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Causal posets, loops and the construction of nets of local algebras for QFT

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Abstract

We provide a model independent construction of a net of C*-algebras satisfying the Haag–Kastler axioms over any spacetime manifold. Such a net, called *the net of causal loops*, is constructed by selecting a suitable base K encoding causal and symmetry properties of the spacetime. Considering K as a partially ordered set (poset) with respect to the inclusion order relation, we define groups of closed paths (loops) formed by the elements of K. These groups come equipped with a causal disjointness relation and an action of the symmetry group of the spacetime. In this way, the local algebras of the net are the group C*-algebras of the groups of loops, quotiented by the causal disjointness relation. We also provide a geometric interpretation of a class of representations of this net in terms of causal and covariant connections of the poset K. In the case of the

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Minkowski spacetime, we prove the existence of Poincaré covariant representations satisfying the spectrum condition. This is obtained by virtue of a remarkable feature of our construction: any Hermitian scalar quantum field defines causal and covariant connections of K. Similar results hold for the chiral spacetime S^1 with conformal symmetry.

Contents

1 Introduction

Quantum field theory (QFT) is nowadays a framework for understanding the physics of relativistic quantum systems, notably elementary particles, and the effects of gravitation on quantum systems, at least in the approximation in which the gravitational field can be treated as a background field. Several mathematical approaches are appropriate for this aim: operator valued distributions; probability measures on the space of distributions; causal and covariant nets of operator algebras. All these approaches, distinguished from others because non-perturbative, are strictly related and result to be, in the Minkowski spacetime and under suitable assumptions, equivalent (see [10, 12, 22]).

The mathematically rigorous construction of quantum fields is object of research since the middle fifties. Many important ideas have been developed throughout these years although the main aim, i.e., the construction of a non-perturbative interacting quantum field in the four-dimensional (4D) Minkowski spacetime, is still an open problem. In the present paper we give a novel construction of quantum fields which applies on any meaningful spacetime. We shall work in the context of the algebraic quantum field theory (AQFT), the last approach among those listed above.

The main idea of AQFT [25], is that the physical content of a theory is encoded in the *observable net*: the correspondence between a suitable family K of bounded regions of the spacetime and the C*-algebras generated by observables measurable within the elements of K. This correspondence is assumed to preserve the inclusion of regions, to satisfy Einstein causality and to be covariant with respect to the symmetry group of the spacetime (Haag–Kastler axioms). In the Minkowski spacetime, the just described scenario turns out to be appropriate for understanding, in terms of superselection sectors of the observable net, the charge structure and the statistics of particle physics [3, 17], as well as the presence of *a global* gauge group acting upon a net of unobservable quantum fields [19].[1]

[1] Charges of electromagnetic type do not fit in superselection sectors analyzed [11,17,19]. Progresses in this direction should appear soon [18].

It is worth pointing out the relevance of the set of regions K of the spacetime: this is usually called *the set of indices* of the observable net and is took to be a base of the topology of the spacetime itself. It is also required that it may codifies the symmetry and the causal structure of the spacetime. Examples of set of indices K are double cones for the Minkowski spacetime; open intervals for the circle S^1; diamonds for an arbitrary globally hyperbolic spacetime. The relevance of K, as a partially ordered set (poset) with respect to the inclusion order relation, has been strengthened by the formulation of the theory of superselection sectors in terms of the non-Abelian cohomology of K (see [30] for a review); this also allowed to extend the analysis of sectors to curved spacetimes [24, 35] and to introduce a generalized notion of sector associated to invariants of the spacetime described by the poset K [11].

Concerning the construction of models in the AQFT framework, apart from the well known approach of the Weyl algebras, i.e., a version of canonical quantization, it is worth pointing out the technique of modular localization [5] (based on the ideas in [38]) allowing to construct a causal and covariant net of operator algebras from a given representation of the Poincaré group. However, important progresses, especially towards the construction of interacting theories, have recently been achieved by the introduction of a new approach based on operator algebras deformation [23] (see also [7]). Finally, it is worth mentioning the abstract construction of Borchers–Uhlmann [9, 42], where a causal and covariant net of *-algebras over the Minkowski spacetime is constructed using merely test functions, and the properties of the localization of their supports. A remarkable feature of this method is that quantum fields satisfying the Wightmann axioms are in bijective correspondence with a class of positive linear functionals of this net.

Our work is motivated by quantum gauge theories, a scenario where meaningful quantities are obtained evaluating fields over paths. The aim of the present paper is to construct a causal and covariant net of C*-algebras over any spacetime, using paths as the main ingredient. This is done in a model independent way, regardless of any specific theory of quantum field over the spacetime itself. In this way, we give a new construction of quantum field by using as input merely the spacetime (to be precise, the base K) and its causal and symmetry structure; at the same time, we hope to capture some structural properties of any reasonable formulation of a quantum gauge theory in the framework of AQFT.

The notion of path that we use here is that used in the context of abstract posets [30, 31]. Given a base K of a spacetime ordered under inclusion, a path in K can be figured out as an approximation of a path γ, intended in the usual topological way, by means of a finite sequence of elements of K covering it. This notion of path is good enough to capture non-trivial invariants of the spacetime, as the fundamental group [35] and the category of flat bundles [33]; moreover, paths are the key for the definition of connection over a poset [32], in essence a group valued map defined on the set of paths of K, designed to give a version of the basic structures of gauge theory in the setting of AQFT.

The first step of our construction is made in Sections 2 and 3. We define the category **Causet** of posets endowed with a causal disjointness relation (*causal posets*) as model of bases of spacetimes. Afterwards, we construct a *causal functor* from **Causet** to the category of C*-algebras.[2] This is done by assigning to any causal poset the group C*-algebra of the free group generated by closed paths (*loops*) quotiented by an induced causal disjointness relation. Using this functor we show that a causal net of C*-algebras can be constructed over any causal poset and, consequently, over any spacetime. This net, called *the net of causal loops*, turns out to be covariant under any symmetry group of the spacetime. We show the non-triviality of the net of causal loops for the circle S^1, for the Minkowski spacetime and, more in general, for an arbitrary globally hyperbolic spacetime.

Concerning the representation theory, the net of causal loops is universal with respect to the notion of connection given in [32]: this means that a connection defined on a Hilbert space which is causal and covariant, in a suitable sense, yields a representation of the net. An interesting point is that it seems that not all the representations of the net arise from connections. Indeed, using a procedure similar to the reconstruction of the potential from the free quantum electromagnetic field, we associate to any representation a *connection system*: a system of connections where causality and covariance arise as properties of the system and not of a single connection, Section 4.3. This also leads to a natural notion of gauge transformations.

The passage from a function of loops, the representation, to a function of paths, the connection, relies on a sort of reference frame for the poset K: the *path-frame*, a choice of a collection of paths joining any element of K to a fixed one, the pole. The resulting connection depends on this choice and turns out to be neither causal nor covariant. However, when the poset admits a covariant family of path-frames, then we obtain the connection system mentioned above. In general, this procedure is subjected to restrictions due to the topology of the spacetime and to the symmetry group. In particular, the Minkowski spacetime with the Poincaré symmetry

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[2] A causal functor generalizes the functor underlying the definition of a locally covariant QFT [4]. The only difference is that the source category of the latter is that of 4D globally hyperbolic spacetimes. Since a causal poset can be associated to any spacetime, a locally covariant QFT can be obtained from any causal functor.

group admits a covariant family of path-frames whilst S^1 with the conformal symmetry group does not.

The construction of the net of causal loops is purely combinatorial and no topological condition is imposed, even if the symmetry group is a topological group. An important result in this regards is that, in the Minkowski spacetime, the net of causal loops admits Poincaré covariant representations fulfilling the spectrum condition, Section 4.5. This is obtained by a remarkable feature of our construction: any Hermitian scalar quantum field yields a causal and Poincaré covariant connection, which results to be manifestly non-trivial in the case of the free scalar field. Similar results hold for the net of causal loops over the chiral spacetime S^1 with conformal symmetry.

2 Causal posets, functors and nets of C*-algebras

The order relation structure enters the theory of quantum fields in the correspondence $o \mapsto \mathcal{A}_o$, known as *the observable net*, associating to the bounded region o of the spacetime the algebra $\mathcal{A}_o$ generated by the observables measurable within o. This correspondence is clearly *isotonous* with respect to the inclusion of regions of the spacetime. On the other hand, for the applications to QFT, it is enough to restrict this correspondence to a family K of regions, usually called *the set of indices*, encoding topological, causal and symmetry properties of the underlying spacetime.[3] These aspects lead to the notion of a causal poset with a symmetry: the set of indices K has to be considered as a poset because of the isotony property of the observable net; symmetry and causality of the spacetime lift to corresponding properties of the poset K. From this point of view, the observable net is nothing but a functor from the poset K, considered as a category, to the category of C*-algebras. This functor preserves, in a suitable sense, the causal and the symmetry property of K.

