburg models to nonlinear sigma models in the same universality class: the resulting correlation functions were isomorphic, as expected, and the computations in Landau–Ginzburg models gave a physical realization of tricks for computing Gromov–Witten invariants, for example. Other steps towards understanding Landau–Ginzburg models on nontrivial spaces were described in [5], where mirror symmetry was described as a duality between Landau–Ginzburg models on nontrivial spaces, generalizing other approaches to the subject.

We begin in Section 2 with a review of elliptic genus computations for nonlinear sigma models and Landau–Ginzburg models over vector spaces. In Section 3, we generalize both of those computations, to compute elliptic genera of both (2,2) and (0,2) supersymmetric Landau–Ginzburg models over nontrivial spaces. As a consistency check, we compare elliptic genera of Landau–Ginzburg models to elliptic genera of nonlinear sigma models in the same universality class. As elliptic genera are indices, they are invariant under renormalization group flow, and using results proven in appendix B, we explicitly verify that elliptic genera of theories in the same universality class do match. Mathematically, that matching is realized via "Thom classes," which also appeared, in another form, in [3, 4]. In Section 4, we conclude with a general discussion of Thom classes and their appearance in physics. In appendix A, we list some handy identities for manipulating elliptic genera, and in appendix B we derive identities for Thom classes in elliptic genera that are used in the bulk of the text.

2 Review

2.1 Elliptic genera of nonlinear sigma models

Elliptic genera of nonlinear sigma models have been discussed extensively elsewhere, so we shall review them only briefly.

An elliptic genus is, physically, the one-loop partition function of a theory with at least (0,2) supersymmetry in which the right-moving fermions are all in a R sector—equivalently, the partition function of a half-twisted theory—and, possibly, the left-moving states are also twisted in some way. More specifically, we shall consider elliptic genera which are of the form

$$\mathrm{Tr}\ (-)^{F_R} \exp(\mathrm{i}\gamma J_L) q^{L_0} \overline{q}^{\overline{L}_0}, \tag{2.1}$$

where q is the modular parameter, and the current J_L is a left-moving $U(1)$ current, which is implicitly assumed to exist.

Computation of such genera has been discussed in many places in the literature, beginning in [6], but let us take a few moments to review the highlights. We shall consider theories of the form of nonlinear sigma models with (0,2) supersymmetry, defined on a complex Kähler manifold X of dimension n with a gauge bundle $\mathcal{E}$ of rank r satisfying

$$\begin{aligned} \Lambda^{\rm top}\mathcal{E} &= K_X, \\ {\rm ch}_2(TX) &= {\rm ch}_2(\mathcal{E}). \end{aligned} \tag{2.2}$$

In addition, we shall usually assume X is Calabi–Yau (although we shall note special cases in which sensible results can be obtained more generally). It can be shown [6, 7] that equation (2.1) is an index, and so is invariant under smooth deformations of the theory. As a result, we can consistently deform the theory to the large-radius limit, where the computation of (2.1) becomes a free-field computation.

Because the right-movers are all in the R sector, the nonzero modes of the right-moving fermions and bosons cancel out, leaving only the left-movers and right-moving zero modes to contribute. The right-moving zero modes are defined by a Fock vacuum transforming as a spinor lift[1] of TX. As a result, all of the states appearing in the trace (2.1) have spinor indices. In particular, the trace (2.1) is the index of the Dirac operator coupled to various bundles defined by the nonzero modes of the fields.

To make this a little more concrete, below we list bosonic oscillators at a few mass levels and corresponding bundles, for a nonlinear sigma model on X:

Mass level	Oscillator	Bundle
1	α^μ_{-1}	TX
2	α^μ_{-2}, $\alpha^\mu_{-1}\alpha^\mu_{-1}$	$TX \oplus {\rm Sym}^2(TX)$
3	α^μ_{-3}, $\alpha^\mu_{-2}\alpha^\nu_{-1}$, $\alpha^\mu_{-1}\alpha^\nu_{-1}\alpha^\rho_{-1}$	$TX \oplus (TX \otimes TX) \oplus {\rm Sym}^3(TX)$

At mass level n, it is straightforward to check that the bundle obtained above is the coefficient of q^n in the following element of the Grothendieck group of vector bundles:

$$\bigotimes_{n=1,2,3,\ldots} S_{q^n}(TX), \tag{2.3}$$

[1]Spinor lifts do not always exist; however, sometimes it is still possible to make sense of such expressions. We shall speak more about this as various special cases arise.

where

$$S_q(TX) = 1 + qTX + q^2\mathrm{Sym}^2(TX) + q^3\mathrm{Sym}^3(TX) + \cdots .$$

Each factor of $S_{q^n}(TX)$ corresponds to a set of states of the form

$$\{1, \alpha^{\mu}_{-n}, \alpha^{\mu_1}_{-n}\alpha^{\mu_2}_{-n}, \alpha^{\mu_1}_{-n}\alpha^{\mu_2}_{-n}\alpha^{\mu_3}_{-n}, \ldots\}$$

and so the tensor product encodes all products of all nonzero oscillator creation operators. Thus, for example, the final result for the elliptic genus will involve computing the index of a bundle which has, among other things, a factor of the tensor product (2.3). Furthermore, because we have been implicitly working with complex manifolds, holomorphic bundles, and we distinguish α^{i}_{-1} from $\alpha^{\bar{i}}_{-1}$, we have

$$\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TX)^{\mathbf{C}} \equiv TX \oplus \overline{TX}\right). \tag{2.4}$$

In the (0,2) supersymmetric nonlinear sigma models we consider, the current J_L exists by virtue of the condition $\Lambda^{\mathrm{top}}\mathcal{E} \cong \mathcal{O}_X$ on $\mathcal{E}$ (and becomes the left R-current in the special case of (2,2) supersymmetry). If the left-moving fermions are in an NS sector, then the trace (2.1) is given by [6, 7]

$$q^{-(1/24)(2n+r)} \int_X \mathrm{Todd}(TX) \wedge \mathrm{ch} \\ \times \left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}((TX)^{\mathbf{C}}) \bigotimes_{n=1/2,3/2,5/2,\ldots} \Lambda_{q^n}\left((\mathrm{e}^{\mathrm{i}\gamma}\mathcal{E})^{\mathbf{C}}\right)\right) \tag{2.5}$$

(compare, e.g., [6] equation (30)) where $S_q(TX)$ is as above, $\Lambda_q(\mathcal{E})$ denotes an element of the Grothendieck group of vector bundles on X defined analogously as the linear combinations

$$\Lambda_q(\mathcal{E}) = 1 + q\mathcal{E} + q^2\mathrm{Alt}^2(\mathcal{E}) + q^3\mathrm{Alt}^3(\mathcal{E}) + \cdots$$

(arising physically from the left-moving fermion oscillator modes, just as the factor (2.3) arose from bosonic oscillator modes), and the **C** symbol indicates complexification:

$$(TX)^{\mathbf{C}} = T^{1,0}X \oplus \overline{T^{1,0}X}, \\ (z\mathcal{E})^{\mathbf{C}} = z\mathcal{E} \oplus \overline{z}\overline{\mathcal{E}}.$$

The prefactor of q is due to the zero energy of the vacuum: each periodic complex boson contributes $-1/12$, and each antiperiodic complex fermion contributes $-1/24$. The fact that the S_{q^n}'s are tensored together for integer n reflects the fact that the bosonic oscillators are integrally moded; the fact that the Λ_{q^n}'s are tensored together for half-integer n's reflects the fact that the fermionic oscillators are half-integrally moded.

If the left-moving fermions are in a R sector rather than an NS sector, then the elliptic genus

$$\mathrm{Tr}_{\mathrm{R,R}}(-)^{F_R} \exp\left(\mathrm{i}\gamma J_L\right) q^{L_0}\overline{q}^{\overline{L}_0}$$

is given by

$$q^{+(1/12)(r-n)} \cdot \int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg(z^{-r/2} \left(\det \mathcal{E}\right)^{+1/2} \Lambda_1\left(z\mathcal{E}^\vee\right) \\ \times \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TX)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{E})^{\mathbf{C}}\right), \Bigg) \tag{2.6}$$

where $z = \exp(-\mathrm{i}\gamma)$. (Compare, e.g., [7] equation (31).) In the case X is Calabi–Yau, $\det \mathcal{E}$ is trivial, since $\Lambda^{\mathrm{top}}\mathcal{E} \cong K_X$, so the expression above is well-defined.

The factor of

$$z^{-r/2} \left(\det \mathcal{E}\right)^{+1/2} \Lambda_1(z\mathcal{E}^\vee) = z^{+r/2} \left(\det \mathcal{E}\right)^{-1/2} \Lambda_1(z^{-1}\mathcal{E}) \tag{2.7}$$

arises above from the zero modes of the left-moving fermions. It reflects the ambiguity in the Fock vacuum: if we define $|0\rangle$ by $\lambda_-^a|0\rangle = 0$, then we have a set of vacua

$$|0\rangle,\ \lambda_-^{\overline{a}}|0\rangle,\ \ldots,\lambda_-^{\overline{a}_1}\ldots\lambda_-^{\overline{a}_r}|0\rangle.$$

Similarly, if instead we define $|0\rangle$ by $\lambda_-^{\overline{a}}|0\rangle = 0$, then we have an equivalent set of vacua

$$|0\rangle,\ \lambda_-^{a}|0\rangle,\ \ldots,\lambda_-^{a_1}\ldots\lambda_-^{a_r}|0\rangle.$$

The existence of these two equivalent characterizations of the Fock vacua corresponds to the two sides of equation (2.7). Furthermore, these vacua correspond to spinor lifts of $\mathcal{E}$: note that we can write

$$\Lambda_1(z\mathcal{E}^\vee) = \mathcal{S}_+(z\mathcal{E}^\vee) \oplus \mathcal{S}_-(z\mathcal{E}^\vee),$$

where $\mathcal{S}_\pm$ denote the two chiral Spin^c lifts of $\mathcal{E}^\vee$, i.e.,

$$\mathcal{S}_+(\mathcal{E}^\vee) \equiv \bigoplus_{n\,\mathrm{even}} \Lambda^n \mathcal{E}^\vee,$$
$$\mathcal{S}_-(\mathcal{E}^\vee) \equiv \bigoplus_{n\,\mathrm{odd}} \Lambda^n \mathcal{E}^\vee,$$

which are made into honest spinors via the $\sqrt{\det \mathcal{E}}$ factors. (Physically, every vector bundle comes with a hermitian fibre metric, so we will often fail to distinguish $\mathcal{E}^\vee$ from $\overline{\mathcal{E}}$.) The prefactor of q is due to the vacuum zero energy: each periodic complex boson contributes $-1/12$, and each periodic complex fermion contributes $+1/12$.

Note that in the spinor lifts of $\mathcal{E}$, $\mathcal{E}$ is not complexified, unlike the nonzero modes. This is because for the ambiguity in the Fock vacuum, we use the relation $\{\psi_0^i, \psi_{0j}\} \propto \delta^i_j$, so we take one of either ψ_0^i, ψ_{0j} to be creation operators and the other to be annihilation operators—the choice does not matter, as the resulting collection of states are the same.

Readers familiar with elliptic genera computations elsewhere should note that the "Witten genus" can be obtained as a special case of the R sector genus above. Specifically, for $z = -1$, the R sector genus above is proportional to

$$\begin{aligned}\mathrm{Tr}_{R,R}(-)^{F_R}(-)^{F_L} q^{L_0} \overline{q}^{\overline{L}_0} &\\ = q^{+(1/12)(r-n)} \int_X \hat{A}(TX) \wedge \mathrm{ch}\Bigg(&(\det \mathcal{E})^{+1/2}\, \Lambda_{-1}\left(\mathcal{E}^\vee\right) \\ \times \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TX)^{\mathbf{C}}) &\bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}\left((\mathcal{E})^{\mathbf{C}}\right)\Bigg)\end{aligned} \tag{2.8}$$

which is precisely the Witten genus, introduced by Witten in [6] (equation (31)). This genus has been shown to play a fundamental role in elliptic cohomology [8]. When $z = 1$, both the genus (2.6) and (2.8) are modular provided the two conditions (2.2) hold; it is not[2] necessary to require that X be Calabi–Yau.

2.2 Elliptic genera of Landau–Ginzburg models over vector spaces

2.2.1 Physical analysis—R sector

Later in this paper, we shall compute elliptic genera of Landau–Ginzburg models over topologically nontrivial spaces. We have already reviewed elliptic genus computations in nonlinear sigma models; next, let us review the computation of elliptic genera in Landau–Ginzburg models on topologically trivial spaces, with quasi-homogeneous superpotentials, as first discussed in [2]. In particular, we will focus on the special case of a Landau–Ginzburg model over the complex line, with a monomial superpotential.

Recall that in that paper, a Landau–Ginzburg model over the complex line $\mathbf{C}$ was considered, with superpotential $W = \Phi^{k+2}$. The elliptic genus was defined there as the trace

$$\mathrm{Tr}\ (-)^{F_R} q^{L_0} \overline{q}^{\overline{L}_0} \exp(\mathrm{i}\gamma J_L)$$

over states in which all fields have R boundary conditions along spacelike directions:

$$\begin{aligned} \phi(x_1+1, x_2) &= \phi(x_1, x_2), \\ \psi_+(x_1+1, x_2) &= \psi_+(x_1, x_2), \\ \psi_-(x_1+1, x_2) &= \psi_-(x_1, x_2). \end{aligned}$$

The boundary conditions along timelike directions require more explanation. First, let us work out the left R-charges of the fields, so as to understand the $\exp(\mathrm{i}\gamma J_L)$ factor in the trace. Because of the superpotential interactions, the left R-symmetry no longer merely rotates the ψ_-'s by a phase, leaving other fields invariant, but rather rotates all of the fields by some phase. It is straightforward to check that the left R-charges are as follows:

[2]Physically, for non-Calabi–Yau cases, one can still imagine computing the genus *at*, although not away from, the extreme large-radius free-field limit of the nonlinear sigma model. The left-moving $U(1)$ current will no longer be nonanomalous, but if we fix $z = \pm 1$, then that is not a concern.