In this section, taking cue from the locally covariant QFT [4], we generalize the above ideas and introduce the notion of a causal functor: a functor from the category formed by causal posets to the one of C*-algebras. We show that if a causal functor is given, then a causal net of C*-algebras is associated to any spacetime. The existence and the construction of a causal functor shall be given in Section 3.

[3]This correspondence underlies the QFT and is the corner stone of AQFT, also known as local quantum physics [1, 2, 6, 22, 25, 30, 41].

2.1 Causal posets

A poset is a non-empty set K endowed with a (binary, reflexive, transitive and antisymmetric) order relation $\leq$. We say that K is *upward directed* whenever for any $a, a' \in K$ there is $o \in K$ such that $a, a' \leq o$. A poset K is *pathwise connected* if for any pair $a, a' \in K$ there are two finite sequences $a_1, \ldots a_{n+1}$ and $o_1, \ldots o_n$ of elements of K, with $a_1 = a$ and $a_{n+1} = a'$, satisfying the relations

$$a_i, a_{i+1} \leq o_i, \quad i = 1, \ldots, n. \tag{2.1}$$

Note that an upward directed poset is pathwise connected.

A *causal disjointness relation* on a poset K is an irreflexive, symmetric binary relation $\perp$ on K stable under the order relation, given $o, a, \tilde{o} \in K$

$$o \perp a, \quad \widetilde{o} \leq o \quad \Rightarrow \quad \widetilde{o} \perp a. \tag{2.2}$$

Note that irreflexivity ($\nexists a \in K$ such that $a \perp a$) and the stability under the order relation imply that two causally disjoint elements $a \perp o$ do not have a common minorant, i.e., $\nexists\ x \in K$ such that $x \leq o, a$.

We call any pathwise connected poset equipped with a causal disjointness relation a **causal poset**.

Remark 2.1. Two observations are in order.

1. The term *causal disjointness relation* is used because usually we deal with a poset arising as a family, ordered under inclusion, of subsets of a globally hyperbolic spacetime. Two events of such spacetimes are *causally disjoint* if they cannot be joined by any non-degenerate causal curve and this relation extends naturally to the subsets of the spacetime. Notice, however, that in particular cases causal disjointness equals the set-disjointness. This happens for instance when one considers subsets laying on a Cauchy surface. Furthermore, this is also the case of the circle S^1, the spacetime of chiral theories, that is obtained as the compactification of one of the lightline of the 2D Minkowski spacetime.

2. It is worth stressing the difference between the notion of a *causal poset* and that of a *causal set*. The causal set approach to quantum gravity, (see [39, 40]), is based on the idea that the spacetime is a discrete set C of events e related by an order relation $\preceq$ describing the causal relation between the events of the spacetime (discrete means that the poset $(C, \preceq)$ is locally finite). Hence, the notion of causal set is more primitive than that of

a causal poset. In fact, let K be the set of the finite collections o of elements of C, ordered under inclusion. Define $o_1 \perp o_2$ if, and only if, for any element $e_1 \in o_1$ there do not exist $e_2 \in o_2$ such that either $e_2 \preceq e_1$ or $e_1 \preceq e_2$. Then $\perp$ is a causal disjointness relation on K.

A *morphism* from a causal poset K to a causal poset K' is a function $\psi : K \to K'$ preserving both the order relation and the causal disjointness relation i.e.,

$$a \leq \tilde{a} \quad \Rightarrow \quad \psi(a) \leq' \psi(\tilde{a}), \quad o \perp \tilde{o} \quad \Leftrightarrow \quad \psi(o) \perp' \psi(\tilde{o}), \tag{2.3}$$

where $\leq'$ and $\perp'$ denote, respectively, the order relation and the causal disjointness relation of K'. A *symmetry group* of a causal poset is noting but a subgroup of the automorphism group of a causal poset.

Note that there is an evident categorical structure underlying these definitions: taking causal posets K as objects and the corresponding morphisms $\psi : K \to K'$ as arrows (K, K'), we get a category **Causet** that we call *the category of causal posets*. In particular, symmetry groups of K are, by definition, subgroups of (K, K).

2.1.1 Some causal posets on spacetimes for AQFT

We give a non-exhaustive list of causal posets associated to different spacetimes, which are of importance in AQFT. Our main aim in the following sections shall be to give constructions that result to be well behaved on such posets.

The poset K used as the set of indices of the net of C*-algebras in AQFT is a family of subsets of the spacetime ordered under inclusion. This family is chosen to best fit the topological, causal and the symmetry, if there are any, properties of the spacetime. To this end, some minimal requests are the following: (1) K is a base for the topology of the spacetime whose elements are connected and simply connected subsets of the spacetime; (2) the causal complements of elements of K are connected; (3) if S is a symmetry group of the spacetime (a group of global isometries, or conformal diffeomorphisms for instance), then K is stable under the action of S, i.e., $s(o) \in K$ for any $o \in K$ and s in S, where $s(o)$ is the image of o by the map s. In this way S is realized as a subgroup of (K, K).

We introduce some causal posets associated to spacetimes, which are of relevance in AQFT.

1. *Globally hyperbolic spacetimes.* Let $\mathcal{M}$ be an arbitrary globally hyperbolic spacetime, the spacetime used in the setting of quantum fields in a

gravitational background field [24, 43]. In such a spacetime two regions are causally disjoints if they cannot be joined by any non-degenerate causal curve. A good choice for the set of indices is the set of *diamonds* of $\mathcal{M}$: roughly, a diamond is the open causal completion of a suitable subset laying on a Cauchy surface of $\mathcal{M}$, (see [11, 35]) for details. The set of diamonds is stable under any global symmetry of the spacetime, if there are any.

2. *The Minkowski spacetime.* A particular globally hyperbolic spacetime is the Minkowski spacetime $\mathbb{M}^4$, the spacetime used in the setting of relativistic QFT. The spacelike separation is the causal disjointness relation and the symmetry group is the Poincaré group $\mathscr{P}_+^\uparrow$. The set of indices used in such theories is that of *double cones* of $\mathbb{M}^4$, a family smaller than that of diamonds of $\mathbb{M}^4$ (see [25]). These may be defined as follows. Given a reference frame of the Minkowski spacetime, let B_R be the open ball with radius $R > 0$ laying on the subspace at time $t = 0$ and centred in the origin of the reference frame. The open causal completion of B_R, we shall denote by o_R, is called a double cone with base B_R. Any other double cone is obtained by letting act the Poincaré group on the double cones o_R for any $R > 0$. For later applications, we note that the Poincaré group does not act freely on the set of double cones. It is easily seen that the stability group of the double cone o_R defined before is the subgroup $SO(3)$ of $\mathscr{P}_+^\uparrow$ of spatial rotations; so the stability group of a generic double cone is isomorphic to $SO(3)$.

3. *The circle S^1.* This is the space used in the setting of chiral conformal QFT, (see [27]) for a friendly introduction. A symmetry group is that of Möbius group (or the larger group of the diffeomorphisms of S^1) and the causal disjointness relation is the set-disjointness. In this case the set of indices is the set of non-empty open intervals of S^1 having a proper closure.[4] For later application, we observe that the Möbius group does not act freely on intervals of S^1. In particular, the stabilizer of an interval o, which is also the stabilizer of its open complement $S^1 \setminus cl(o)$, is isomorphic to the modular group.

The causal posets listed above, although defined on different spacetimes, share some important properties (see the references quoted above):

(1) For any o there exist a, x such that $a \Subset o \Subset x$.
(2) If $a \Subset o$, then exists $\tilde{a}$ such that $a \Subset \tilde{a} \Subset o$.
(3) For any o there exists a such that $\mathrm{cl}(a) \perp o$.
(4) If $\mathrm{cl}(a) \perp o$ there exists x such that $a \Subset x$ and $\mathrm{cl}(x) \perp o$.

[4] The set of non-empty open intervals of $\mathbb{R}^1$ is also considered, both for the analysis on the chiral lines and for the time-zero formulation of a theory on the 2-d Minkowski space, see [15] for the example of the massless scalar free field.

Here, $\mathrm{cl}(a)$ stands for the closure of a in the spacetime topology, and $a \Subset o$ stands for $\mathrm{cl}(a) \subset o$. Note that any such a poset is infinite non-countable. Furthermore, as can be easily deduced by the above relations, any such poset has neither maximal nor minimal elements.

2.2 Causal functors

Roughly speaking a *causal functor* is a functor from **Causet** to a target category in which causality is realized in terms of commutation relations. For our purposes, natural choices of the target category shall be the ones of groups and of C*-algebras; in the C*-case, the notion of causal functor generalizes the one of locally covariant QFT (see [4]).

As a preliminary step, we introduce target categories of the causal functor, generically denoted by **T**:

(i) the category **Grp** whose objects are groups and whose arrows are group monomorphisms;
(ii) the category **C*Alg** whose objects are unital C*-algebras and whose arrows are unital *-monomorphisms.

We now are ready to give the definition of a causal functor.

Definition 2.2. A **causal functor** is a functor $\mathcal{A} : \mathbf{Causet} \to \mathbf{T}$ fulfilling the following property: given $\psi_i \in (K_i, K)$, $i = 1, 2$, if there are $o_1, o_2 \in K$ such that

$$o_1 \perp o_2 \quad \textit{and} \quad \psi_1(K_1) \leq o_1 \quad \psi_2(K_2) \leq o_2, \tag{2.4}$$

then

$$\mathcal{A}_{\psi_1}(A)\, \mathcal{A}_{\psi_2}(B) = \mathcal{A}_{\psi_2}(B)\, \mathcal{A}_{\psi_1}(A), \tag{2.5}$$

for any $A \in \mathcal{A}(K_1)$ and $B \in \mathcal{A}(K_2)$. When **T** is **Grp** (**C*Alg**) we shall say that $\mathcal{A}$ is a group- (C*-) causal functor.

Some comments about the above definition. First, the functor $\mathcal{A}$ is denoted by $\mathcal{A}(K)$, $K \in \mathbf{obj}(\mathbf{Causet})$, at the level of objects and by $\mathcal{A}_\psi$, $\psi \in \mathbf{arr}(\mathbf{Causet})$ at the level of arrows. Secondly, $\psi_i(K_i) \leq o_i$ means that any element of $\psi_i(K_i)$ is smaller that o_i, for $i = 1, 2$. So if two posets have causally disjoint embeddings in a larger poset, then the corresponding images, via the causal functor, commute elementwise.

The functor assigning the full group C*-algebra to any discrete group allows us to assign a C*-causal functor to any group causal functor. To this end, a key rôle is played by the discreteness of groups involved. Very briefly, given the discrete group G we denote the convolution algebra by $\ell^1(G)$ and its enveloping C*-algebra by C^*G. A dense *-algebra of $\ell^1(G)$ is given by the set $\mathbb{C}G$ of functions $f : G \to \mathbb{C}$, which can be expressed as linear combinations $f = \sum_g f(g)\,\delta_g$ of delta functions. Given the discrete group morphism $\sigma : G \to H$ there are natural induced morphisms

$$\tilde{\sigma} : \mathbb{C}G \to \mathbb{C}H, \quad \sigma_* : \mathrm{C}^*G \to \mathrm{C}^*H,$$

defining the functor

$$\mathrm{C}^* : \mathbf{Grp} \to \mathbf{C^*Alg}, \quad G \mapsto \mathrm{C}^*G,\ \sigma \mapsto \sigma_*.$$

A crucial point is that σ_* is injective for any monomorphism σ; this is not ensured when G is not discrete (for details see the discussion in [36, Section 3.2.2]).

Remark 2.3. Two observations are in order.

1. There is a bijective correspondence between unitary representations ρ of G and representations π of C^*G. In fact $\tilde{\rho}(f) := \sum f(g)\rho(g)$ extends to a continuous representation of the algebra $\ell^1(G)$. This in turn induces a representation ρ_* of C^*G. Conversely, note that there is a group monomorphism $G \in g \to \delta_{*,g} \in \mathrm{C}^*G$, where $\delta_{*,g}$ is the embedding of δ_g into C^*G. Therefore if π is a representation of $\mathrm{C}^*(G)$, then $\pi^*(g) := \pi(\delta_{*,g})$ is a unitary representation of G. These two procedures are each other's inverses.

2. Given injective group morphisms $\sigma_i : G_i \to H$, $i = 1, 2$, such that the groups $\sigma_1(G_1)$ and $\sigma_2(G_2)$ commute elementwise within H, then the C*-algebras $\sigma_{*,1}(\mathrm{C}^*G_1)$ and $\sigma_{*,2}(\mathrm{C}^*G_2)$ commute elementwise within C^*H. This is quite evident in terms of the delta functions: given $\delta_{g_i} \in \mathbb{C}G_i$ for $i = 1, 2$, then $\tilde{\sigma}_1(\delta_{g_1}) * \tilde{\sigma}_2(\delta_{g_2}) = \delta_{\sigma_1(g_1)\sigma_2(g_2)} = \delta_{\sigma_2(g_2)\sigma_1(g_1)} = \tilde{\sigma}_2(\delta_{g_2}) * \tilde{\sigma}_1(\delta_{g_1})$ and the general proof follows by density.

On these grounds, if $\mathcal{G}$ is a group-causal functor, then the composition

$$\mathcal{G}_* := \mathrm{C}^* \circ \mathcal{G} : \mathbf{Causet} \to \mathrm{C}^*\mathbf{Alg} \tag{2.6}$$

is a C*-causal functor. Covariance is obvious while causality follows from the previous observation. We shall call *group* C*-*causal functor* such a causal functor.

Remark 2.4. Any group C*-causal functor $\mathcal{G}_*$ has a non-trivial cyclic representation. Let π_K be the left regular representation of $\mathcal{G}(K)$ on the Hilbert space $\ell^2(\mathcal{G}(K))$ and denote its extension to the C*-algebra $\mathrm{C}^*\mathcal{G}(K)$

by $\pi_{*,K}$. Given $\psi \in (K, K')$, let $V_\psi : \ell^2_K \to \ell^2_{K'}$ be the isometry defined by $V_\psi(\delta_g) := \delta_{\psi(g)}$, for any $g \in \mathcal{G}_(K)$. Then one can easily see that

$$V_\psi \, \pi_{*,K} = \pi_{*,K'} \circ \mathcal{G}_{*,\psi} \, V_\psi, \;\; V_{\psi' \circ \psi} = V_{\psi'} V_\psi, \;\; \forall \psi \in (K, K'), \;\; \psi' \in (K', K''). \tag{2.7}$$

So, the pair (π_*, V) is a representation of $\mathcal{G}_*$.[5] Moreover, let δ_{1_K} be the delta function of the identity 1_K of $\mathcal{G}_K$. Since $\psi 1_{K_1} = 1_{K_2}$ we have, according to the definition of V_ψ, that $V_\psi \, \delta_{1_{K_1}} = \delta_{1_{K_2}}$, hence the collection $\delta_{1_K} \in \ell^2(\mathcal{G}_K)$, for $K \in$ **Causets**, is a cyclic invariant vector.

2.2.1 A comment on the causality condition (2.4) and the relation with locally covariant QFT

We now show a relation between C*-causal functors and the locally covariant QFT. We refer the reader to the original paper [4] for a complete description and physical motivation of the theory. We also recall some new developments of the theory: [28] for a generalization to the conformally covariant case and [20, 21] for an analysis of theories describing the same physics in all spacetimes, the so called SPASs condition.

We briefly recall the axioms of a locally covariant QFT. To this end, let **Loc** denote the category whose objects are 4D globally hyperbolic spacetimes $\mathcal{M}$ and whose arrows $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ are isometric embeddings, preserving the orientation and the time orientation, and such that if $x, y \in \psi(\mathcal{M}_2)$, then $J^+(x) \cap J^-(y) \subseteq \psi(\mathcal{M}_2)$, where $J^{+/-}(x)$ is the causal future/past of the point x.

A *locally covariant QFT* is nothing but a functor $\mathscr{A} : \mathbf{Loc} \to \mathbf{C^*Alg}$. We say that $\mathscr{A}$ is *causal* whenever given $\psi_i : \mathcal{M}_i \to \mathcal{M}$, $i = 1, 2$, such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally disjoint in $\mathcal{M}$, then

$$\mathscr{A}_{\psi_1}(A) \, \mathscr{A}_{\psi_2}(B) = \mathscr{A}_{\psi_2}(B) \, \mathscr{A}_{\psi_1}(A),$$

for any $A \in \mathscr{A}(\mathcal{M}_1)$ and $B \in \mathscr{A}(\mathcal{M}_2)$; $\mathscr{A}$ is said to satisfy the *time-slice axiom* whenever

$$\mathscr{A}_\psi(\mathscr{A}(\mathcal{M}_1)) = \mathscr{A}(\mathcal{M}_2)$$

for any $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\psi(\mathcal{M}_1)$ contains a Cauchy surface of $\mathcal{M}_2$.