Field	R-charge
ϕ	1
ψ_+	1
ψ_-	$-(k+1)$

Furthermore, also because of the superpotential interactions, $(-)^{F_R}$ no longer merely corresponds to a sign on ψ_+'s; rather, it generates a sign on both ψ_+ and ψ_- simultaneously.

We do not list here the timelike boundary conditions, but the attentive reader should recall, for example, that fields with Ramond boundary conditions along timelike directions correspond to traces with $(-)^F$ factors.

The zero modes of ψ_- contribute a factor of

$$\exp(-i\gamma(k+1)/2) \, - \, \exp(+i\gamma(k+1)/2)$$

the zero modes of ψ_+ contribute a factor of

$$\exp(i\gamma/2) \, - \, \exp(-i\gamma/2).$$

The nonzero modes of the fermions contribute

$$\prod_{n=1}^{\infty} \left(1 - z^{k+1} q^n\right) \left(1 - z^{-(k+1)} q^n\right) \left(1 - z^{-1} \overline{q}^n\right) \left(1 - z \overline{q}^n\right),$$

where

$$z = \exp(-i\gamma)$$

and where the minus signs are due to the $(-)^{F_R}$ factor in the trace (and the fact that because of the superpotential interactions, $(-)^{F_R}$ multiplies both ψ_- and ψ_+ simultaneously by a sign). We can rewrite this in the form of the index of a Dirac operator. Using the notation of appendix A, if we let L denote the tangent bundle of $\mathbf{C}$ restricted to the origin, then the expression above for the contribution from the nonzero modes of the fermions is of the form

$$\mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}\left((z^{k+1}L)^{\mathbf{C}}\right) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-\overline{q}^n}\left((z^{-1}L)^{\mathbf{C}}\right)\right).$$

The nonzero modes of the bosons contribute

$$\prod_{n=1}^{\infty} \frac{1}{1-z^{-1}q^n} \frac{1}{1-zq^n} \frac{1}{1-z^{-1}\overline{q}^n} \frac{1}{1-z\overline{q}^n},$$

which we can rewrite as

$$\mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((z^{-1}L)^{\mathbf{C}}\right) \bigotimes_{n=1,2,3,\ldots} S_{\overline{q}^n}\left((z^{-1}L)^{\mathbf{C}}\right) \right).$$

Note that the $\overline{q}$ contributions from the bosons and fermions cancel out—this can be seen either directly from the product formula above or using the identities in appendix A.

Finally, in this special case, the zero modes[3] ϕ_0, $\overline{\phi}_0$ also contribute a factor of

$$\left(\frac{1}{1-z}\right)\left(\frac{1}{1-z^{-1}}\right)$$

as discussed in [2].

Putting this together, we get the genus

$$f(z,q) = \frac{z^{-1/2}\left(z^{(k+1)/2} - z^{-(k+1)/2}\right)}{(1-z^{-1})} \prod_{n=1}^{\infty} \frac{\left(1-z^{k+1}q^n\right)\left(1-z^{-(k+1)}q^n\right)}{(1-z^{-1}q^n)\,(1-zq^n)}. \tag{2.9}$$

We can interpret this as index theory over a point, the fixed-point locus of a $U(1)$ action, i.e., the space of bosonic zero modes satisfying the boundary condition

$$\phi(x_1, x_2+1) = \exp(\mathrm{i}\gamma)\phi(x_1,x_2)$$

for generic γ, which is to say, $\phi_0 = \{0\}$.

[3] Taking into account the timelike boundary conditions, there are no zero modes from the point of view of path integral quantization. The ϕ_0, $\overline{\phi}_0$ referred to here are solely an artifact of the periodic moding in canonical quantization.

Note that the expression for the nonzero modes is given by

$$\mathrm{ch}\left(\bigotimes_{n\geq 1} S_{z^{-1}q^n}(L) \otimes S_{zq^n}(\overline{L}) \otimes \Lambda_{-z^{k+1}q^n}(L) \otimes \Lambda_{-z^{-(k+1)}q^n}(\overline{L}) \right).$$

Instead of working with fields with R boundary conditions along spacelike directions, one could instead try to compute elliptic genera in which the ψ_- have NS boundary conditions in spacelike directions. Note, however, that because of the $\psi_+\psi_-\phi^k$ Yukawa coupling in the theory, if the ψ_- have NS boundary conditions, then so too must the ψ_+, and then the right-moving contributions would no longer cancel out.

2.2.2 Physical analysis—NS sector

In the last subsection we reviewed the results of [2] on computing elliptic genera of Landau–Ginzburg models over vector spaces, in the R sector. In this subsection we will extend the results of [2] to the NS sector.

From the table of left R-charges in the last subsection, we see that fields in the NS sector have spacelike boundary conditions

$$\begin{aligned} \phi(x_1+1, x_2) &= -\phi(x_1, x_2), \\ \psi_+(x_1+1, x_2) &= -\psi_+(x_1, x_2), \\ \psi_-(x_1+1, x_2) &= (-)^{k+1}\psi_-(x_1, x_2). \end{aligned}$$

From the spacelike boundary conditions above, we see that we must consider the cases of k even and odd separately.

In the case k is even, there are no zero modes at all. The fermions contribute

$$\prod_{n=1/2,3/2,\ldots} \left[1 - z^{k+1}q^n\right] \left[1 - z^{-(k+1)}q^n\right] \left[1 - z^{-1}\overline{q}^n\right] \left[1 - z\overline{q}^n\right]$$

and the bosons contribute

$$\prod_{n=1/2,3/2,\ldots} \left[1 - z^{-1}q^n\right]^{-1} \left[1 - zq^n\right]^{-1} \left[1 - z^{-1}\overline{q}^n\right]^{-1} \left[1 - z\overline{q}^n\right]^{-1}.$$

Putting this together, we see that for k even, the elliptic genus is given by

$$\prod_{n=1/2,3/2,\ldots} \left[1 - z^{k+1}q^n\right] \left[1 - z^{-(k+1)}q^n\right] \left[1 - z^{-1}q^n\right]^{-1} \left[1 - zq^n\right]^{-1}.$$

In the case k is odd, there are ψ_- zero modes. In this case, the total contribution from the fermions is

$$\left(z^{(k+1)/2} - z^{-(k+1)/2}\right) \prod_{n=1,2,\dots} \left[1 - z^{k+1}q^n\right]\left[1 - z^{-(k+1)}q^n\right] \times \prod_{n=1/2,3/2,\dots} \left[1 - z^{-1}\overline{q}^n\right]\left[1 - z\overline{q}^n\right]$$

and the bosons contribute

$$\prod_{n=1/2,3/2,\dots} \left[1 - z^{-1}q^n\right]^{-1}\left[1 - zq^n\right]^{-1}\left[1 - z^{-1}\overline{q}^n\right]^{-1}\left[1 - z\overline{q}^n\right]^{-1}.$$

Putting this together, we see that for k odd, the elliptic genus is given by

$$\left(z^{(k+1)/2} - z^{-(k+1)/2}\right) \prod_{n=1,2,\dots} \left[1 - z^{k+1}q^n\right]\left[1 - z^{-(k+1)}q^n\right] \times \prod_{n=1/2,3/2,\dots} \left[1 - z^{-1}q^n\right]^{-1}\left[1 - zq^n\right]^{-1}.$$

2.2.3 Mathematical interpretation

It is striking that the Landau–Ginzburg elliptic genus (2.9) has a natural meaning in the equivariant elliptic cohomology of [9,10]: it is the equivariant genus or Euler class of a virtual representation of $U(1)$, associated to the sigma orientation [8,11,12].

Let h be a generalized cohomology theory. By the suspension isomorphism, the reduced h-theory of the k-sphere contributes to the cohomology of a point: precisely, we have

$$\tilde{h}^0(S^k) \cong h^{-k}(*).$$

In stable homotopy theory, this statement even makes sense when k is negative.

This has the following generalization in the equivariant setting. Let G be a (compact Lie) group, and let E be a G-equivariant generalized cohomology theory. If V is a representation of G, let S^V be its one-point compactification. Then

$$\tilde{E}^0(S^V) \cong E^{-V}(*).$$

Moreover, this statement makes sense even when V is a virtual representation of G.

Let E be the $U(1)$-equivariant elliptic cohomology associated to an elliptic curve C. If V is a representation of $U(1)$, it turns out that $\tilde{E}^0(S^V) \cong E^{-V}(*)$ is the sections of a line bundle $\mathbb{L}(V)$ over the elliptic curve C. More precisely, let L_k be the one-dimensional representation of $U(1)$ given by $z \mapsto z^k$. Let $C[k]$ be the divisor of points of order k of C, and let $\mathcal{O}(-C[k]) = I(C[k])$ be the ideal sheaf of functions which vanish to first order at $C[k]$. Then one has [10]

$$\mathbb{L}(L_k) \cong \mathcal{O}(-C[k])$$

and

$$\mathbb{L}(L_{k+1} - L_1) \cong \mathcal{O}(-C[k+1] + C[1]) \cong \mathcal{O}(-[C[k+1] + (0)). \qquad (2.10)$$

For convenience, let us write V_k for the virtual representation $L_k - L$, and let D denote the divisor $(0) - C[k+1]$, so (2.10) says $\mathbb{L}(V_k) \cong \mathcal{O}(D)$.

As we recalled at (2.8), in [6], Witten introduced the genus of spin manifolds

$$M \mapsto \int \hat{A}(M) ch \left(\bigotimes_{n \geq 1} S_{q^n}(T^{\mathbb{C}}) \right);$$

its K-theory characteristic series is

$$\sigma(L, q) = (L^{1/2} - L^{-1/2}) \prod_{n \geq 1} \frac{(1 - q^n L)(1 - q^n L^{-1})}{(1 - q^n)^2}. \qquad (2.11)$$

In [8,13–15], it was shown that this genus plays a fundamental role in elliptic cohomology. Expression (2.11) gives an equivariant genus simply by taking L to be an equivariant line bundle, and in [11, 12] it is shown that this equivariant genus plays an equally fundamental role in equivariant elliptic cohomology.

Let τ be complex number with positive imaginary part, let Λ be the lattice $2\pi\mathrm{i}\mathbb{Z} + 2\pi\mathrm{i}\tau\mathbb{Z}$, let $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, and let C be the elliptic curve

$$C = \mathbb{C}/\Lambda \cong \mathbb{C}^{\times}/q^{\mathbb{Z}}.$$

Letting $z \in \mathbb{C}^{\times}$, the expression $f(z,q) = \sigma(z^k,q)/\sigma(z,q)$ of (2.9) is easily seen to be the equivariant elliptic cohomology Euler class of the representation of the virtual representation $L_{k+1} - L_1$ of $U(1)$, associated to the characteristic series (2.11). It vanishes to first order at the points of order $k+1$ except the origin, so it is trying to be a section of $\mathcal{O}(D)$.

The only apparent problem is that $f(z,q)$ does not descend to a function on C. Instead

$$f(zq^n,q) = (-1)^{nk} z^{-n[(k+1)^2-1]} q^{-\frac{n^2}{2}[(k+1)^2-1]} f(q,z). \tag{2.12}$$

That is, f is a section of the line bundle over C given by this transformation rule.

This means that f lives in the *twisted* equivariant elliptic cohomology of a point, and it is an important fact the twist is controlled by the second Stiefel Whitney class and the first Pontrjagin class.

Recall that if $u \in H^2(BU(1))$ is the generator, then a virtual representation of $U(1)$ has equivariant Stiefel-Whitney, Chern and Pontrjagin classes, which are elements of $H^*(BU(1))$ with $\mathbb{Z}/2$, $\mathbb{Z}$, and $\mathbb{Z}$ coefficients. One checks easily (recall that $V_k = L_{k+1} - L_1$) that

$$\begin{aligned} &w_2^{U(1)}(V_k) \equiv ku \mod 2, \\ &p_1^{U(1)}(V_k) = ((k+1)^2 - 1)u^2 = -\deg(D)u^2. \end{aligned}$$

Thus, we can rewrite the transformation rule (2.12) as

$$f(zq^n,q) = (-1)^{nw_2^{U(1)}(V_k)} z^{-np_1^{U(1)}(V_k)} q^{-\frac{n^2}{2}p_1^{U(1)}(V_k)} f(z,q). \tag{2.13}$$

This is a general phenomenon: if V is a virtual spin $U(1)$-equivariant bundle over an $U(1)$-space X, then it is essentially a result of [11, 12] that we can use $\lambda(V) = \frac{p_1}{2}(V)$ to form a twisted form of the equivariant elliptic cohomology of X, say $E(X)_{\lambda(V)}$. We can also form the reduced equivariant elliptic cohomology of X^V, the one-point compactification of V, and

$$\tilde{E}(X^V) \cong E(*)_{\lambda(V)}.$$

In the case X is a point (and assuming k even so we can ignore w_2), this reduces to

$$\tilde{E}(S^V) = E^{-V}(*) \cong E^0(*)_{\lambda(V)}.$$

Physically, the main observation is that the transformation rule (2.13) satisfied by the elliptic genus of the Landaug–Ginzburg model is controlled by the equivariant w_2 and p_1 of the representation $L_{k+1} - L_1$.

In the next section we shall consider more complicated examples of Landau–Ginzburg models.

3 Landau–Ginzburg models over nontrivial spaces

In this section, we generalize both of the computations of the previous section to discuss elliptic genera of Landau–Ginzburg models over nontrivial spaces. To compute elliptic genera of Landau–Ginzburg models over nontrivial spaces, we need to assume the spaces have $\mathbf{C}^\times$ actions with respect to which the superpotential is quasi-homogeneous, much as in [3, 4]. That $\mathbf{C}^\times$ action weights the contributions of the various oscillator modes to the genus, via the $\exp(\mathrm{i}\gamma J_L)$ factor in the trace. Moreover, because of the twisted boundary conditions, the integral appearing in the genus, the integral over the space of bosonic zero modes, will be an integral over the fixed-point locus of that $\mathbf{C}^\times$ action.

In particular, we will primarily focus on total spaces of complex vector bundles, for which the $\mathbf{C}^\times$ action in question will be a rotation of the fibres.