[5]This is a slight generalization of the notion of representation used in [11, 36], where unitarity of the analogues of the V_ψ is required.

In order to get a connection between causal functors and the locally covariant QFT (which are causal) we must relax the causality condition (2.4). Let $\mathcal{A}' : \mathbf{Causet} \to \mathbf{C^*Alg}$ be a functor such that given $\psi_i \in (K_i, K)$ for $i = 1, 2$, if

$$\psi_1(K_1) \perp \psi_2(K_2), \tag{2.8}$$

then (2.5) holds, where $\psi_1(K_1) \perp \psi_2(K_2)$ means that $\psi_1(o_1)$ is causally disjoint from $\psi_2(o_2)$ for any $o_1 \in K_1$, $o_2 \in K_2$. This clearly ensures that $\mathcal{A}'$ is actually a causal functor. Let K denote the functor assigning to any globally hyperbolic spacetime $\mathcal{M}$ the set of diamonds $K(\mathcal{M})$ of $\mathcal{M}$ ordered under inclusion, and to any arrow $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ the induced map $K(\mathcal{M}_1) \ni o \to \psi(o) \in K(\mathcal{M}_2)$ (diamonds are stable under the isometric embeddings). Then, using (2.8), it is easily seen that

$$\mathscr{A} : \mathcal{A}' \circ \mathrm{K} : \mathbf{Loc} \to \mathbf{C^*Alg},$$

is a causal locally covariant QFT. Obviously, because of the generality of the notion of causal functor, nothing can be said about the time-slice axiom.

The causality relation (2.4), which actually characterizes the notion of a causal functor and results to be too restrictive for locally covariant QFT, is mainly motivated by gauge theories and in particular by a remark due to D. Buchholz and J.E. Roberts in the setting of quantum electrodynamics (see [29] for details). In fact, assume that the vector potential A_μ is realized as a quantum field in the Minkowski spacetime. It can be easily seen that a suitable smoothing of A_μ around a closed curve is an *observable* since it is related to the electromagnetic field *via* the Stokes theorem, similarly to the classical case. Clearly, this observable is not localized around the curve but it can be localized around any surface having that curve as a boundary. Consequently, if we take two causally disjoint curves γ_1 and γ_2 *forming a link*, the smoothing of A_μ around these two curves do not commute. The causality relation (2.4) forbids this situation. Actually, as observed in Section 2.1.1, the posets that we use as indices for nets over spacetimes have connected and *simply connected* regions as elements, so two curves localized within two causally disjoint regions cannot form a link.

2.3 Nets of C*-algebras

Mimicking the locally covariant QFT, we now show that once a C*-causal functor $\mathcal{A}$ is given, it is possible to construct a causal net of C*-algebras over any causal poset K which is, in particular, covariant with respect to any symmetry group of K.

Let us first recall some basic definitions. A net of C*-algebras over K is a pair $(\mathcal{A}, \jmath)_K$ where $\mathcal{A}$ is a correspondence $o \mapsto \mathcal{A}_o$, $o \in K$, of unital C*-algebras, the *fibres* of the net, and $\jmath$ is a correspondence $o \leq a \mapsto \jmath_{oa}$ of unital *-monomorphisms $\jmath_{ao} : \mathcal{A}_o \to \mathcal{A}_a$, the *inclusion maps*, satisfying the *net relations*

$$\jmath_{o''o'} \circ \jmath_{o'o} = \jmath_{o''o}, \quad o'' \geq o' \geq o. \tag{2.9}$$

The net is said to be *causal* whenever

$$[\jmath_{ao_1}(\mathcal{A}_{o_1}), \jmath_{ao_2}(\mathcal{A}_{o_2})] = 0, \quad a \geq o_1, o_2, \; o_1 \perp o_2. \tag{2.10}$$

If G is a symmetry group of K the net is said to be *G-covariant* whenever for any $o \in K$ and $g \in G$ there is an isomorphism $\alpha_g^o : \mathcal{A}_o \to \mathcal{A}_{go}$ subjected to the following conditions:

$$\alpha_g^{ho} \circ \alpha_h^o = \alpha_{gh}^o, \quad \alpha_g^a \circ \jmath_{ao} = \jmath_{ga\,go} \circ \alpha_g^o, \quad a \geq o, \; h, g \in G. \tag{2.11}$$

We now show how a net of C*-algebras is defined from a C*-causal functor $\mathcal{A}$ after fixing a causal poset K. As a first step, for any $o \in K$ we define the set

$$(K|o) := \{a \in K \mid a \leq o\} \; ;$$

this, equipped with the ambient order relation $\leq$ and the ambient causal disjointness relation $\perp$, is a causal poset too, so the inclusion $\iota_{K,(K|o)} : (K|o) \to K$ is an arrow of $((K|o), K)$. We define

$$\mathcal{A}_o := \mathcal{A}_{\iota_{K,(K|o)}}(\mathcal{A}(K|o)), \quad o \in K. \tag{2.12}$$

Any $\mathcal{A}_o$ is a C*-subalgebra of $\mathcal{A}(K)$ and

$$\mathcal{A}_o \subseteq \mathcal{A}_a, \quad o \leq a. \tag{2.13}$$

To prove (2.13), observe that according to the definition (2.12) the inclusion of $\mathcal{A}_o$ into $\mathcal{A}_a$ is given by $\mathcal{A}_{\iota_{K,(K|a)}} \circ \mathcal{A}_{\iota_{(K|a),(K|o)}} \circ \mathcal{A}^{-1}_{\iota_{K,(K|o)}}$ where $\iota_{(K|a),(K|o)} \in ((K|o), (K|a))$. Using functoriality, we have that

$$\begin{aligned}
\mathcal{A}_{\iota_{K,(K|a)}} \circ \mathcal{A}_{\iota_{(K|a),(K|o)}} \circ \mathcal{A}^{-1}_{\iota_{K,(K|o)}} &= \mathcal{A}_{\iota_{K,(K|a)} \circ \iota_{(K|a),(K|o)}} \circ \mathcal{A}^{-1}_{\iota_{K,(K|o)}} \\
&= \mathcal{A}_{\iota_{K,(K|o)}} \circ \mathcal{A}^{-1}_{\iota_{K,(K|o)}} = \mathcal{A}_{\iota_{K,(K|o)}} \circ \mathcal{A}^{-1}_{\iota_{K,(K|o)}} \\
&= \mathrm{id}_{\mathcal{A}(K)},
\end{aligned}$$

which gives the desired inclusion (2.13), that we denote by $\mathrm{id}_{ao} : \mathcal{A}_o \to \mathcal{A}_a$. This yields the net of C*-algebras $(\mathcal{A}, \mathrm{id})_K$. To prove causality we note that, since the image of $\iota_{K,(K|o)}$ is $(K|o) \leq o$, and since $\mathcal{A}$ is causal, we have, as desired,

$$[\mathcal{A}_o, \mathcal{A}_a] = 0, \quad o \perp a. \tag{2.14}$$

Finally, let $S \subseteq (K, K)$ be a symmetry group of K. Define

$$\alpha_s := \mathcal{A}_s, \quad s \in S. \tag{2.15}$$

Then $\alpha_s : \mathcal{A}(K) \to \mathcal{A}(K)$ is a *-automorphism for any $s \in S$ and $\iota^{-1}_{K,(K|go)} \circ s \circ \iota_{K,(K|o)} : (K|o) \to (K|so)$ is an isomorphism of causal posets, therefore $\alpha_s : \mathcal{A}_o \to \mathcal{A}_{so}$ is a *-isomorphism. Since the inclusion maps are constant, it easily follows that $(\mathcal{A}, \mathrm{id}, \alpha)_K$ is a S-covariant causal net of C*-algebras over K. Summing up:

Theorem 2.5. *Given a* C**-causal functor* $\mathcal{A}$*, for any causal poset* K *we have:*

(i) $(\mathcal{A}, \mathrm{id})_K$ *is a causal net of* C**-subalgebras of* $\mathcal{A}(K)$*;*
(ii) *If* S *is a symmetry group of* K*, then* $(\mathcal{A}, \mathrm{id}, \alpha)_K$ *is* S*-covariant.*

3 Causal functors from groups of loops

In this section we construct a causal functor using the structure of causal posets only. We first associate a (free) group with any poset and show that this association yields a functor from the category of causal posets to groups. Afterwards we introduce causal commutators. These are normal subgroups of the free group. We then perform the quotient and show that this leads to a group-causal functor hence, according to the results of the previous section, to a C*-causal functor.

3.1 Simplices

A simplicial set $\Sigma_*(K)$ can be defined over any poset K, in the following way [30, 32]. As a first step we set $\Sigma_0(K) := K$ and then, inductively, we define elements of $\Sigma_n(K)$, the *n-simplices*, as follows,

$$x := (|x|; \partial_0 x, \ldots, \partial_n x) \ : \ \begin{cases} \partial_i x \in \Sigma_{n-1}(K), \ \ |x| \in K, \ \ \forall \mathrm{i} = 0, \ldots, n, \ \ n \geq 1, \\ |\partial_i x| \leq |x|, \ \ \forall \mathrm{i} = 0, \ldots, n, \ \ n \geq 1, \\ \partial_{ij} x := \partial_i(\partial_j x) = \partial_{j,i+1} x, \ \ \forall \mathrm{i} \geq j, \ \ n \geq 2. \end{cases}$$

The $(n-1)$-simplices $\partial_i x$ are called the *faces*, whilst $|x|$ is called the *support* of x. Note that the previous definition also yields the face maps $\partial_i : \Sigma_n(K) \to \Sigma_{n-1}(K)$, $\mathrm{i} = 0, \ldots, n$, $n \geq 1$. There are also *degeneracies operators* $\sigma_i : \Sigma_n(K) \to \Sigma_{n+1}(K)$, $\mathrm{i} = 0, \ldots, n$, $n \geq 0$ (see the cited references for the definition), making $\Sigma_*(K)$ a simplicial set.

The simplicial set $\Sigma_*(K)$ is symmetric since it is closed under any permutation of the vertices of any of its simplex. Anyway since in what follows we shall mainly use 1-simplices of $\Sigma_*(K)$, we focus on the properties of $\Sigma_1(K)$. Any 1-simplex b is formed by a triple of elements of K,

$$b = (|b|; \partial_0 b, \partial_1 b) : \partial_0 b, \partial_1 b \leq |b|.$$

The intuitive content of the 1–simplex b is that it is a segment starting at the “point” $\partial_1 b$ and ending at $\partial_0 b$. A *degenerate* 1-simplex of the form $\sigma_0 a$ for $a \in K$, is the 1-simplex having the support and faces equal to a.

Since $\Sigma_*(K)$ is symmetric we can associate to any 1-simplex b its opposite $\overline{b}$: this is the 1-simplex having the same support as b and inverted faces:

$$|\overline{b}| := |b|, \quad \partial_1 \overline{b} := \partial_0 b, \quad \partial_0 \overline{b} := \partial_1 b.$$

Another simplicial sets, which shall play a role in what follows is the *nerve* of K. This is the subsimplicial set $\mathrm{N}_*(K)$ of those simplices of $\Sigma_*(K)$ whose vertices are totally ordered and whose support coincide with the greatest vertex. In particular for 1-simplices we have that $b \in \mathrm{N}_1(K)$ if, and only if, $\partial_1 b \leq \partial_0 b = |b|$.

3.2 The free group of loops on a poset

A first step towards the construction of a causal functor is to associate a free group and its subgroup of loops to a suited subset of the set of 1-simplices of a poset. The subgroup of loops will play a more relevant rôle concerning the construction of the target category of the functor. As a reference for free group theory (see [26]).

Given a causal poset K we define[6]

$$\mathrm{T}_1(K) := \Sigma_1(K) \setminus \mathrm{N}_1(K). \tag{3.1}$$

[6]We are following a different idea from that used in [36] to define causal net of C^*-algebras. This because we want to remove any topological effect from the definition of the group associated with a causal poset. In this regard, we recall that the first homotopy group of a poset is encoded in the nerve $\mathrm{N}_1(K)$, (see [33]).

So $\mathrm{T}_1(K)$ is the set of 1-simplices of K which do not belong to the nerve of K, or, equivalently, $b \in \mathrm{T}_1(K)$ if, and only if, $|b| > \partial_0 b, \partial_1 b$. This set is closed under the operation of making the opposite and, in general is not empty. Moreover, it is sufficient to describe path connectedness of the poset (this will clear soon).

Now, let us consider the group generated by $\mathrm{T}_1(K)$ with relation $b^{-1} = \overline{b}$, i.e.,

$$\mathrm{F}(K) := \{b \in \mathrm{T}_1(K) \,|\, b^{-1} = \overline{b}\}. \tag{3.2}$$

$\mathrm{F}(K)$ is (non-canonically) isomorphic to a free group. To this end let $\sim_{\mathrm{opp}}$ denote the equivalence relation $b \sim_{\mathrm{opp}} b' \Leftrightarrow\ b' = \overline{b}$ for any $b \in \mathrm{T}_1(K)$. Then by the axiom of choice there is a section $s : \mathrm{T}_1(K)/\sim_{\mathrm{opp}} \longrightarrow \mathrm{T}_1(K)$ and it is easily seen that $\mathrm{F}(K)$ is isomorphic to the free group generated by $s(\mathrm{T}_1(K)/\sim_{\mathrm{opp}})$.

Thus, elements of $\mathrm{F}(K)$ are words

$$w := b_n\, b_{n-1}\, \ldots\, b_1, \quad b_i \in \mathrm{T}_1(K).$$

The inverse of a word is given by $\overline{w} := \overline{b}_1, \ldots, \overline{b}_n$, where $\overline{b}$ is the opposite of the 1-simplex b of $\mathrm{T}_1(K)$. The empty word 1 is the identity of the group. The *support* $|w|$ of a word w is the collection of the supports of the generators of w, as a reduced word w^{r}. For instance if $w = b_2 b\, \overline{b}\, b_1$ and is $b_1 \neq \overline{b}_2$ then $|w| = \{|b_2|, |b_1|\}$ since the reduced word w^{r} equals $b_2 b_1$. A word $w = b_n\, \ldots\, b_1$ is said to be a *path* whenever its generators satisfy the relation

$$\partial_0 b_{i+1} = \partial_1 b_i, \quad i = 1, \ldots, n-1. \tag{3.3}$$

We set $\partial_1 w := \partial_1 b_1$ and $\partial_0 w := \partial_0 b_n$ and call these 0-simplices, respectively, the *starting* and the *ending point* of the path w. We shall also use the notation $w : a \to o$ to denote a path from a to o. Finally, a path $w : o \to o$ is said to be a *loop over* o. Note that if $w : a \to o$ is a path, then the reduced word w^{r} is still a path from a to o.

Causality is extended from K to $\mathrm{F}(K)$ as follows: two words w_1 and w_2 are *causally disjoint*,

$$w_1 \perp w_2 \qquad \Longleftrightarrow \qquad \exists o_1, o_2 \in K, \quad |w_1| \leq o_1, \quad |w_2| \leq o_2, \quad o_1 \perp o_2. \tag{3.4}$$

We stress that the above relation is given on the reduced words w_1^{r} and w_2^{r} according to the definition of the support of a word.

We now show that the correspondence associating $K \mapsto \mathrm{F}(K)$ is a functor. Let $K_1, K_2 \in$ **Causets** and $\psi \in (K_1, K_2)$. Extend the action of ψ from the poset K_1 to $\mathrm{T}_1(K_1)$ by setting $\psi(b)$, by a little abuse of notation, as

$$|\psi(b)| := \psi(|b|), \quad \partial_1\psi(b) := \psi(\partial_1 b), \quad \partial_0\psi(b) := \psi(\partial_0 b). \tag{3.5}$$

Since ψ is order preserving an injective, $\psi(b)$ is a 1-simplex of $\mathrm{T}_1(K_2)$. Moreover, $\overline{\psi(b)} = \psi(\bar{b})$. Finally if $w = b_n \cdots b_1$ is any word of F_{K_1} we have that

$$\psi(w) := \psi(b_n) \cdots \psi(b_1) \tag{3.6}$$

is a word of $\mathrm{F}(K_2)$. Clearly, if w is a reduced word of $\mathrm{F}(K_1)$, then $\psi(w)$ is a reduced word of $\mathrm{F}(K_2)$; if $w : o \to o$ is a loop of $\mathrm{F}(K_1)$, then $\psi(w) : \psi(o) \to \psi(o)$ is a loop of $\mathrm{F}(K_2)$; if $w \perp_1 v$ are causally disjoint words in $\mathrm{F}(K_1)$, then $\psi(w) \perp_2 \psi(v)$ in $\mathrm{F}(K_2)$.