We will check our results in cases that the Landau–Ginzburg models are in the same universality classes as nonlinear sigma models, by comparing elliptic genera. The expressions we will derive for genera in the two representatives of the universality class will typically look different, but will turn out to match, often via Thom classes (much as in A-twisted Landau–Ginzburg models [3, 4]).

3.1 The (2,2) quintic and other complete intersections

Consider a Landau–Ginzburg model over the space

$$X = \mathrm{Tot}\left(\mathcal{O}(-5) \xrightarrow{\pi} \mathbf{P}^4\right)$$

with superpotential $W = pG$. Using the same prescription as in [2], we would like to compute the elliptic genus of this Landau–Ginzburg model. Furthermore, since this Landau–Ginzburg model flows under the renormalization group [3] to a nonlinear sigma model on the quintic $Q = \{G = 0\} \subset \mathbf{P}^4$, the elliptic genus of the Landau–Ginzburg model should match that of the quintic.

First, let us review the elliptic genera of the quintic; then, we shall compute the corresponding elliptic genera in the Landau–Ginzburg model, and check that they are the same.

Specializing the results in Section 2.1, the elliptic genus of the quintic,

$$\mathrm{Tr}_{\mathrm{NS,R}}(-)^{F_R} \exp(\mathrm{i}\gamma J_L) q^{L_0} \overline{q}^{\overline{L}_0}$$

with NS boundary conditions along spacelike directions on the left-moving fermions, is given by

$$q^{-(1/24)(9)} \int_Q \hat{A}(TQ) \wedge \mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TQ)^{\mathbf{C}}\right) \right.$$
$$\left. \times \bigotimes_{k=1/2,3/2,5/2,\ldots} \Lambda_{q^k}\left(\left(\mathrm{e}^{\mathrm{i}\gamma} TQ\right)^{\mathbf{C}}\right)\right).$$

Now, let us compute the corresponding elliptic genus in the Landau–Ginzburg model, and check that they match. In the case of the Landau–Ginzburg model, the left-moving R-charge acts differently on the p multiplet than on the rest of the fields. From [3], the fields have left R-charges as follows:

Field	Q_L	Field	Q_L
ϕ^i	0	p	-1
ψ_+^i	0	ψ_+^p	-1
ψ_-^i	1	ψ_-^p	0

(It is straightforward to check this symmetry is anomaly-free.) Imposing boundary conditions on the p field will remove its zero modes except for $p = 0$, and so will force the space of bosonic zero modes to be the zero section of the total space of $\mathcal{O}(-5)$. When we restrict to that zero section, there is a short exact sequence of holomorphic bundles

$$0 \longrightarrow T\mathbf{P}^4 \longrightarrow T\mathcal{O}(-5)|_{\mathbf{P}^4} \longrightarrow \mathcal{O}(-5) \longrightarrow 0$$

and so as smooth bundles,

$$T\mathcal{O}(-5)|_{\mathbf{P}^4} \cong T\mathbf{P}^4 \oplus \mathcal{O}(-5).$$

However, the different components of the restriction of the tangent bundle have different boundary conditions along the time axis.

Let us next figure out what spacelike boundary conditions the Landau–Ginzburg model fields should possess so as to RG flow to the NS sector theory. To that end, note that if some of the fields have NS boundary conditions, then interactions in the theory can demand that other fields also have NS boundary conditions. After all, the boundary conditions must act by a symmetry of the theory, else the form of the action will be coordinate-dependent. (Put another way, NS and R boundary conditions arise in the GSO $\mathbf{Z}_2$ orbifold, but one can only orbifold by a symmetry.) In particular, because of the interaction terms

$$\psi_+^i \psi_-^p D_i G + \psi_+^i \psi_-^j p D_i D_j G + \psi_+^p \psi_-^i D_i G + \cdots ,$$

we see that if the fermions ψ_+^i have R boundary conditions and ψ_-^i have NS boundary conditions, then the p field must also have NS boundary conditions, as too must its superpartner ψ_+^p, though its other superpartners ψ_-^p must have R boundary conditions. For later use in computing the genus matching the R sector case, note that if the fermions ψ_+^i and ψ_-^i both have R boundary conditions, then so too must p and both its superpartners $\psi_\pm^p$.

Let us also work out the action of $(-)^{F_R}$, since it appears in the trace defining the elliptic genus [6]. Under the right R-charge, ϕ^i and ψ_-^i have charge 0, and ψ_+^i charge 1. In order to be consistent with interaction terms, as above, p and ψ_-^p must have charge -1, and ψ_+^p charge 0.

Let us check this analysis by writing out the contributions of the nonzero right-moving modes (in the sector which flows to the NS theory), which should all cancel out. The right-moving nonzero modes of the ϕ and p fields should contribute a factor of

$$\mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{\overline{q}^n}\left((T\mathbf{P}^4)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} S_{-\overline{q}^n}\left(z\mathcal{O}(-5)^{\mathbf{C}}\right)\right)$$

(where the $-q^n$ on the p contribution is due to the presence of the $(-)^{F_R}$ in the trace, as discussed above, and $z = \exp(-\mathrm{i}\gamma)$), and the nonzero modes of ψ_+^i, ψ_+^p contribute a factor of

$$\mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} \Lambda_{-\overline{q}^n}\left((T\mathbf{P}^4)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{\overline{q}^n}\left(z\mathcal{O}(-5)^{\mathbf{C}}\right).\right)$$

However, it can be shown (see appendix A) that

$$S_q(\mathcal{E}) = (\Lambda_{-q}(\mathcal{E}))^{-1}$$

and so we see that the total contribution of the right-moving nonzero modes cancels out. Note in particular that the factor of $(-)^{F_R}$ in the trace was essential in order for this to happen.

Putting this together, and using the multiplicative properties of the S_{q^n} and Λ_{q^n}, we get that the elliptic genus of the Landau–Ginzburg model that RG flows to the NS theory is given by

$$\begin{aligned} &q^{-(1/24)(-2+2(4)+(1)(4)-1)}\overline{q}^{-(1/24)(2(4)-2(4)-1+1)} \int_{\mathbf{P}^4} \mathrm{Todd}(T\mathbf{P}^4) \wedge \mathrm{ch} \\ &\times \Bigg((\mathcal{O} \ominus \mathcal{O}(-5)) \bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((T\mathbf{P}^4)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z\mathcal{O}(-5))^{\mathbf{C}}\right) \\ &\times \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}T\mathbf{P}^4)^{\mathbf{C}}\right) \bigotimes_{n=1,2,\ldots} \Lambda_{-q^n}\left((\mathcal{O}(-5))^{\mathbf{C}}\right)\Bigg). \end{aligned}$$

The factor of $\mathcal{O} \ominus \mathcal{O}(-5)$ arises from the zero modes of ψ_-^p, which is in the R sector, and odd under $(-)^{F_R}$. The S_{q^n} factor arises from the ϕ modes, and the S_{-q^n} factor from the p modes. It has a $-q^n$ instead of q^n because the p field has $(-)^{F_R} = -1$. Similarly, the Λ_{q^n} factor comes from the ψ_-^i modes (which have $(-)^{F_R} = +1$), and the Λ_{-q^n} from the ψ_-^p modes (which have $(-)^{F_R} = -1$). The overall factors of q, $\overline{q}$ are determined by the zero energy of a set of free p, $\psi_\pm^p$, ϕ^i, $\psi_\pm^i$ fields, subject to the boundary conditions discussed previously.

Next, we shall check that the Landau–Ginzburg elliptic genera match the corresponding nonlinear sigma model elliptic genera, which should follow physically from the fact that they are in the same universality class. We will begin with the Landau–Ginzburg elliptic genus that RG flows to the NS sector nonlinear sigma model elliptic genus. To do this, we will use results on Thom classes for elliptic genera which are derived in appendix B. Using Thom classes, the NS sector elliptic genus of the nonlinear sigma model on the quintic should match the integral over $\mathbf{P}^4$ of

$$\mathrm{Todd}(T\mathbf{P}^4) \wedge \mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((T\mathbf{P}^4)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,5/2,\ldots} \Lambda_{q^n}\left((z^{-1}T\mathbf{P}^4)^{\mathbf{C}}\right) \right)$$

times the Thom class for the embedding, which is (appendix B.5)

$$\mathrm{ch}\left(\bigotimes_{n=1/2,3/2,5/2,\ldots} S_{-q^n}\left((z\mathcal{O}(-5))^{\mathbf{C}}\right) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}\left((\mathcal{O}(-5))^{\mathbf{C}}\right) \right)$$

which exactly matches the physics computation in the Landau–Ginzburg model above. Thus, we see explicitly that in the NS sector, the elliptic genus of the Landau–Ginzburg model on the total space of $\mathcal{O}(-5) \to \mathbf{P}^4$ matches that of the quintic nonlinear sigma model, as expected.

This form of the Thom class may look rather complicated, but can be understood relatively easily (and naively) as the S^1-equivariant Thom class on the loop space, following the language and ideas of [7].

A similar analysis can be done for the R sector genera. Here, the genus

$$\mathrm{Tr}_{R,R}(-)^{F_R} \exp(\mathrm{i}\gamma J_L) q^{L_0} \overline{q}^{\overline{L}_0}$$

is given by

$$\int_Q \mathrm{Todd}(TQ) \wedge \mathrm{ch}\left(z^{-3/2} \Lambda_1\left(zT^*Q\right) \bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TQ)^{\mathbf{C}}\right) \right. \\ \left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}\left((z^{-1}TQ)^{\mathbf{C}}\right) \right) \tag{3.1}$$

(using the fact that $\det TQ \cong \mathcal{O}$).

Next, let us compute the corresponding elliptic genus in the Landau–Ginzburg model. We can follow a very similar analysis to the NS sector case.

Half-integral modings become integral modings, and there are additional contributions from zero modes. Specifically, from the bosonic p field zero modes, there is a contribution

$$S_{-1}\left((z\mathcal{O}(-5))^{\mathbf{C}}\right)$$

from the fermionic ψ_-^i zero modes, a contribution

$$z^{+4/2}\Lambda_1(z^{-1}T\mathbf{P}^4)$$

from the fermionic ψ_-^p zero modes, a contribution

$$\Lambda_{-1}(\mathcal{O}(-5))$$

and from the fermionic ψ_+^p zero modes, a contribution

$$z^{-1/2}\Lambda_1(z\mathcal{O}(-5)).$$

Putting this all together, we find

$$\begin{aligned} q^0 \int_{\mathbf{P}^4} & \mathrm{Todd}(T\mathbf{P}^4)\wedge \\ & \times \mathrm{ch}\Bigg(z^{+4/2}z^{-1/2}\Lambda_1(z^{-1}T\mathbf{P}^4)\otimes\Lambda_{-1}(\mathcal{O}(-5))\otimes\Lambda_1(z\mathcal{O}(-5)) \\ & \times \bigotimes_{n=1,2,3,\ldots} S_{q^n}((T\mathbf{P}^4)^{\mathbf{C}}) \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z\mathcal{O}(-5))^{\mathbf{C}}) \\ & \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}T\mathbf{P}^4)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}((\mathcal{O}(-5))^{\mathbf{C}}) \Bigg). \end{aligned} \tag{3.2}$$

The mathematical demonstration that (3.1) and (3.2) coincide is given in Section B.7.

So far, we have discussed only the quintic hypersurface in $\mathbf{P}^4$, but the analysis trivially extends to more general complete intersections. Consider an nonlinear sigma model (NLSM) on a complete intersection $Y \equiv \{G_\mu = 0\} \subset B$ defined by $G_\mu \in \Gamma(\mathcal{G})$, $\mathcal{G}$ a holomorphic vector bundle $\mathcal{E}$. Assume B has complex dimension b, and Y has complex dimension y (so $\mathcal{G}$ has rank $b - y$). (We assume Y is Calabi–Yau.) The elliptic genus of this nonlinear sigma model, with NS boundary conditions along spacelike directions on the left-moving fermions, is given by

$$q^{-(1/24)(3y)} \int_Y \hat{A}(TY) \wedge \\ \times \mathrm{ch} \left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TY)^{\mathbf{C}}\right) \bigotimes_{k=1/2,3/2,5/2,\ldots} \Lambda_{q^k}\left((z^{-1}TY)^{\mathbf{C}}\right) \right) \tag{3.3}$$

Corresponding to this nonlinear sigma model is a Landau–Ginzburg model on

$$X = \mathrm{Tot}\left(\mathcal{G}^\vee \stackrel{\pi}{\longrightarrow} B\right)$$

with superpotential $W = p^\mu G_\mu$. It is straightforward to check that the left-moving $U(1)$ symmetry J_L will be anomaly-free so long as[4]

$$\left(\Lambda^{\mathrm{top}} TB\right) \otimes \left(\Lambda^{\mathrm{top}} \mathcal{G}^\vee\right) \cong K_X^{-1}$$

is trivializable, hence X is Calabi–Yau. (We will get analogous results in other cases, but from now on will for brevity usually omit this part of the analysis.) Following almost exactly the same analysis as for the quintic, the corresponding elliptic genus of this Landau–Ginzburg model is given by

$$q^{-(1/24)(2b+b-(b-y)-2(b-y))} \int_B \mathrm{Todd}(TB) \\ \wedge \mathrm{ch} \left(\Lambda_{-1}\left(\mathcal{G}^\vee\right) \bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TB)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z\mathcal{G}^\vee)^{\mathbf{C}}\right) \right. \\ \left. \times \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}TB)^{\mathbf{C}}\right) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}\left((\mathcal{G}^\vee)^{\mathbf{C}}\right). \right) \tag{3.4}$$

This matches equation (B.24) in appendix B.5, using the fact that, for example,

$$(z\mathcal{G}^\vee)^{\mathbf{C}} = z\mathcal{G}^\vee \oplus z^{-1}\mathcal{G} = (z^{-1}\mathcal{G})^{\mathbf{C}}.$$

As discussed in appendix B.5, this matches equation (3.3).