According to the above definition for any morphism $\psi : K_1 \to K_2$, we have that $\psi : \mathrm{F}(K_1) \to \mathrm{F}(K_2)$ is an injective group morphism as well. So

$$\mathrm{F} : \mathbf{Causet} \to \mathbf{Grp}, \quad K \mapsto \mathrm{F}(K), \quad \psi \mapsto \psi,$$

is a functor. We now give the following key

Definition 3.1. We call the **group of loops** of K the subgroup $\mathrm{L}(K)$ of $\mathrm{F}(K)$ generated by loops.

In general $\mathrm{L}(K)$ is not a normal subgroup of $\mathrm{F}(K)$. An element of $\mathrm{L}(K)$ is, by definition, a word of the form $w = p_n p_{n-1} \cdots p_1$ where any p_i is a loop over a 0-simplex o_i. The reduced word w^{r} of w may have the following forms: if $o_i \neq o_{i+1}$ for any i, since no cancellation is possible between loops defined over different base point, then $w^{\mathrm{r}} = p_n^{\mathrm{r}} p_{n-1}^{\mathrm{r}} \cdots p_1^{\mathrm{r}}$, where p_i^{r} is the reduced word of p_i. If, for instance $o_1 = o_2$, $o_i \neq o_{i+1}$ for $i \geq 2$, since $p_2 p_1$ is a loop over o_1, then $w^{\mathrm{r}} = p_n^{\mathrm{r}} p_{n-1}^{\mathrm{r}} \cdots p_3^{\mathrm{r}} p_{21}^{\mathrm{r}}$, where p_{21}^{r} is a loop obtained by reducing the loop $p_2 p_1$. Hence, $\mathrm{L}(K)$ is stable under reduction of words.

Since, as observed before, any $\psi \in (K_1, K_2)$ preserves loops, we have that $\psi : \mathrm{L}(K_1) \to \mathrm{L}(K_2)$, implying, as before, that the assignment

$$\mathrm{L} : \mathbf{Causet} \to \mathbf{Grp}, \quad K \mapsto \mathrm{L}(K), \quad \psi \mapsto \psi,$$

is a functor.

3.3 Causality

The final step towards the construction of a causal functor is to introduce causality. This will be done in terms of commutators of the groups introduced in the previous section. We shall see that the quotient of the group of loops by commutators gives rise to a causal functor. To start, we denote as usual the commutator of two words w_1, w_2 of $\mathrm{F}(K)$ by $[w_1, w_2] = w_1 w_2 \overline{w}_1 \overline{w}_2$.

Definition 3.2. A **causal commutator of** $\mathrm{F}(K)$ is a word of the form

$$w\,[p_1, p_2]\,\overline{w} \quad \text{where} \quad p_i : o_i \to o_i, \quad i = 1, 2, \quad p_1 \perp p_2, \quad w \in \mathrm{F}(K),$$

and it is a **causal commutator of** $\mathrm{L}(K)$ when $w \in \mathrm{L}(K)$. We denote the group generated by causal commutators of $\mathrm{F}(K)$ and $\mathrm{L}(K)$, respectively, by $\mathrm{CF}(K)$ and $\mathrm{CL}(K)$.

Note that, by definition, $\mathrm{CF}(K)$ is a normal subgroup of $\mathrm{F}(K)$ and $\mathrm{CL}(K)$ is a normal subgroup of $\mathrm{L}(K)$. Moreover, $\mathrm{CL}(K)$ is a subgroup, in general not normal, of $\mathrm{CF}(K)$. An important relation between these two groups is given by the following

Lemma 3.3. $\mathrm{CL}(K) = \mathrm{CF}(K) \cap \mathrm{L}(K)$ *for any causal poset* K.

Proof. Clearly $\mathrm{CL}(K) \subseteq \mathrm{CF}(K) \cap \mathrm{L}(K)$. So, consider a generic element C of $\mathrm{CF}(K)$. By Definition 3.2, C has the form

$$C = w_1\,[q_1, p_1]\,\overline{w}_1\,w_2\,[q_2, p_2]\,\overline{w}_2 \ldots w_n\,[q_n, p_n]\,\overline{w}_n,$$

where $q_i : o_i \to o_i$ and $p_i : a_i \to a_i$ are causally disjoint loops for $i = 1, \ldots, n$ and $w_i \in \mathrm{F}(K)$. Assume that $C \in \mathrm{L}(K)$. Then we must show that $w_i \in \mathrm{L}(K)$ for any i. Assume that it is not the case. Then it is enough to consider the case where $w_n \notin \mathrm{L}(K)$. In fact if $w_n \in \mathrm{L}(K)$, then $w_n\,[q_n, p_n]\,\overline{w}_n \in \mathrm{L}(K)$. So

$$w_1\,[q_1, p_1]\,\overline{w}_1\ \ldots\ w_{n-1}\,[q_{n-1}, p_{n-1}]\,\overline{w}_{n-1} \in \mathrm{L}(K)$$

and we may repeat the same reasoning with respect to w_{n-1}.

So assume that $w_n \notin \mathrm{L}(K)$. It is enough to consider the case where w_n is a path from x_n to y_n with $x_n \neq y_n$. Clearly $y_n \neq a_n$, where a_n is the basepoint of p_n, leads to a contradiction. But also the equality $y_n = a_n$ leads to a contradiction. In fact in this case $\overline{p}_n \overline{w}_n$ is a path from x_n to y_n and no reduction is possible in order to get a loop since $\overline{q}_n$ is causally disjoint from p_n (recall that causal disjointness is irreflexive, Section 2.1). So the only possibility is that w_n is a loop. This, and the previous observation, complete the proof. □

Again, since any $\psi \in (K_1, K_2)$ preserves inclusions, loops, and causal disjointness, we have that $\psi : \mathrm{C}\sharp(K_1) \to \mathrm{C}\sharp(K_2)$ is an injective group morphism for $\sharp = \mathrm{L}, \mathrm{F}$. So, as before, the assignment

$$\mathrm{C}\sharp : \mathbf{Causet} \to \mathbf{Grp}, \quad K \mapsto \mathrm{C}\sharp(K), \quad \psi \mapsto \psi,$$

with $\sharp = \mathrm{L}, \mathrm{F}$, is a functor.

We now are in a position to define the desired causal functor. As observed $\mathrm{CL}(K)$ and $\mathrm{CF}(K)$ are, respectively, a normal subgroup of $\mathrm{L}(K)$ and a normal subgroup of $\mathrm{F}(K)$. So consider the quotient groups

$$\widehat{\mathrm{F}}(K) := \mathrm{F}(K)/\mathrm{CF}(K) \quad \text{and} \quad \widehat{\mathrm{L}}(K) := \mathrm{L}(K)/\mathrm{CL}(K), \tag{3.7}$$

and denote the quotient maps, respectively, by the symbols Q^{F}_K and Q^{L}_K. Furthermore let

$$\imath_K \circ Q^{\mathrm{L}}_K(w) := Q^{\mathrm{F}}_K(w), \quad w \in \mathrm{L}(K). \tag{3.8}$$

Since $\mathrm{CL}(K) \subseteq \mathrm{CF}(K)$ this definition is well posed, i.e., if $Q^{\mathrm{L}}_K(w) = 1$ then $Q^{\mathrm{F}}_K(w) = 1$, and yields a group monomorphism $\imath_K : \widehat{\mathrm{L}}(K) \to \widehat{\mathrm{F}}(K)$. Injectivity follows by observing that if $Q^{\mathrm{F}}_K(w) = 1$, then $w \in \mathrm{CF}(K)$. Since $w \in \mathrm{L}(K)$, by Lemma 3.3, we have that $w \in \mathrm{CL}(K)$, so $Q^{\mathrm{L}}_K(w) = 1$. Similarly, by setting for any $\psi \in (K_1, K_2)$

$$\widehat{\mathrm{F}}_\psi \circ Q^{\mathrm{F}}_{K_1} := Q^{\mathrm{F}}_{K_2} \circ \psi, \quad \widehat{\mathrm{L}}_\psi \circ Q^{\mathrm{L}}_{K_1} := Q^{\mathrm{L}}_{K_2} \circ \psi, \tag{3.9}$$

we get that $\widehat{\mathrm{F}}_\psi : \widehat{\mathrm{F}}(K_1) \to \widehat{\mathrm{F}}(K_2)$ and $\widehat{\mathrm{L}}_\psi : \widehat{\mathrm{L}}(K_1) \to \widehat{\mathrm{L}}(K_2)$ are injective group morphisms. This leads to the commuting diagram

$$\begin{tikzcd}
\mathrm{F}(K_1) \arrow[rrr, "\psi"] \arrow[dr, "Q^{\mathrm{F}}_{K_1}"] & & & \mathrm{F}(K_2) \arrow[dl, "Q^{\mathrm{F}}_{K_2}"'] \\
& \widehat{\mathrm{F}}(K_1) \arrow[r, "\widehat{\mathrm{F}}_\psi"] & \widehat{\mathrm{F}}(K_2) & \\
& \widehat{\mathrm{L}}(K_1) \arrow[r, "\widehat{\mathrm{L}}_\psi"'] \arrow[u, "\imath_{K_1}"'] & \widehat{\mathrm{L}}(K_2) \arrow[u, "\imath_{K_2}"] & \\
\mathrm{L}(K_1) \arrow[rrr, "\psi"] \arrow[ur, "Q^{\mathrm{L}}_{K_1}"'] \arrow[uuu, "\subseteq"] & & & \mathrm{L}(K_2) \arrow[ul, "Q^{\mathrm{L}}_{K_2}"] \arrow[uuu, "\subseteq"']
\end{tikzcd} \tag{3.10}$$

and to the existence of a group-causal functor.