[4]We are implicitly using the somewhat obscure fact that if p is a coordinate on the total space of a bundle $\mathcal{F}$, say, then ψ^p zero modes are sections of $\mathcal{F}^\vee$.

Similarly, in the R sector, the nonlinear sigma model genus is

$$\int_Y \mathrm{Todd}(TY) \wedge \mathrm{ch}\left(z^{-y/2} \left(\det TY\right)^{1/2} \Lambda_1(zT^*Y) \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TY)^{\mathbf{C}}) \right.$$
$$\left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}TY)^{\mathbf{C}}) \right). \tag{3.5}$$

The corresponding R sector Landau–Ginzburg genus is given by

$$q^0 \int_B \mathrm{Todd}(TB) \wedge \mathrm{ch}\left(z^{+b/2} z^{-(b-y)/2} \Lambda_1(z^{-1}TB) \otimes \Lambda_{-1}(\mathcal{G}^\vee) \otimes \Lambda_1(z\mathcal{G}^\vee) \right.$$
$$\times \left(\det T^*B\right)^{1/2} \left(\det \mathcal{G}^\vee\right)^{-1/2} \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}})$$
$$\times \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z\mathcal{G}^\vee)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}TB)^{\mathbf{C}})$$
$$\left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}((\mathcal{G}^\vee)^{\mathbf{C}}) \right). \tag{3.6}$$

Note that

$$K_X = (\det T^*B) \otimes (\det \mathcal{G}),$$

so when X is Calabi–Yau, the middle row of determinant factors is trivial. In Section B.7, we show that the genera (3.5) and (3.6) coincide.

3.2 Models realizing cokernels of maps

Beginning here and in the next several subsections, we will compute elliptic genera of (0,2) Landau–Ginzburg models that renormalization-group flow to heterotic NLSM's, as described in [4].

In this subsection, we will study a heterotic Landau–Ginzburg model that should flow under the renormalization group to a heterotic NLSM on a space

B with a bundle $\mathcal{E}'$ defined as the cokernel of an injective map:

$$\mathcal{E}' = \text{coker} \left\{ \mathcal{F}_1 \stackrel{\tilde{E}}{\longrightarrow} \mathcal{F}_2 \right\}.$$

The corresponding heterotic Landau–Ginzburg model will be defined over the space

$$X = \text{Tot} \left(\mathcal{F}_1 \stackrel{\pi}{\longrightarrow} B \right),$$

with gauge bundle $\mathcal{E} \equiv \pi^* \mathcal{F}_2$, all $F_a \equiv 0$, and $E^a = p\tilde{E}^a$ for p fibre coordinates along $\mathcal{F}_1$.

The NS sector elliptic genus of the NLSM is given by

$$q^{-(1/24)(2n+r)} \int_B \text{Todd}(TB) \wedge \text{ch} \left(\bigotimes_{n=1,2,3,\ldots} S_{q^n} \left((TB)^{\mathbf{C}} \right) \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n} \left((z^{-1}\mathcal{E}')^{\mathbf{C}} \right) \right), \tag{3.7}$$

where n is the dimension of B, and r is the rank of $\mathcal{E}'$. We are implicitly assuming that B is a spin manifold (otherwise we cannot make sense of the right-moving R sector vacuum). This is computed as the one-loop partition function in which the left-moving fermions λ_- have NS boundary conditions along the spatial direction:

$$\lambda_-(x_1 + 1, x_2) = -\lambda_-(x_1, x_2),$$

This is why the Λ_{q^n} factors are half-integrally moded—reflecting the NS boundary conditions along the spatial axis.

In principle, this should match the corresponding elliptic genus of the Landau–Ginzburg model, which we shall describe next. As before, to define the genus, we must twist the boundary conditions of the fields by a left-moving $U(1)$ current that commutes with the right-moving supercharge. Consistent choices must leave the interaction terms

$$\psi_+^i \lambda_-^{\overline{a}} p D_i \tilde{E}^b h_{\overline{a}b} + \psi_+^p \lambda_-^{\overline{a}} \tilde{E}^b h_{\overline{a}b} + \text{cc}$$

invariant. For example, if we give the λ_-^a NS boundary conditions along the spacelike axis, then we must also give p and ψ_+^p NS boundary conditions along the spacelike axis. More generally, the fields will have charges as follows:

Field	Q_L	Field	Q_L
ϕ^i	0	p	+1
ψ_+^i	0	ψ_+^p	+1
λ_-^a	+1		

(Note that in order for this symmetry to RG flow to the corresponding NLSM symmetry, λ_-^a must have charge +1, which then determines the other charges.) Because of the interaction terms above[5] , $(-)^{F_R}$ will be -1 for ψ_+^i and p. The nontrivial boundary condition on the p field, but not the ϕ^i fields, means that the space of bosonic zero modes will be the zero section of the vector bundle X, i.e., a copy of B. Furthermore, as smooth bundles,

$$TX|_B \cong TB \oplus \mathcal{F}_1$$

in which the fermions coupling to the two factors will have different boundary conditions.

Now, we shall compute the NS sector elliptic genus in the Landau–Ginzburg model, and compare to the corresponding genus in the NLSM. Using the charges above, we see that the elliptic genus of the Landau–Ginzburg model is

$$q^{-(1/24)(+2n-r_1+r_2)}\overline{q}^{-(1/24)(+2n-r_1-2n+r_1)} \int_B \mathrm{Todd}(TB)$$
$$\wedge\, \mathrm{ch}\left(\bigotimes_{n=1,2,3\ldots} S_{q^n}\left((TB)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z^{-1}\mathcal{F}_1)^{\mathbf{C}}\right) \right.$$
$$\left. \times \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{F}_2)^{\mathbf{C}}\right) \right), \tag{3.8}$$

where the two S_{q^n} factors arise from modes of the ϕ and p fields, and the Λ_{q^n} factor from modes of λ_-^a. (Contributions from ψ_+^i and ψ_+^p are easily checked to cancel out against right-moving contributions from ϕ^i and p.) The r_i are the ranks of the bundles $\mathcal{F}_i$.

[5]This choice is not unique. Another choice compatible with the interaction terms is to take $(-)^{F_R}$ to be -1 for ψ_+^i, ψ_+^p and λ_-^a. This choice also yields to all $\overline{q}$ contributions cancelling out, and RG flows to a nonlinear sigma model in which the trace over states has a $(-)^{F_R}(-)^{F_L}$ instead of just $(-)^{F_R}$. Our choice above is made to reproduce the trace over states given earlier.

Let us compare this to the elliptic genus in equation (3.7), using the identities in appendix A. From the definition of $\mathcal{E}'$ and the identities in appendix A we have immediately that

$$\Lambda_q(z^{-1}\mathcal{E}') = \Lambda_q(z^{-1}\mathcal{F}_2)\left(\Lambda_q(z^{-1}\mathcal{F}_1)\right)^{-1} = \Lambda_q(z^{-1}\mathcal{F}_2)S_{-q}(z^{-1}\mathcal{F}_1)$$

from which it immediately follows that the elliptic genus of the Landau–Ginzburg model (3.8) matches that of the nonlinear sigma model (3.7) to which it flows under the renormalization group [4], as expected.

Next, we shall work through the corresponding computations for R sector genera. The R sector elliptic genus of the NLSM is given by

$$\begin{aligned} q^{+(1/12)(r_2-r_1-n)} &\int_B \mathrm{Todd}(TB) \\ &\wedge \mathrm{ch}\left(z^{-(r_2-r_1)/2}\left(\det \mathcal{E}'\right)^{1/2}\Lambda_1(z\mathcal{E}'^{\vee}) \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}}) \right. \\ &\left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{E}')^{\mathbf{C}}) \right). \end{aligned}$$

The corresponding R sector Landau–Ginzburg genus is given by

$$\begin{aligned} q^{-(1/24)(2n+2r_1-2r_2)} &\int_B \mathrm{Todd}(TB) \\ \wedge \mathrm{ch}&\left(z^{+r_2/2}\Lambda_1(z^{-1}\mathcal{F}_2)z^{+r_1/2}\Lambda_1(z^{-1}\mathcal{F}_1)\left(\det\mathcal{F}_2\right)^{-1/2}\left(\det\mathcal{F}_1\right)^{-1/2} \right. \\ &\times \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}}) \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z^{-1}\mathcal{F}_1)^{\mathbf{C}}) \\ &\left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{F}_2)^{\mathbf{C}}) \right), \end{aligned} \tag{3.9}$$

where the

$$z^{+r_2/2}\left(\det\mathcal{F}_2\right)^{-1/2}\Lambda_1(z^{-1}\mathcal{F}_2)$$

factor is from λ_- zero modes, the

$$z^{+r_1/2}\left(\det\mathcal{F}_1\right)^{-1/2}\Lambda_1(z^{-1}\mathcal{F}_1)$$

factor from ψ_+^p zero modes, and the $S_{-1}((z^{-1}\mathcal{F}_1)^{\mathbf{C}})$ factor from p zero modes.

One can show that the R sector Landau–Ginzburg genus matches that of the NLSM in the same fashion as before. The factors from nonzero modes combine in exactly the same form as before, and for the zero modes, note that

$$
\begin{aligned}
&z^{+r_2/2} \left(\det \mathcal{F}_2\right)^{-1/2} \Lambda_1(z^{-1}\mathcal{F}_2) z^{+r_1/2} \left(\det \mathcal{F}_1\right)^{-1/2} \\
&\quad \times \Lambda_1(z^{-1}\mathcal{F}_1) S_{-1}\left(z^{-1}\mathcal{F}_1\right) S_{-1}\left(z\mathcal{F}_1^\vee\right) \\
&\qquad = z^{-r_2/2} \left(\det \mathcal{F}_2\right)^{1/2} \Lambda_1(z\mathcal{F}_2^\vee) z^{+r_1/2} \left(\det \mathcal{F}_1\right)^{-1/2} S_{-1}\left(z\mathcal{F}_1^\vee\right) \\
&\qquad = z^{-(r_2-r_1)/2} \left(\det \mathcal{E}'\right)^{1/2} \Lambda_1\left(z\mathcal{E}'^\vee\right),
\end{aligned}
$$

which shows that the R sector Landau–Ginzburg genus matches the R sector NLSM genus.

3.3 Models realized as kernels of maps

Suppose we have a heterotic NLSM on a space B with gauge bundle given by the kernel $\mathcal{E}'$ of the short exact sequence

$$
0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}_1 \xrightarrow{F_a} \mathcal{F}_2 \longrightarrow 0.
$$

Applying ideas from [4, 16], this heterotic NLSM should be in the same universality class as a heterotic Landau–Ginzburg model on

$$
X = \mathrm{Tot}\left(\mathcal{F}_2^\vee \xrightarrow{\pi} B\right),
$$

with gauge bundle $\mathcal{E} = \pi^* \mathcal{F}_1$, $E^a \equiv 0$, and $F_a = p\tilde{F}_a$ defined by the map $\tilde{F}_a : \mathcal{F}_1 \to \mathcal{F}_2$ defining $\mathcal{E}'$ and p fibre coordinates on $\mathcal{F}_2$.

The NS sector elliptic genus of the NLSM is

$$
\begin{aligned}
&q^{-(1/24)(2n+r)} \int_B \mathrm{Todd}(TB) \\
&\quad \wedge \mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TB)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{E}')^{\mathbf{C}}\right) \right),
\end{aligned}
\tag{3.10}
$$

where n is the dimension of B and r is the rank of $\mathcal{E}'$. As before, we are implicitly assuming that B is a spin manifold.

Next, let us compute the elliptic genus of the Landau–Ginzburg model. As before, we must twist by a left-moving $U(1)$ symmetry, which is determined by the interactions

$$\psi_+^i \lambda_-^a p D_i \tilde{F}_a + \psi_+^p \lambda_-^a \tilde{F}_a + \mathrm{cc}.$$

This determines the charges to be

Field	Q_L	Field	Q_L
ϕ^i	0	p	-1
ψ_+^i	0	ψ_+^p	-1
λ_-^a	$+1$		

(almost, but not quite, the same as for the previous example of a cokernel). In addition, because of the interaction terms, if we put NS boundary conditions on the λ_-^a along spacelike directions, then we must also put NS boundary conditions on p and ψ_+^p. For the same reason, $(-)^{F_R}$ must have value -1 for[6] ψ_+^i and p. The resulting elliptic genus of the Landau–Ginzburg model is

$$q^{-(1/24)(2n-r_2+r_1)}\overline{q}^{-(1/24)(2n-r_2-2n+r_2)} \int_B \mathrm{Todd}(TB)$$
$$\wedge\, \mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TB)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z\mathcal{F}_2^\vee)^{\mathbf{C}}\right) \right.$$
$$\left. \times \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{F}_1)^{\mathbf{C}}\right) \right), \qquad (3.11)$$

where the first S_{q^n} factor is from modes of the ϕ^i field, the second from modes of the p field, and the Λ_{-q^n} factor is from modes of the λ_-^a field. The r_i are the ranks of the $\mathcal{F}_i$, and the overall q and $\overline{q}$ factors are determined by the zero energy contributions of the fields.

Finally, let us compare these two elliptic genera. From the definition of $\mathcal{E}'$ and the identities in appendix A, we have that

$$\Lambda_q(z\mathcal{E}') = \Lambda_q(z\mathcal{F}_1)\left(\Lambda_q(z\mathcal{F}_2)\right)^{-1} = \Lambda_q(z\mathcal{F}_1) S_{-q}(z\mathcal{F}_2),$$

so the elliptic genera of the two representatives of the same universality class match, as expected.

[6] As in the last section, this choice is ambiguous; we make the choice that RG flows to the NLSM elliptic genus given earlier, with the state trace containing $(-)^{F_R}$ not $(-)^{F_R+F_L}$.

Next, we shall work through the corresponding computations for R sector genera. The R sector elliptic genus of the NLSM is given by

$$q^{+(1/12)(r_1-r_2-n)} \int_B \mathrm{Todd}(TB) \wedge \mathrm{ch}\left(z^{-(r_1-r_2)/2} \left(\det \mathcal{E}'\right)^{1/2} \Lambda_1(z\mathcal{E}'^{\vee}) \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}}) \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{E}')^{\mathbf{C}}) \right).$$

The corresponding R sector Landau–Ginzburg genus is given by

$$q^{-(1/24)(2n+2r_2-2r_1)} \int_B \mathrm{Todd}(TB) \wedge \mathrm{ch}\left(z^{+r_1/2}\Lambda_1(z^{-1}\mathcal{F}_1) z^{-r_2/2}\Lambda_1(z\mathcal{F}_2^{\vee}) (\det \mathcal{F}_1)^{-1/2} \left(\det \mathcal{F}_2^{\vee}\right)^{1/2} \times \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}}) \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z\mathcal{F}_2^{\vee})^{\mathbf{C}}) \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{F}_1)^{\mathbf{C}}) \right). \tag{3.12}$$

We can pair up the nonzero modes to match the R sector NLSM genus in the same fashion as for the NS sector genera. The zero modes are related as follows:

$$\begin{aligned} & z^{+r_1/2}\Lambda_1(z^{-1}\mathcal{F}_1) (\det \mathcal{F}_1)^{-1/2} z^{-r_2/2}\Lambda_1(z\mathcal{F}_2^{\vee}) \\ & \quad \times \left(\det \mathcal{F}_2^{\vee}\right)^{1/2} S_{-1}(z\mathcal{F}_2^{\vee}) S_{-1}(z^{-1}\mathcal{F}_2) \\ & \quad = z^{-(r_2-r_1)/2}\Lambda_1(z^{-1}\mathcal{F}_1) S_{-1}(z^{-1}\mathcal{F}_2) (\det \mathcal{F}_1)^{-1/2} \left(\det \mathcal{F}_2^{\vee}\right)^{1/2} \\ & \quad = z^{-(r_2-r_1)/2}\Lambda_1(z^{-1}\mathcal{E}') \left(\det \mathcal{E}'\right)^{-1/2} \\ & \quad = z^{-(r_1-r_2)/2}\Lambda_1(z\mathcal{E}'^{\vee}) \left(\det \mathcal{E}'\right)^{1/2}. \end{aligned}$$

Thus, we see that the R sector Landau–Ginzburg genus does indeed match the R sector NLSM genus, as predicted by renormalization group flow.

3.4 Models realized as cohomologies of monads

Suppose we have a heterotic NLSM on a space B with gauge bundle $\mathcal{E}'$ given by the cohomology of the short complex

$$0 \longrightarrow \mathcal{F}_0 \xrightarrow{\tilde{E}^a} \mathcal{F}_1 \xrightarrow{\tilde{F}_a} \mathcal{F}_2 \longrightarrow 0$$

at the middle term. Judging from related examples and standard analyses in previous sections here and in [4, 16], this heterotic NLSM should be in the same universality class as a heterotic Landau–Ginzburg model on

$$X = \mathrm{Tot}\left(\mathcal{F}_0 \oplus \mathcal{F}_2^\vee \xrightarrow{\pi} B\right),$$

with $\mathcal{E} \equiv \pi^* \mathcal{F}_1$, and $E^a = p' \tilde{E}^a$, $F_a = p \tilde{F}_a$, where p are fibre coordinates along $\mathcal{F}_2^\vee$ and p' fibre coordinates along $\mathcal{F}_0$.

The elliptic genus of the NLSM is

$$q^{-(1/24)(2n+r)} \int_B \mathrm{Todd}(TB) \wedge \mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TB)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{E}')^{\mathbf{C}}\right), \right) \tag{3.13}$$

where n is the dimension of B and r is the rank of $\mathcal{E}'$. As before, we are assuming that B is a spin manifold.

Next, let us compute the elliptic genus of the Landau–Ginzburg model. As before, we must twist by a left-moving $U(1)$ symmetry, which is determined by the interaction terms

$$\psi_+^i \lambda_-^a p D_i \tilde{F}_a + \psi_+^p \lambda_-^a \tilde{F}_a + \psi_+^i \lambda_-^{\overline{a}} p' D_i \tilde{E}^b h_{\overline{a}b} + \psi_+^{p'} \lambda_-^{\overline{a}} \tilde{E}^b h_{\overline{a}b} + \mathrm{cc}$$

and is given by the charges

Field	Q_L	Field	Q_L
ϕ^i	0	ψ_+^i	0
p	-1	ψ_+^p	-1
p'	$+1$	$\psi_+^{p'}$	$+1$
λ_-^a	$+1$		

From the interaction terms, the fields λ_-^a, p, p', ψ_+^p, and $\psi_+^{p'}$ will be in the NS sector, where ϕ^i, ψ_+^i will be in the R sector. Furthermore, $(-)^{F_R}$ acts by[7] -1 on ψ_+^i, p, and p'.

The resulting NS sector elliptic genus of the Landau–Ginzburg model is

$$
\begin{aligned}
&q^{-(1/24)(2n-r_2-r_0+r_1)}\overline{q}^{-(1/24)(2n-r_2-r_0-2n+r_2+r_0)}\int_B \mathrm{Todd}(TB) \\
&\wedge \mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TB)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z\mathcal{F}_2^\vee)^{\mathbf{C}}\right)\right. \\
&\quad \left. \times \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z^{-1}\mathcal{F}_0)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{F}_1)^{\mathbf{C}}\right)\right),
\end{aligned}
$$

where the S_{q^n} factors come from ϕ, p, and p' left-moving modes, and the λ_{-q^n} from λ_-^a modes. The r_i are the ranks of the $\mathcal{F}_i$.

Let us compare the NS elliptic genus in the Landau–Ginzburg model above to that of the nonlinear sigma model, in equation (3.13). Since these two theories are in the same universality class [4], the elliptic genera should match. Now, from the definition of $\mathcal{E}'$ and the identities in appendix A, we have that

$$
\Lambda_q(z\mathcal{E}) = \Lambda_q(z\mathcal{F}_1)\left(\Lambda_q(z\mathcal{F}_0)\right)^{-1}\left(\Lambda_q(z\mathcal{F}_2)\right)^{-1} = \Lambda_q(z\mathcal{F}_1)S_{-q}(z\mathcal{F}_0)S_{-q}(z\mathcal{F}_2)
$$

from which we see that, as expected, the elliptic genera match.

Next, we shall work through the corresponding computations for R sector genera. The R sector elliptic genus of the NLSM is given by

$$
\begin{aligned}
&q^{+(1/12)(r_1-r_0-r_2-n)}\int_B \mathrm{Todd}(TB) \\
&\wedge \mathrm{ch}\left(z^{-(r_1-r_0-r_2)/2}\left(\det\mathcal{E}'\right)^{1/2}\Lambda_1(z\mathcal{E}'^\vee) \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}})\right. \\
&\quad \left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{E}')^{\mathbf{C}})\right).
\end{aligned}
$$

[7] As before, there is an ambiguity, and we make the choice that flows to standard conventions in the IR.

The corresponding R sector Landau–Ginzburg genus is given by

$$
\begin{aligned}
& q^{-(1/24)(2n+2r_0+2r_2-2r_1)} \int_B \mathrm{Todd}(TB) \\
& \quad \wedge \mathrm{ch} \left(z^{+r_1/2} \Lambda_1(z^{-1}\mathcal{F}_1) z^{+r_0/2} \Lambda_1(z^{-1}\mathcal{F}_0) z^{-r_2/2} \Lambda_1(z\mathcal{F}_2^\vee) \right. \\
& \qquad \times (\det \mathcal{F}_1)^{-1/2} (\det \mathcal{F}_0)^{-1/2} (\det \mathcal{F}_2)^{+1/2} \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}}) \\
& \qquad \times \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z^{-1}\mathcal{F}_0)^{\mathbf{C}}) \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z\mathcal{F}_2^\vee)^{\mathbf{C}}) \\
& \qquad \left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{F}_1)^{\mathbf{C}}) \right). \qquad (3.14)
\end{aligned}
$$

We can pair up the nonzero modes to match the R sector NLSM genus in the same fashion as for the NS sector genera. The zero modes are related as follows:

$$
\begin{aligned}
& z^{+r_1/2} \Lambda_1(z^{-1}\mathcal{F}_1) (\det \mathcal{F}_1)^{-1/2} z^{-r_0/2} \Lambda_1(z\mathcal{F}_0^\vee) (\det \mathcal{F}_0)^{1/2} z^{-r_2/2} \\
& \quad \times \Lambda_1(z\mathcal{F}_2^\vee) (\det \mathcal{F}_2)^{+1/2} S_{-1}(z^{-1}\mathcal{F}_0) S_{-1}(z\mathcal{F}_0^\vee) S_{-1}(z\mathcal{F}_2^\vee) S_{-1}(z^{-1}\mathcal{F}_2) \\
& \qquad = z^{+r_1/2} \Lambda_1(z^{-1}\mathcal{F}_1) (\det \mathcal{F}_1)^{-1/2} z^{-r_0/2} (\det \mathcal{F}_0)^{1/2} z^{-r_2/2} \\
& \qquad \quad \times (\det \mathcal{F}_2)^{+1/2} S_{-1}(z^{-1}\mathcal{F}_0) S_{-1}(z^{-1}\mathcal{F}_2) \\
& \qquad = z^{(r_1-r_0-r_2)/2} \left(\det \mathcal{E}'\right)^{-1/2} \Lambda_1(z^{-1}\mathcal{E}') \\
& \qquad = z^{-(r_1-r_0-r_2)/2} \left(\det \mathcal{E}'\right)^{+1/2} \Lambda_1(z\mathcal{E}'^\vee).
\end{aligned}
$$

Thus, we see that the R sector Landau–Ginzburg genus does indeed match the R sector NLSM genus, as predicted by renormalization group flow.

3.5 Models realized as cohomologies of monads over complete intersections

Here, we consider the most general case. Suppose we want a heterotic Landau–Ginzburg model that will flow to a heterotic NLSM on a complete intersection $Y \equiv \{G_\mu = 0\} \subset B$ defined by $G_\mu \in \Gamma(\mathcal{G})$, $\mathcal{G}$ a holomorphic vector bundle on B, with a gauge bundle $\mathcal{E}'$ given by the cohomology of the complex of holomorphic vector bundles

$$0 \longrightarrow \mathcal{F}_0|_Y \stackrel{\tilde{E}^a|_Y}{\longrightarrow} \mathcal{F}_1|_Y \stackrel{\tilde{F}_a|_Y}{\longrightarrow} \mathcal{F}_2|_Y \longrightarrow 0,$$

where $\tilde{E}_a : \mathcal{F}_0 \to \mathcal{F}_1$ and $\tilde{F}_a : \mathcal{F}_1 \to \mathcal{F}_2$ are defined over all of B, but the sequence above only necessarily becomes a complex over $Y \subset B$. (Furthermore, the complex is exact everywhere except at the $\mathcal{F}_1$ term.) Explicitly,

$$\begin{aligned} \mathrm{Ker}\, \tilde{E}^a &= 0, \\ \mathrm{Coker}\, \tilde{F}_a &= 0, \\ \mathcal{E}' &= \mathrm{Ker}\, \tilde{F}_a / \mathrm{Img}\, \tilde{E}^a. \end{aligned}$$

Generalizing the (0,2) gauged linear sigma model (GLSM) description in [4, 16], the corresponding Landau–Ginzburg model is defined over the space

$$X = \mathrm{Tot}\left(\mathcal{F}_0 \oplus \mathcal{F}_2^{\vee} \stackrel{\pi}{\longrightarrow} B \right),$$

with gauge bundle $\mathcal{E}$ an extension[8] of $\pi^* \mathcal{F}_1$ by $\pi^* \mathcal{G}^{\vee}$:

$$0 \longrightarrow \pi^* \mathcal{G}^{\vee} \longrightarrow \hat{\mathcal{E}} \longrightarrow \pi^* \mathcal{F}_1 \longrightarrow 0.$$

The $F_a \in \Gamma(\mathcal{E}^{\vee})$ are partially determined by $G \in \Gamma(\mathcal{G})$ and

$$F_a|_{\pi^* \mathcal{F}_1^{\vee}} = p \tilde{F}_a,$$

where p are fibre coordinates on $\mathcal{F}_2^{\vee}$ and $\tilde{F}_a$ is the map $\mathcal{F}_1 \to \mathcal{F}_2$. The $E^a \in \Gamma(\mathcal{E})$ are partially determined by $p' \tilde{E}^a$, where p' are fibre coordinates on $\mathcal{F}_0$ and $\tilde{E}^a$ is the map $\mathcal{F}_0 \to \mathcal{F}_1$.

[8]In general, the extension will be nontrivial, as an example we will discuss momentarily will make clear. Aside from that, we have not found a way to uniquely determine the extension in terms of data of the IR NLSM. In fact, since renormalization group flow is a lossy process, it is not completely clear that the Landau–Ginzburg model should be uniquely determined by the NLSM – perhaps several different extensions defining different $\mathcal{E}$'s in the Landau–Ginzburg model all flow to the same NLSM. We have no such examples, but neither can we rule out the possibility.

The NS sector elliptic genus of the NLSM is

$$q^{-(1/24)(2n+r)} \int_Y \mathrm{Todd}(TY) \wedge \mathrm{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TY)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{E}')^{\mathbf{C}}\right) \right), \tag{3.15}$$

where n is the dimension of Y and r is the rank of $\mathcal{E}'$. As before, we assume B is a spin manifold.