Theorem 3.4. *The following assertions hold.*

(i) *The assignment* $\widehat{\mathrm{L}} : \mathbf{Causet} \to \mathbf{Grp}$ *is a group-causal functor endowed with the natural transformation* $Q^{\mathrm{L}} : \mathrm{L} \to \widehat{\mathrm{L}}$ *defined by* $K \mapsto Q^{\mathrm{L}}_K$.

(ii) *The assignment* $\widehat{\mathrm{F}} : \mathbf{Causet} \to \mathbf{Grp}$ *is a functor endowed with a natural transformation* $Q^{\mathrm{F}} : \mathrm{F} \to \widehat{\mathrm{F}}$ *defined by* $K \mapsto Q^{\mathrm{F}}_K$.

These two functors are related by a natural transformation $\imath : \widehat{\mathrm{L}} \to \widehat{\mathrm{F}}$ *defined by* $K \to \imath_K$ *and satisfying the relation* $\imath \circ Q^{\mathrm{L}} = Q^{\mathrm{F}}$*, where on the r.h.s. of this equation,* L *is identified as a subfunctor of* F.

Proof. We only have to show the causal properties of $\widehat{\mathrm{L}}$, that is, equation (2.5). Consider $K_i, K \in \mathbf{Causet}$ and $\psi_i \in (K_i, K)$ for $i = 1, 2$ and assume that $\psi_i(K_i) \leq o_i \in K$, for $i = 1, 2$, and $o_1 \perp o_2$. Given $l_i \in \mathrm{L}(K_i)$, by equation (3.9) we have that

$$\widehat{\mathrm{L}}_{\psi_i} \circ Q^{\mathrm{L}}_{K_i}(l_i) = Q^{L}_K \circ \psi_i(l_i), \quad i = 1, 2,$$

and the proof follows since $\psi_1(l_1)$ and $\psi_2(l_2)$ are causally disjoint loops of $\mathrm{L}(K)$. □

Using the functor C^* introduced in Section 2.2 and, in particular, the equation (2.6), we derive from Theorem 3.4, the C^*-causal functor.

Corollary 3.5. *The following assertions hold.*

(i) *The assignment* $\widehat{\mathrm{L}}_* := \mathrm{C}^* \circ \widehat{\mathrm{L}} : \mathbf{Causet} \to \mathbf{C^*Alg}$ *is a* C^**-causal functor.*

(ii) *The assignment* $\widehat{\mathrm{F}}_* := \mathrm{C}^* \circ \widehat{\mathrm{F}} : \mathbf{Causet} \to \mathbf{C^*Alg}$ *is a* C^**-functor.*

There is a natural transformation $\imath_* := \mathrm{C}^* \circ \imath : \widehat{\mathrm{L}}_* \to \widehat{\mathrm{F}}_*$.

3.4 The net of causal loops

Finally, using the C^*-causal functor $\widehat{\mathrm{L}}_*$ and the C^*-functor $\widehat{\mathrm{F}}_*$ of Corollary 3.5, we arrive to the main result of this section, i.e., the definition of the causal and covariant net of C^*-algebras associated to a causal poset.

Let K be a causal poset with a symmetry group S. We consider the causal and covariant net of C^*-algebras associated with the C^*-causal functor $\widehat{\mathrm{L}}_*$.

Namely, remembering the construction of Section 2.3, the fibres of the net are defined from equation (2.12) as

$$\mathcal{A}_o := \widehat{\mathrm{L}}_{*,\iota_{K,(K|o)}}(\widehat{\mathrm{L}}_*(K|o)), \quad o \in K. \tag{3.11}$$

Correspondingly, the C*-algebra associated with K is defined as $\mathcal{A}(K) := \widehat{\mathrm{L}}_*(K)$; this yields a causal net $(\mathcal{A}, \mathrm{id})_K$ of C*-subalgebras of $\mathcal{A}(K)$. Finally, the action of the symmetry group S of K on this net is defined by

$$\alpha_s := \widehat{\mathrm{L}}_{*,s}, \quad s \in S. \tag{3.12}$$

Then, according to Theorem 2.5, $(\mathcal{A}, \mathrm{id}, \alpha)_K$ is a causal and S-covariant net of C*-algebras over K, that we call **the C*-net of causal loops over** K.

In a similar way, the C*-functor $\widehat{\mathrm{F}}_*$ yields the C*-dynamical system $(\mathcal{F}(K), S, \varphi)$, where $\mathcal{F}(K) := \widehat{\mathrm{F}}_*(K)$ and $\varphi_s := \widehat{\mathrm{F}}_{*,s}$ for any $s \in S$. Then, as the fibres of the net of causal loops are subalgebras of $\mathcal{A}(K)$ and using the natural transformation $\rho := \imath_*$ given in Corollary 3.5, we have that

$$(\mathcal{A}, \mathrm{id}, \alpha)_K \xrightarrow{\subseteq} (\mathcal{A}(K), S, \alpha) \xrightarrow{\rho_K} (\mathcal{F}(K), S, \varphi) \tag{3.13}$$

is a sequence of S-covariant monomorphisms. These observations lead to the following

Corollary 3.6. *Let K be a causal poset with a symmetry group S. Then,*

(i) *The net of causal loops $(\mathcal{A}, \mathrm{id}, \alpha)_K$ is a causal and S-covariant net of C*-subalgebras of $\mathcal{A}(K)$.*
(ii) *$\rho_K : (\mathcal{A}(K), S, \alpha) \to (\mathcal{F}(K), S, \varphi)$ is an S-covariant monomorphism of C*-dynamical systems.*

This corollary represents one of the main results of the present paper. In fact, for the class of spacetimes discussed in Section 2.1.1, the corresponding net of causal loops is a causal net of non-Abelian C*-algebras, which is covariant under the symmetry group of the spacetime.

Theorem 3.7. *The following assertions hold:*

(i) *the net of causal loops over the set of **double cones** of the Minkowski spacetime $\mathbb{M}^4$ is covariant under the Poincaré group;*
(ii) *the causal net of loops over the set of **open intervals** of S^1 is covariant under the Möbius group (or the diffeomorphism group of S^1);*
(iii) *the causal net of loops over the set of **diamonds** of a globally hyperbolic spacetime $\mathcal{M}$ is covariant under any global isometry or under any conformal diffeomorphism.*

In all these cases the fibres of the net of causal loops are ***non-Abelian*** C^**-algebras.*

Proof. We have to prove only that the fibres are non-Abelian C^*-algebras. According to the Definition (3.11) and to the definition of the functor $\widehat{\mathrm{L}}_*$ given in Corollary 3.5, it is enough to prove that $\widehat{\mathrm{L}}(K|o)$ is a non-Abelian group. To this end, recall that $\widehat{\mathrm{L}}(K|o) = \mathrm{L}(K|o)/\mathrm{CL}(K|o)$. We now make use of the properties of the posets associated to the above spacetimes, outlined at the end of the Section 2.1.1. By the properties 1 and 2 follows that $\mathrm{L}(K|o)$ is a free group with infinite generators. Since the subgroup $\mathrm{CL}(K|o)$ of causal commutators $\mathrm{L}(K|o)$ is a proper subgroup of the commutator subgroup of $\mathrm{L}(K|o)$, the quotient $\widehat{\mathrm{L}}(K|o)$ is a non-Abelian group. □

4 Representations of the net of causal loops

We have seen that a causal net of C^*-algebras, the net of loops, is associated to any space(time) manifold; this is covariant with respect to any symmetry group of the space(time). In this section we shall deal with the representations of the net of loops. On the one hand, we want to understand how the combinatorial structure which underlies the net of loops reflects on its representations. On the other hand, we want to understand, in the case of a topological symmetry group, whether non-trivial continuously covariant representations of the covariant net exist.