Next, let us compute the elliptic genus of the Landau–Ginzburg model. As before, we must twist by a left-moving $U(1)$ symmetry, and the charges under that symmetry are determined in part by interaction terms. We need to distinguish λ_- coupling to $\pi^*\mathcal{F}_1$ from $\pi^*\mathcal{G}^\vee$; we shall, schematically (ignoring for the moment the extension), denote the former by $\lambda_-^{\underline{a}}$ and the latter by $\lambda_-^{\underline{\alpha}}$. Then, equally schematically, the interactions will be of the form

$$\psi_+^i \lambda_-^{\underline{a}} p D_i \tilde{F}_a, \quad \psi_+^i \lambda_-^{\underline{\alpha}} D_i G_\alpha, \quad \psi_+^i \lambda_-^{\overline{a}} p' \left(D_i \tilde{E}^b \right) h_{\overline{a}b}.$$

From these, we can see that if ϕ^i, ψ_+^i are neutral under J_L, and $\lambda_-^{\underline{a}}$ has charge $+1$, then p must have charge -1, in order for the interactions above to remain neutral. To be a left R-symmetry, all ψ_+'s must have the same left charge as the corresponding scalars. Proceeding in this fashion, we find that the charges determining that left $U(1)$ symmetry are

Field	Q_L	Field	Q_L
ϕ^i	0	ψ_+^i	0
p	-1	ψ_+^p	-1
p'	$+1$	$\psi_+^{p'}$	$+1$
$\lambda_-^{\underline{a}}$	$+1$		
$\lambda_-^{\underline{\alpha}}$	0		

It is straightforward to check that this symmetry is anomaly-free so long as

$$\left(\Lambda^{\mathrm{top}}\mathcal{F}_1\right) \otimes \left(\Lambda^{\mathrm{top}}\mathcal{F}_2\right)^\vee \otimes \left(\Lambda^{\mathrm{top}}\mathcal{F}_0^\vee\right)$$

is trivializable, which implies that $\Lambda^{\mathrm{top}}\mathcal{E}'$ is trivializable, and the Calabi–Yau condition.

We also see that ϕ^i, ψ_+^i, and λ_-^α are in the R sector, and λ_-^a, p, p', ψ_+^p, $\psi_+^{p'}$ are in the NS sector. Furthermore, as before, $(-)^{F_R}$ should act by -1 on ψ_+^i, p, p', and λ_-^α.

The resulting elliptic genus of the Landau–Ginzburg model is

$$q^{-(1/24)(2m-r_2-r_0-2s+r_1)}\overline{q}^{-(1/24)(2m-r_2-r_0-2m+r_2+r_0)}\int_B \mathrm{Todd}(B)$$
$$\wedge\, \mathrm{ch}\left(\left(\mathcal{S}_+(\mathcal{G}^\vee)\ominus \mathcal{S}_-(\mathcal{G}^\vee)\right) \bigotimes_{n=1,2,3,\ldots} S_{q^n}\left((TB)^{\mathbf{C}}\right)\right.$$
$$\times \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z\mathcal{F}_2^\vee)^{\mathbf{C}}\right) \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}\left((z^{-1}\mathcal{F}_0)^{\mathbf{C}}\right)$$
$$\left.\times \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}\left((z^{-1}\mathcal{F}_1)^{\mathbf{C}}\right) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}\left((\mathcal{G}^\vee)^{\mathbf{C}}\right)\right), \qquad (3.16)$$

where m is the dimension of B, s the rank of $\mathcal{G}$, and r_i the rank of $\mathcal{F}_i$.

Let us compare the NS sector Landau–Ginzburg elliptic genus above to the NS sector elliptic genus of the nonlinear sigma model (3.15). Since the two theories are in the same universality class, the two genera ought to match. In appendix B.6 we show mathematically that they do match, as expected.

As a consistency check, let us explicitly describe how to recover the results on (2,2) complete intersections in Section 3.1 from the expression above. The (2,2) locus is described by taking [4] $\mathcal{F}_0 = 0$, $\mathcal{F}_2 = \mathcal{G}$, and $\mathcal{F}_1 = TB$, so that $\mathcal{E} = TX$. It is easy to check that in this case, the expression above reduces to equation (3.4), as expected.

Next, we shall work through the corresponding computations for R sector genera. The R sector elliptic genus of the NLSM is given by

$$q^{+(1/12)(r_1-r_0-r_2-(m-s))}\int_Y \mathrm{Todd}(TY)$$
$$\wedge\, \mathrm{ch}\left(z^{-(r_1-r_0-r_2)/2}\left(\det \mathcal{E}'\right)^{1/2}\Lambda_1(z\mathcal{E}'^\vee) \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TY)^{\mathbf{C}})\right.$$
$$\left.\times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{E}')^{\mathbf{C}})\right).$$

The corresponding R sector Landau–Ginzburg genus is given by

$$q^{-(1/24)(2m+2r_0+2r_2-2s-2r_1)} \int_B \mathrm{Todd}(TB)$$
$$\wedge \, \mathrm{ch} \Big(z^{+r_1/2} \Lambda_1(z^{-1}\mathcal{F}_1) \Lambda_{-1}(\mathcal{G}^\vee) z^{+r_0/2} \Lambda_1(z^{-1}\mathcal{F}_0) z^{-r_2/2} \Lambda_1(z\mathcal{F}_2^\vee)$$
$$\times (\det \mathcal{F}_1)^{-1/2} (\det \mathcal{F}_0)^{-1/2} (\det \mathcal{F}_2)^{+1/2} \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}})$$
$$\times \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z^{-1}\mathcal{F}_0)^{\mathbf{C}}) \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z\mathcal{F}_2^\vee)^{\mathbf{C}})$$
$$\times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}\mathcal{F}_1)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}((\mathcal{G}^\vee)^{\mathbf{C}}) \Big). \qquad (3.17)$$

It is not difficult to adapt the methods of Appendix B to prove that the genera (3.16) and (3.17) coincide.

4 General remarks on Thom classes

In this paper we have seen that for Landau–Ginzburg models in the same universality class as nonlinear sigma models, the elliptic genera of the Landau–Ginzburg models match those of the nonlinear sigma models via a Thom class computation—the two expressions look very different, but the Landau–Ginzburg model computation has the form of an integral over a larger space of something that localizes onto the smaller space over which the nonlinear sigma model genus is defined.

This particular property is not specific to elliptic genera, but crops up in other contexts as well. For example, in [3], A-twisted Landau–Ginzburg models were discussed, and it was observed there that correlation functions in A-twisted Landau–Ginzburg models in the same universality class as nonlinear sigma models, matched by virtue of a Thom form, represented specifically by a Mathai–Quillen form.

5 Conclusions

In this paper, we have discussed elliptic genera of both (2,2) and (0,2) supersymmetric Landau–Ginzburg models over nontrivial spaces, generalizing methods of [2] for Landau–Ginzburg models over vector spaces. We checked our results using the renormalization group: Landau–Ginzburg models in the same universality class as ordinary nonlinear sigma models should have matching elliptic genera, which we were able to confirm explicitly. In those computations, just as in the A-twisted Landau–Ginzburg model computations of [3, 4], we saw that the renormalization group is realized mathematically via Thom classes.

One direction for future work lies in understanding elliptic genera and other properties of Landau–Ginzburg models over nontrivial *stacks* [17], in addition to nontrivial spaces. One application would be to complete our knowledge of elliptic genera at various limits of gauged linear sigma models, as 'typical' limits of Kähler moduli space are "hybrid Landau–Ginzburg models," which are precisely Landau–Ginzburg models over stacks. Another application would be to compute elliptic genera of noncommutative spaces (in the sense of Kontsevich and others, as opposed to [18]), as described in [19] (where they appeared as part of a general discussion of novel geometries and non-birational phases of abelian GLSM's, realizing Kuznetsov's "homological projective duality" [20]) Those examples of new CFT's are realized as IR limits of certain hybrid Landau–Ginzburg models appearing in GLSMs, hence, one way to compute their elliptic genera would be to compute the elliptic genus of a corresponding Landau–Ginzburg model on a stack.

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Appendix A Some useful identities

Define

$$\begin{aligned} S_q\left(z\mathcal{E}\right) &= 1 \,\oplus\, zq\mathcal{E} \,\oplus\, z^2q^2\mathrm{Sym}^2\mathcal{E} \,\oplus\, z^3q^3\mathrm{Sym}^3\mathcal{E} \,\oplus\, \cdots \\ &= S_{zq}\mathcal{E}, \\ \Lambda_q\left(z\mathcal{E}\right) &= 1 \,\oplus\, zq\mathcal{E} \,\oplus\, z^2q^2\mathrm{Alt}^2\mathcal{E} \,\oplus\, z^3q^3\mathrm{Alt}^3\mathcal{E} \,\oplus\, \cdots \\ &= \Lambda_{zq}\mathcal{E} \end{aligned}$$

and similarly

$$\begin{aligned} S_q\left(z\mathcal{E}\right)^{\mathbf{C}} &= S_q\left(z\mathcal{E}\right) \otimes S_q\left(\overline{z}\overline{\mathcal{E}}\right), \\ \Lambda_q\left(z\mathcal{E}\right)^{\mathbf{C}} &= \Lambda_q\left(z\mathcal{E}\right) \otimes \Lambda_q\left(\overline{z}\overline{\mathcal{E}}\right), \end{aligned}$$

These expressions should be understood as elements of K-theory of the underlying space.

These expressions have good multiplicative properties:

$$\begin{aligned}
S_q(\mathcal{E}\oplus\mathcal{F}) &= (S_q\mathcal{E})\otimes(S_q\mathcal{F}),\\
S_q(\mathcal{E}\ominus\mathcal{F}) &= (S_q\mathcal{E})\otimes(S_q\mathcal{F})^{-1},\\
\Lambda_q(\mathcal{E}\oplus\mathcal{F}) &= (\Lambda_q\mathcal{E})\otimes(\Lambda_q\mathcal{F}),\\
\Lambda_q(\mathcal{E}\ominus\mathcal{F}) &= (\Lambda_q\mathcal{E})\otimes(\Lambda_q\mathcal{F})^{-1},
\end{aligned}$$

where we have used the facts that

$$\begin{aligned}
\mathrm{Sym}^n(\mathcal{E}\oplus\mathcal{F}) &= \bigoplus_{i=0}^{n}\mathrm{Sym}^i(\mathcal{E})\otimes\mathrm{Sym}^{n-i}(\mathcal{F}),\\
\mathrm{Alt}^n(\mathcal{E}\oplus\mathcal{F}) &= \bigoplus_{i=0}^{n}\mathrm{Alt}^i(\mathcal{E})\otimes\mathrm{Alt}^{n-i}(\mathcal{F}).
\end{aligned}$$

Furthermore, the inverses are straightforward to compute. Using the multiplicative properties above and the splitting principle, it suffices to consider the action on line bundles. For a line bundle $\mathcal{L}$,

$$S_q\mathcal{L} = 1 \,\oplus\, q\mathcal{L} \,\oplus\, q^2\mathcal{L}^2 \,\oplus\, \cdots = \frac{1}{1\,\ominus\, q\mathcal{L}} = (\Lambda_{-q}\mathcal{L})^{-1},$$

so we see that

$$(S_q\mathcal{E})^{-1} = \Lambda_{-q}\mathcal{E}$$

for any vector bundle $\mathcal{E}$, and similarly

$$(\Lambda_q\mathcal{E})^{-1} = S_{-q}\mathcal{E}.$$

Appendix B Thom class formulas

B.1 Umkehr maps and the Riemann–Roch formula

We briefly recall the yoga of Umkehr maps in K-theory and ordinary cohomology, and their relationship through the Riemann–Roch formula. This story arose in the work of Atiyah, Hirzebruch, and Singer as part of the development of index theory, and makes essential use of a construction of Pontrjagin and Thom.

B.2 Thom space

Let $V \to X$ be a vector bundle. The *Thom space* of X is the space

$$X^V = D(V)/S(V), \tag{B.1}$$

the disk bundle modulo the sphere bundle. One needs a metric on V to make sense of $D(V)$ and $S(V)$, but any two metrics give the same homotopy type for X^V. If X is compact, then we can take X^V to be the one-point compactification of V. Here are some important points about this construction.

First, if $\underline{n}$ denotes the trivial bundle of rank n over X, then

$$X^{\underline{n}} = \frac{D^n \times X}{S^{n-1} \times X} \cong \frac{S^n \times X}{* \times X}.$$

If Y is a pointed space, let Σ^n denote the n-fold "reduced suspension" of Y:

$$\Sigma^n Y = \frac{S^n \times Y}{* \times Y \cup S^n \times *}.$$

If X_+ refers to the space X with a disjoint point added, then

$$\Sigma^n(X_+) \cong \frac{S^n \times X}{* \times X} \cong X^{\underline{n}}.$$

Thus, the construction $V/X \mapsto X^V$ is a generalization of reduced suspension.

Second, formation of the Thom spectrum is natural with respect to pullbacks: given

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ Y & \xrightarrow{f} & X, \end{array}$$

we have an induced map of Thom spaces

$$f : Y^{f^*V} \to X^V. \tag{B.2}$$

Third, the Thom space construction is exponential in the sense that for $V \to X$ and $W \to Y$, we have

$$(X \times Y)^{V \hat{\oplus} W} \approx X^V \wedge Y^W. \tag{B.3}$$

Here $\wedge$ denotes the "smash product": if A and B are two pointed spaces, then

$$A \wedge B = \frac{A \times B}{A \times * \cup * \times B}.$$

Since the internal Whitney sum $V \oplus W$ is the pull-back along the diagonal

$$\begin{array}{ccc} V \oplus W & \longrightarrow & V \hat{\oplus} W \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X, \end{array}$$

by combining (B.2) and (B.3) we conclude that if V and W are vector bundles over X, then we have a map of Thom spectra

$$X^{V \oplus W} \to X^V \wedge X^W.$$

From now on we will not distinguish in notation between the internal and external Whitney sums.

Fourth, there is no projection map $X^V \to X$, because all of $S(V)$ was crushed to a single point. However, the "relative" diagonal map

$$V \to V \times V \to X \times V$$

does induce a map

$$X^V \to X_+ \wedge X^V. \tag{B.4}$$

Note that the zero section $\zeta : X \to V$ does give rise to a map $\zeta : X \to X^V$, also called the zero section.

Let E be a generalized cohomology theory. If Y is a pointed space, then we can use the inclusion $* \to Y$ to form the associated reduced cohomology

$$\bar{E}^*(Y) = \operatorname{Ker} E^*(Y) \to E^*(*).$$

The Mayer–Vietoris sequence implies that we have the so-called "suspension isomorphism"

$$\bar{E}^*(Y) \cong \bar{E}^{*+n}(\Sigma^n Y).$$

If X an unpointed space then

$$E^*(X) \cong \bar{E}^*(X_+),$$

so the suspension isomorphism can be read to say that

$$E^*(X) \cong \bar{E}^*(X_+) \cong \bar{E}^{*+n}(X^{\underline{n}}). \tag{B.5}$$

An *orientation* or *Thom isomorphism* for V in E-theory is an isomorphism

$$E^*(X) \cong \bar{E}^{*+r}(X^V), \tag{B.6}$$

where r is the rank of V. Comparing (B.5), we see that an orientation of V is a generalization of the suspension isomorphism.

Typically this terminology arises when E is a ring spectrum: that is, $E^*(X)$ is a graded ring, rather than merely a graded abelian group. In that case, the relative diagonal (B.4) gives us a map

$$E^*(X) \otimes \bar{E}^*(X^V) \to \bar{E}^*(X^V),$$

so $\bar{E}^*(X^V)$ is a module over the ring $E^*(X)$. We then require the Thom isomorphism (B.6) to be an isomorphism of $E^*(X)$-modules. Since $E^*(X)$ is free of rank 1 as a module over itself, the Thom isomorphism is completely determined by a choice of generator $U \in \bar{E}^r(X^V)$, called the *Thom class*.

The pull-back of the Thom class U along the zero section $\zeta : X \to X^V$ is called the "Euler class"

$$e(V) = \zeta^* U \in E^r(X).$$

The composition

$$E^*(X) \xrightarrow{\text{Thom}} E^{*+r}(X^V) \xrightarrow{\zeta^*} E^{*+r}(X)$$

is multiplication by the Euler class.

The name "orientation" arises from the fact that in the case that E is ordinary cohomology with integer coefficients, a choice of Thom class is equivalent to a choice of orientation for the vector bundle V, in the classical sense. If M is a compact oriented manifold with tangent bundle TM, then the Euler class $e(TM)$ has the property that

$$\chi(M) = \int_M e(TM),$$

where $\chi(M)$ is the Euler characteristic

$$\chi(M) = \sum_i (-1)^i \dim H^i(M;\mathbb{Q}).$$

When E is a ring spectrum, one often asks not just for one orientation of the single bundle V/X, but a compatible family of orientations of a family of bundles.

Let $\mathcal{V}$ be a family of vector bundles which contains the trivial bundles and is closed under pull-back and Whitney sum. For example, $\mathcal{V}$ could be the family of complex vector bundles, or spin vector bundles, etc. A *(stable exponential) orientation* of $\mathcal{V}$ in E-theory is a rule Φ which assigns to $V/X \in \mathcal{V}$ of rank r a Thom class

$$\Phi(V/X) \in E^r(X^V),$$

which is

(1) natural, in the sense that

$$f^*\Phi(V/X) = \phi(f^*V/Y) \in E^r(Y^{f^*V}),$$

if V/X and $f : Y \to X$;

(2) exponential, in the sense that

$$\Phi(V \oplus W/(X \times Y)) = \phi(V/X)\phi(W/Y) \in E^{r+s}((X \times Y)^{V \oplus W}),$$

using the equivalence (B.3); and

(3) unital, meaning that the induced Thom isomorphism

$$E^*(X) \cong \bar{E}^{*+n}(X^{\underline{n}})$$

coincides with the suspension isomorphism (B.5).

B.3 The Pontrjagin–Thom construction

Orientations in a generalized cohomology theory are an important ingredient in the theory of generalized integration/intersections. The other important ingredient is the Pontrjagin–Thom construction.

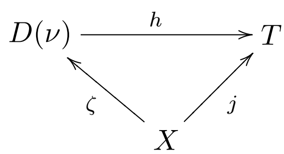
First suppose that

$$j : X \to Y$$

is an embedding, with normal bundle ν. By the Tubular Neighbourhood Theorem, there is a neighbourhood N of X in Y and a homeomorphism

$$h : D(\nu) \cong T$$

making the diagram



commute (ζ is the zero section of ν).

By collapsing the complement of T to a point, we get a map of pointed spaces

$$\tau(j) : Y_+ \to X^\nu \tag{B.7}$$

from the one-point compactification of Y to the Thom space of ν. In our applications, Y will be compact, and in that case Y_+ indicates Y, with a disjoint basepoint added. This is called the "Pontrjagin–Thom collapse."

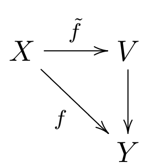
Now suppose that $f : X \to Y$ is a proper map (for example, a fibre bundle with compact fibre). Let us first "convert it to an embedding," for example by choosing an embedding $X \to \mathbb{R}^N$ and then considering

$$\tilde{f} : X \to Y \times \mathbb{R}^N.$$

More generally, you could find some vector bundle V/Y and an embedding

$$\tilde{f} : X \to V$$

making the diagram



commute. Now apply the Pontrjagin–Thom construction: we get a map

$$Y^{\underline{N}} \to X^{\nu(\tilde{f})}$$

or more generally,

$$Y^V \to X^{\nu(\tilde{f})}. \tag{B.8}$$

All this appears to depend on the choice of $\tilde{f}$, but not as much as you might think. Note that

$$V + TY = TX + \nu(\tilde{f}),$$

and experience with K-theory makes you willing to rewrite this as

$$\nu(\tilde{f}) - V = TY - TX.$$

If $X \to Y$ is a fibre bundle then, with Tf the bundle of tangents along the fibre, we have

$$TX = TY + Tf$$

and so

$$TY - TX = -Tf$$

so in the end we can write

$$\nu(\tilde{f}) - V = -Tf.$$

A stable homotopy theorist will then tell you that the map (B.8) gives rise to stable map

$$Y_+ = Y^{\underline{0}} \xrightarrow{\tau(f)} X^{-Tf} \tag{B.9}$$

whose homotopy class depends only on $f : X \to Y$.

This is a generalization of the Pontrjagin–Thom map of an embedding: if $j : X \to Y$ is an embedding then

$$Tj = -\nu(j)$$

and so the map in (B.9) can equivalently be written

$$\tau(j) : Y_+ \to X^{\nu(j)},$$

and as such is the same as the map in (B.7).

B.4 The Umkehr map

Suppose that

$$f : X \to Y$$

is a proper map and E is a ring spectrum. Let $d = \dim X - \dim Y$ to $\dim Tf = d$. The Pontrjagin–Thom construction gives a map

$$\tau(f) : Y_+ \to X^{-Tf},$$

and so in E-theory we get a homomorphism

$$\tau(f)^* : \bar{E}^*(X^{-Tf}) \to \bar{E}^*(Y_+) \cong E^*(Y). \tag{B.10}$$

If $-Tf$ or equivalently Tf is *oriented* in E-theory, then we have a Thom isomorphism

$$\Phi : E^*(X) \cong \bar{E}^{*-d}(X^{-Tf})$$

and composing with the Pontrjagin–Thom map we get finally a homomorphism

$$f_\Phi = f_! : E^*(X) \to E^{*-d}(Y). \tag{B.11}$$

This is the "generalized integration map" in E-theory. When E is ordinary cohomology, then via the De Rham isomorphism $f_!$ corresponds to integration of differential forms over the fibre. The Atiyah–Singer index theorem interprets K-theory's $f_!$ as the index of a families elliptic operator.

The notation $f_!$ or f_* is fairly standard: see for example [21, 22]. One problem with the notation is that it does not indicate the dependence on the orientation Φ: this is like writing $\int f$ without indicating the volume form. In this paper we'll instead write f_Φ, to indicate the choice of Thom isomorphism.

In our applications, we will suppose that Φ is a stable, natural, and exponential family of orientations for complex vector bundles in K-theory. The compatibility of the family of orientations Φ implies that the associated Umkehr maps enjoy the following properties.

(1) Suppose $f : X \to Y$ and $g : Y \to Z$ are complex-oriented maps. We have an exact sequence

$$0 \to Tf \to Tgf \to f^*Tg \to 0.$$

If we allow Tgf to inherit a complex structure from Tf and Tg, then

$$g_\Phi f_\Phi = (gf)_\Phi. \tag{B.12}$$

(2) If

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{j} & Z \end{array}$$

is a pull-back diagram, then

$$g_\Phi i^* = j^* f_\Phi : K(X) \to K(Y; R). \tag{B.13}$$

(3) If $V \to X$ is a complex vector bundle, and

$$\zeta : X \to V$$

is its zero section, then

$$\zeta_\Phi : E^*(X) \to \bar{E}^{*+r}(X^V)$$

is the Thom isomorphism, and so

$$\zeta^* \zeta_\Phi(u) = u \cdot e_\Phi(V), \tag{B.14}$$

where $e_\Phi(V) = \zeta^*\zeta_\Phi(1)$ is the Euler class associated to the vector bundle V and orientation Φ.

If X is a manifold of dimension d, let π^X be the map to a point $\pi^X : X \to *$. A complex structure on π^X is a complex structure on TX, and given such a complex structure we can form

$$\pi_{\Phi}^{X} : E^{*}(X) \to E^{*-d}(*),$$

and so

$$\pi_{\Phi}^{X}(1) \in E^{-d}(*).$$

By Thom's theory of bordism, the rule

$$X \mapsto \pi_{\Phi}^{X}(1)$$

is a ring homomorphism

$$\Omega_{*}^{U} \to E^{-*}(*)$$

from the bordism ring of stably complex manifolds to $E^{*}(*)$. It is called the *genus* associated to the orientation Φ.

Let $V \to X$ be a complex vector bundle over X. Let s be a section of V, which intersects the zero section ζ transversely. Let $Z = s^{-1}(0)$, so we have a pull-back diagram

$$\begin{array}{ccc} Z & \xrightarrow{j} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \zeta} \\ Y & \xrightarrow{s} & V. \end{array}$$

Using the rules above we have

$$j_{\Phi} j^{*} u = s^{*} \zeta_{\Phi} u. \tag{B.15}$$

However, $s^{*} = \zeta^{*}$ since s and ζ are homotopic, and so we have

$$j_{\Phi} j^{*} u = \zeta^{*} \zeta_{!} u = u \cdot e_{\Phi}(V). \tag{B.16}$$

The "topological Riemann–Roch formula" studies how the Umkehr changes as the family of orientations Φ changes. Usually this is expressed in terms of a change of cohomology theories. We shall only need the case of ordinary cohomology and K-theory, so we state it in that case. Let

$$\mathrm{ch} : K(X) \to H^{\mathrm{even}}(X; \mathbb{Q})$$

be the Chern character.

Let Φ be a multiplicative family of orientations in K-theory. By the splitting principle, the following rules determine an exponential characteristic class for complex vector bundles

$$V/X \mapsto \mathcal{C}(V/X) \in H^{\mathrm{even}}(X;\mathbb{Q}):$$

(1) $\mathcal{C}(\underline{n}) = 1$;
(2) $\mathcal{C}(V \oplus W) = \mathcal{C}(V)\mathcal{C}(W)$ for any complex V, W over X;
(3) If L is a complex line bundle, then

$$\mathcal{C}(L) = \frac{c_1 L}{\operatorname{ch} e_\Phi(L)}.$$

Then we have the following, which you can glean from [23] and which is stated in explicit form in [22].

Proposition B.17. *Let $f : X \to Y$ be a proper complex oriented, and let Φ be a stable exponential family of complex orientations in K-theory. Then*

$$\operatorname{ch} f_\Phi(u) = \int_f \mathcal{C}(Tf) \operatorname{ch} u.$$

Example B.18. There is a family λ of complex orientations for K-theory with the property that

$$e_\lambda(V) = \Lambda_{-1}(\overline{V}).$$

The associated characteristic class is the Todd class

$$\operatorname{Todd}(V) = \prod_i \frac{x_i}{1 - \mathrm{e}^{-x_i}},$$

where the x_i are defined by

$$c(V) = \prod_i (1 + x_i).$$

The genus

$$\pi_\lambda^X(1) = \int_X \operatorname{Todd}(TX)$$

is the Todd genus.

Example B.19. Now suppose that R is a ring, and

$$\mu : K(X) \to K(X; R)^{\times}$$

is a characteristic class of complex vector bundles satisfying

$$\mu(\underline{n}) = 1$$

and

$$\mu(V + W) = \mu(V)\mu(W).$$

Then μ determines a stable exponential family Φ of orientations for complex vector bundles in K-theory by the formula

$$\Phi(V) = \lambda(V)\mu(V).$$

Now the Riemann–Roch formula gives

$$\operatorname{ch} f_{\Phi}(u) = \int_f \operatorname{Todd}(Tf) \operatorname{ch}(\mu(V)^{-1} u). \tag{B.20}$$

Example B.21. For example, we can set $R = \mathbb{Z}[\![q]\!]$ and

$$\mu(V) = \prod_{n \geq 1} \Lambda_{-q^n}(V^{\mathbb{C}}).$$

We write σ for the family of orientations of complex vector bundles given by

$$\sigma(V) = \lambda(V) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}(V^{\mathbb{C}}).$$

Suppose that

$$c(TX) = \prod_i (1 + x_i).$$

The resulting genus is

$$\begin{aligned} \pi_{\Phi}^{X}(1) &= \int_X \operatorname{Todd}(X) \operatorname{ch}\left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}((TX)^{\mathbb{C}}) \right) \\ &= \int_X \operatorname{Todd}(X) \prod_i \prod_{n \geq 1} \frac{1}{(1 - q^n e^{x_i})(1 - q^n \mathrm{e}^{-x_i})}. \end{aligned}$$

If X is an SU-manifold so that its A-hat class and Todd class coincide, then (up to factor depending only on the dimension of X) this is the "Witten genus" of X.

B.5 Thom classes for Landau–Ginzburg models

Let z denote the standard complex representation of S^1, and so the complex representation ring of S^1 is $R[S^1] \cong \mathbb{Z}[z, z^{-1}]$. Let A denote S^1-equivariant K-theory with coefficients in the ring $\mathbb{Z}[\![q^{1/2}]\!]$, so

$$A(*) \cong R[S^1][\![q^{1/2}]\!] \cong \mathbb{Z}[z, z^{-1}][\![q^{1/2}]\!].$$

If V is a complex vector bundle, let $z^n V$ be V considered as an S^1-equivariant vector bundle with the circle acting by $z \mapsto z^n$.

Let

$$\mu : K(X) \to A(X)^{\times}$$

be the exponential characteristic class given by

$$\mu(V) = \bigotimes_{k=1/2,3/2,\ldots} S_{-q^n}((z^{-1}V)^{\mathbb{C}}).$$

Φ be the orientation for complex vector bundles in A-theory given by

$$\Phi(V) = \sigma(V)\mu(V),$$

where σ is the orientation of Example B.21. If Y is a manifold with a complex tangent bundle, then the Riemann–Roch formula gives

$$\pi^Y_{\Phi}(x) = \int_Y \mathrm{Todd}(TY)\,\mathrm{ch} \times \left(x \bigotimes_{n=1,2,3,\ldots} S_{q^n}(TY^{\mathbb{C}}) \bigotimes_{n=1/2,3/2,\ldots} \Lambda_{q^n}((z^{-1}TY)^{\mathbb{C}}) \right). \tag{B.22}$$

Setting $x = 1$ gives the elliptic genus on Y discussed in Section 3.1; see (3.3).

Now suppose that $\mathcal{G}$ is a complex vector bundle over a manifold B. Let ζ denote its zero section, and suppose that G is another section which intersects ζ transversely. Let $Y = G^{-1}(0)$ so that we have a pull-back diagram of the form

$$\begin{array}{ccc} Y & \xrightarrow{j} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \zeta} \\ B & \xrightarrow{s} & \mathcal{G}, \end{array} \tag{B.23}$$

In this situation,

$$\nu(j) = j^*\nu(\zeta) = \mathcal{G}|_Y.$$

We have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & B \\ & {\scriptstyle \pi^Y}\searrow & \downarrow{\scriptstyle \pi^B} \\ & & *, \end{array}$$

and so equation (B.12) implies that

$$\pi^Y_\Phi(x) = \pi^B_\Phi j_\Phi(x).$$

We shall apply this formula in the case that

$$x = j^*u,$$

in which case (B.16) implies that

$$\begin{aligned} j_\Phi j^* u = u e_\Phi(\mathcal{G}) = u e_\sigma(\mathcal{G})\mu(\mathcal{G}) &= u\Lambda_{-1}(\overline{\mathcal{G}}) \bigotimes_{n=1,2,\ldots} \Lambda_{-q^n}(\mathcal{G}^{\mathbb{C}}) \\ &\times \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}((z^{-1}\mathcal{G})^{\mathbb{C}}), \end{aligned}$$

so that

$$\begin{aligned} \pi^Y_\Phi(j^*u) &= \pi^B_\Phi j_\Phi(j^*u) \\ &= \pi^B_\Phi \left(u\Lambda_{-1}(\overline{\mathcal{G}})\Phi(\mathcal{G})\right) \end{aligned}$$

$$= \int_B \text{Todd}(TB)$$
$$\times \wedge \text{ch} \left(u \bigotimes_{n=1,2,3,\ldots} S_{q^n}(TB^{\mathbb{C}}) \bigotimes_{k=1/2,3/2,\ldots} \Lambda_{q^n}((z^{-1}TB)^{\mathbb{C}}) \right)$$
$$\times \wedge \text{ch} \left(\Lambda_{-1}(\overline{\mathcal{G}}) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}(\mathcal{G}^{\mathbb{C}}) \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}((z^{-1}\mathcal{G})^{\mathbb{C}}) \right). \tag{B.24}$$

Recalling that for a complex vector bundle V we have $V^{\mathbb{C}} \cong V \oplus \bar{V}$, we see that this expression matches (3.4).

Example B.25. The situation in Section 3.1 arises when Y is a generic quintic hypersurface in $P = \mathbb{P}^4$: so s is a generic quintic and $\mathcal{G} = \mathcal{O}(5)$. Then

$$\overline{\mathcal{G}} = \mathcal{O}(-5)$$

and

$$\lambda(\mathcal{G}) = \Lambda_{-1}(\overline{\mathcal{G}}) = 1 - \mathcal{O}(-5).$$

Also setting $u = 1$ and comparing with (B.22), (B.24) becomes

$$\int_Y \text{Todd}(TY)\,\text{ch} \left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}(TY^{\mathbb{C}}) \bigotimes_{k=1/2,3/2,\ldots} \Lambda_{q^n}((zTY)^{\mathbb{C}}) \right)$$
$$= \int_P \text{Todd}(TP)\,\text{ch} \left(\bigotimes_{n=1,2,3,\ldots} S_{q^n}(TP^{\mathbb{C}}) \bigotimes_{k=1/2,3/2,\ldots} \Lambda_{q^n}((zTP)^{\mathbb{C}}) \right)$$
$$\times\, \text{ch} \left((1 - \mathcal{O}(-5)) \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}(\mathcal{O}(5)^{\mathbb{C}}) \bigotimes_{k=1/2,3/2,\ldots} S_{-q^n}(z(\mathcal{O}(5))^{\mathbb{C}}) \right). \tag{B.26}$$

One checks easily that (B.26) implies that the genera (B.22) and (B.24) coincide.

B.6 The formula for a complex of vector bundles

We continue to suppose that $\mathcal{G}$ is a line bundle on B, with a section G which intersects ζ transversely. We let $Y = G^{-1}(0)$, and we write $j : Y \to B$ for the inclusion. Once again we have $\nu(j) = \mathcal{G}$.

Now we suppose that

$$\mathcal{F} = \left(\mathcal{F}_0 \xrightarrow{s_0} \mathcal{F}_1 \xrightarrow{s_1} \mathcal{F}_2\right)$$

is a complex of vector bundles on B : we suppose that $s_i s_{i-1} = 0$, but we do not suppose that the complex is exact. Let

$$\begin{aligned}\mathcal{E}_0 &= \operatorname{Ker} s_0,\\ \mathcal{E}_1 &= \operatorname{Ker} s_1 / \operatorname{Img} s_0,\\ \mathcal{E}_2 &= \mathcal{F}_2 / \operatorname{Img} s_1.\end{aligned}$$

In the physical situation studied in Section 3.5 it is convenient to reverse signs and set

$$\mathcal{E}' = -\mathcal{E}_0 + \mathcal{E}_1 - \mathcal{E}_2.$$

If $V_0, \ldots, V_n$ is a sequence of complex vector bundles, then we define

$$\Lambda_t(V) = \Lambda_t\left(\sum_i (-1)^i V_i\right) \cong \bigotimes \Lambda_t(V_i)^{(-1)^i}$$

and similarly for $S_t V$.

Lemma B.27. *We have*

$$\mathcal{F}_0 - \mathcal{F}_1 + \mathcal{F}_2 = \mathcal{E}_0 - \mathcal{E}_1 + \mathcal{E}_2$$

in $K(B)$, and so

$$S_{-q^n}(z^{-1}\mathcal{F}) = S_{-q^n}(z^{-1}\mathcal{E}) = \Lambda_{q^n}(z^{-1}\mathcal{E}') \qquad \text{(B.28)}$$

in $K_{S^1}(B; \mathbb{Z}[\![q]\!])$.

Proof. The sections s_i lead to decompositions

$$\begin{aligned}
\mathcal{F}_0 &\cong \operatorname{Ker} s_0 + (\operatorname{Ker} s_0)^{\perp} \cong \mathcal{E}_0 + \operatorname{Img} s_{0,} \\
\mathcal{F}_1 &\cong \operatorname{Ker} s_1 + (\operatorname{Ker} s_1)^{\perp} \cong \operatorname{Img} s_0 + \mathcal{E}_1 + \operatorname{Img} s_1, \\
\mathcal{F}_2 &\cong \operatorname{Img} s_1 + \mathcal{E}_2.
\end{aligned}$$

□

Proposition B.29. *In this situation, we have*

$$\begin{aligned}
&\int_Y \operatorname{Todd}(Y) \wedge \operatorname{ch}\left(\bigotimes_{n=1,2,\cdots} S_{q^n}\left((TY)^{\mathbb{C}}\right) \bigotimes_{n=1/2,3/2,\cdots} \Lambda_{q^n}((z^{-1}\mathcal{E}')^{\mathbb{C}}) \right) \\
&= \int_B \operatorname{Todd}(B) \wedge \operatorname{ch}\left(\bigotimes_{n=1,2,\cdots} S_{q^n}\left((TB)^{\mathbb{C}}\right) \bigotimes \Lambda_{-1}(\bar{\mathcal{G}}) \bigotimes_{n=1,2,\ldots} \Lambda_{-q^n}(\mathcal{G}^{\mathbb{C}}) \right. \\
&\quad \left. \times \bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}((z^{-1}\mathcal{F})^{\mathbb{C}}) \right).
\end{aligned}$$

In particular, this shows the elliptic genera (3.15) and (3.16) coincide.

Proof. We calculate the pushforward using the multiplicative orientation σ of Example B.21, whose Euler class is

$$e_\sigma(V) = \Lambda_{-1}(\bar{V}) \bigotimes_{n\geq 1} \Lambda_{-q^n}(V^{\mathbb{C}}).$$

As always we have

$$\pi^Y_\sigma(x) = \pi^B_\sigma(j_\sigma(x)). \tag{B.30}$$

We apply this formula to

$$x = j^* u = j^* \left(\bigotimes_{n=1/2,3/2,\ldots} S_{-q^n}(z^{-1}\mathcal{E})^{\mathbb{C}} \right),$$

so

$$j_\sigma j^* u = u e_\sigma(\mathcal{G}). \tag{B.31}$$

using (B.16). Thus

$$\pi_\sigma^Y \left(\bigotimes_{n=1/2,3/2,\dots} S_{-q^n}(z^{-1}\mathcal{E})^{\mathbb{C}} \right) \\ = \pi_\sigma^B \left(\Lambda_{-1}(\bar{\mathcal{G}}) \bigotimes_{n\geq 1} \Lambda_{-q^n}(\mathcal{G}^{\mathbb{C}}) \bigotimes_{n=1/2,3/2,\dots} S_{-q^n}(z\mathcal{E})^{\mathbb{C}} \right). \tag{B.32}$$

Using the Lemma, we may replace $S_{-q^n}((z^{-1}\mathcal{E})^{\mathbb{C}})$ with $S_{-q^n}((z^{-1}\mathcal{F})^{\mathbb{C}})$ or $\Lambda_{q^n}((z^{-1}\mathcal{E}')^{\mathbb{C}})$. Doing so and using the Riemann–Roch formula (Proposition B.17) to rewrite both sides as integrals after applying the Chern character gives the equation in the statement of the Proposition. □

B.7 R-sector genera

It is easy to adapt the methods of the preceding sections to the R-sector genera studied in this paper. For definiteness we compare the genera (3.5) and (3.6), and we describe the essential idea in a slightly different way from the preceding sections. Let Y be a submanifold of B with normal bundle $\mathcal{G}$. The intersection theory argument we have been using shows in general that

$$\int_Y \mathrm{Todd}(Y)\,\mathrm{ch}\,(\xi(TY)) = \int_B \mathrm{Todd}(B)\,\mathrm{ch}\left(\Lambda_{-1}(\mathcal{G}^\vee)\xi(\mathcal{G})^{-1}\xi(TB)\right),$$

and

$$\int_Y \hat{A}(Y)\,\mathrm{ch}\,(\xi(TY)) = \int_B \hat{A}(B)\,\mathrm{ch}\left((\Delta_+(\mathcal{G}^\vee) - \Delta_-(\mathcal{G}^\vee))\xi(\mathcal{G}^\vee)^{-1}\xi(TB)\right).$$

Here, ξ is an exponential function

$$KO(X) \to K(X;R)^\times$$

for some ring of coefficients R.

For example, consider the R-sector NLSM genus (3.5)

$$\int_Y \mathrm{Todd}(TY) \wedge \mathrm{ch}\left(z^{-y/2} \, (\det TY)^{1/2} \, \Lambda_1(zT^*Y) \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TY)^{\mathbf{C}}) \right.$$
$$\left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}TY)^{\mathbf{C}}) \right)$$

and its Landau–Ginzburg couterpart (3.6)

$$q^0 \int_B \mathrm{Todd}(TB) \wedge \mathrm{ch}\left(z^{+b/2} z^{-(b-y)/2} \Lambda_1(z^{-1}TB) \otimes \Lambda_{-1}(\mathcal{G}^\vee) \otimes \Lambda_1(z\mathcal{G}^\vee) \right.$$
$$\times (\det T^*B)^{1/2} \left(\det \mathcal{G}^\vee\right)^{-1/2} \bigotimes_{n=1,2,3,\ldots} S_{q^n}((TB)^{\mathbf{C}})$$
$$\times \bigotimes_{n=0,1,2,\ldots} S_{-q^n}((z\mathcal{G}^\vee)^{\mathbf{C}}) \bigotimes_{n=1,2,3,\ldots} \Lambda_{q^n}((z^{-1}TB)^{\mathbf{C}})$$
$$\left. \times \bigotimes_{n=1,2,3,\ldots} \Lambda_{-q^n}((\mathcal{G}^\vee)^{\mathbf{C}}) \right). \tag{B.33}$$

The factor

$$z^{-y/2} \det(TY)^{1/2} \Lambda_1(zT^*Y) \cong z^{y/2} \det(T^*Y)^{1/2} \Lambda_1(z^{-1}TY)$$

in the Y integral corresponds to a factor

$$z^{b/2} \det(T^*B)^{1/2} \Lambda_1(z^{-1}TB) z^{-(b-y)/2} \det(\mathcal{G}^\vee)^{-1/2} S_{-1}(z^{-1}\mathcal{G}),$$

in the B integral. The S_{-1} part comes from the $n = 0$ case of $S_{-q^n}((z\mathcal{G}^\vee)^{\mathbb{C}})$, which contributes

$$S_{-1}(z\mathcal{G}^\vee) \otimes S_{-1}(z^{-1}\mathcal{G}).$$

The second factor is the one we need, while the first is cancelled by the factor $\Lambda_{-1}(z\mathcal{G}^\vee)$ in the first line of the B integral.

Similar remarks apply to the R-sector genera involving complexes of vector bundles; the proof is left to the interested reader.

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