We shall focus on two classes of representations, both of them admitting an interpretation in terms of cohomology of posets.

The first class is the one of representations induced by the dynamical system $(\mathcal{F}(K), S, \varphi)$, equation (3.13). These correspond to connection 1-cochains of the poset K, according to the definition given in [32], which in addition satisfy some causal and covariant properties. By means of this correspondence we shall point out, at the end of this section, a sort of "physical universal property" of the net of loops: any Bosonic free quantum field on a suited class of space(time)s, including the Minkowski spacetime and the circle S^1, provides a representation of this net. Moreover, in these cases, the construction implies the existence of non-trivial, continuously covariant representations of the net.

The second class is one of the representations induced by the dynamical system $(\mathcal{A}(K), S, \alpha)$, equation (3.13). Any such a representation results to be associated to a causal and covariant *connection system*, i.e., a collection of connections where causality and covariance are properties of the system and not of the single connection. Actually, the system associated to a representation is not unique; two systems associated to the same representation are related by a suitable notion of *gauge* transformation. In this sense, we may establish a link with quantum gauge theories, interpreting such a representation as a gauge field and any connection system associated to the representation as a gauge potential.

From now on, being interested to the applications to QFT, the causal posets we consider are those associated to physical spacetimes, i.e., the ones listed in Section 2.1.1.

4.1 Preliminaries

Fix a causal poset K with a symmetry group S. Then S acts on K by causal automorphisms, i.e., $s \in (K, K)$, for any $s \in S$, and extending by functoriality the above maps we get S-actions by group automorphisms on $\mathrm{F}(K)$ and $\mathrm{L}(K)$ which, since causality and order are preserved, factorize through the S-actions $\widehat{\mathrm{F}}_s$, $\widehat{\mathrm{L}}_s$, $s \in S$, on $\widehat{\mathrm{F}}(K)$, $\widehat{\mathrm{L}}(K)$ respectively. So we get actions on the associated C*-algebras

$$\alpha_s = \widehat{\mathrm{L}}_{*,s} \in \mathbf{aut}\mathcal{A}(K), \quad \varphi_s = \widehat{\mathrm{F}}_{*,s} \in \mathbf{aut}\mathcal{F}(K), \quad s \in S.$$

In this way the results of the previous section are summarized as follows,

$$\begin{tikzcd}
(\mathrm{F}(K), S) \arrow[r, "Q^F_K"] & (\widehat{\mathrm{F}}(K), S) \arrow[r, dotted, "\mathrm{C}^*"] & (\mathcal{F}(K), S, \varphi) \\
(\mathrm{L}(K), S) \arrow[u, "\subseteq"'] \arrow[r, "Q^L_K"'] & (\widehat{\mathrm{L}}(K), S) \arrow[u, "\imath_K"] \arrow[r, dotted, "\mathrm{C}^*"'] & (\mathcal{A}(K), S, \alpha) \arrow[u, "\rho_K"'] \\
 & & (\mathcal{A}, \mathrm{id}, \alpha)_K \arrow[u, "\subseteq"']
\end{tikzcd} \tag{4.1}$$

Here, on the l.h.s. we have a commuting diagram of groups carrying an S-action, whilst on the r.h.s. we have an S-covariant inclusion of the net $(\mathcal{A}, \mathrm{id}, \alpha)_K$ into the C*-dynamical system $(\mathcal{A}(K), S, \alpha)$ which, in turn, embeds in the C*-dynamical system $(\mathcal{F}(K), S, \varphi)$ defined by $(\widehat{\mathrm{F}}(K), S)$.

Now, a *covariant representation* of the net $(\mathcal{A}, \mathrm{id}, \alpha)_K$ is a pair (π, Γ) where $\pi = \{\pi_o : \mathcal{A}_o \to \mathcal{BH} \,|\, o \in K\}$ is a family of representations on a fixed

Hilbert space $\mathcal{H}$ satisfying

$$\pi_a \upharpoonright \mathcal{A}_o = \pi_o, \quad o \leq a, \tag{4.2}$$

and Γ is a unitary representation of the symmetry group S on $\mathcal{H}$ such that

$$\mathrm{Ad}_{\Gamma_L} \circ \pi_o = \pi_{Lo}, \quad o \in K, L \in S. \tag{4.3}$$

If Γ is strongly continuous, then we say that (π, Γ) is *continuously covariant.*

The net of causal loops $(\mathcal{A}, \mathrm{id}, \alpha)_K$ has non-trivial covariant representations since it is embedded in $(\mathcal{A}(K), S, \alpha)$, and any covariant representation of this dynamical system induces a covariant representation of the net. Notice however that $(\mathcal{F}(K), S, \varphi)$ induces, via the morphism ρ_K, representations of the net as well. In what follows, we shall analyze these two classes of representations in terms of the cohomology of posets and, as a by-product, we shall show at the end the existence of continuously covariant representations.

Remark 4.1. A net of C*-algebras admits more general representations when defined over a non-simply-connected poset [36]. The form of the representations we are considering in the present paper are called *Hilbert space representations* in [36]. We restrict to this set of representations because any representation of the net of causal loops induced by the above dynamical systems has this form.

4.1.1 A cohomological description of connections over posets

The notion of a connection over a poset was introduced in the context of (non-Abelian) cohomology of a poset, with the aim of finding a suitable framework for studying gauge theories in the setting of AQFT [32].

A *connection* 1-cochain of a poset K is a field $u : \Sigma_1(K) \ni b \to u(b) \in \mathcal{U}(\mathcal{H})$ of unitary operators of a Hilbert space $\mathcal{H}$ satisfying $u(b)^{-1} = u(\overline{b})$, $\forall b \in \Sigma_1(K)$, and the 1-cocyle equation on the nerve of K:

$$u(\partial_0 c)\, u(\partial_2 c) = u(\partial_1 c), \quad c \in \mathrm{N}_2(K). \tag{4.4}$$

A connection is *flat* if it is a 1-cocycle, i.e., if the 1-cocycle equation is verified on all $\Sigma_2(K)$. This definition finds its motivation in several properties that such a notion share with the corresponding one on fibred bundles over manifolds. The curvature of a connection, its 2-coboundary, satisfies an equation analogous of the Bianchi identity; a connection is flat if, and only if, its curvature is trivial; any flat connection defines a representation of the fundamental group of the poset K; there is an analogue of the Ambrose–Singer theorem. Furthermore, any connection identifies the fibred bundle where it lives: precisely, a (unique) 1-cocycle is associated with any connection, and the former is nothing but that a fibred bundle over the poset K. So, a gauge transformation of a connection turns out to be an automorphism of the associated flat connection. Finally, any fibred bundle over K can be interpreted as a flat bundle in the sense of differential geometry when K is a base of a manifold (see [33]).

4.2 Connections

We now show that representations of the net of causal loops induced by the dynamical system $(\mathcal{F}(K), S, \varphi)$ corresponds to a class of connections of K. To start, we generalize the notion of connection 1-cochain of a poset, recalled above, taking into account the properties of causality and covariance.

Definition 4.2. Given a causal poset K with a symmetry group S. A **causal and covariant connection over** K is a pair (u, Γ) where u is a field $\mathrm{T}_1(K) \ni b \to u(b) \in \mathcal{U}(\mathcal{H})$ of unitary operators of a Hilbert space $\mathcal{H}$ and Γ is a unitary representation of S in $\mathcal{H}$ satisfying the relations:

(1) $u(\bar{b}) = u(b)^{-1}$ for any $b \in \mathrm{T}_1(K)$;
(2) $\Gamma_s u(b) = u(s(b))\,\Gamma_s$ for any $s \in S$ and $b \in \mathrm{T}_1(K)$;
(3) $u(w_1)u(w_2) = u(w_2)u(w_1)$ for any $w_1, w_2 \in \mathrm{L}(K)$ such that $w_1 \perp w_2$.

If Γ is strongly continuous, then we say that (u, Γ) is **continuously covariant**.

Note that although u is defined only on $\mathrm{T}_1(K)$ it admits an obvious extension u' on all $\Sigma_1(K)$,

$$u'(b) := \begin{cases} u(b), & b \in \mathrm{T}_1(K), \\ 1, & b \in \mathrm{N}_1(K), \end{cases}$$

since $\mathrm{T}_1(K)$ and $\mathrm{N}_1(K)$ are disjoint subsets of $\Sigma_1(K)$; so u' is a connection in the sense of [32] (see Section 4.1.1 above) and the results of that paper applies.

The relation between causal and covariant connections of K and covariant representations of the net $(\mathcal{A}, \mathrm{id}, \alpha)_K$ induced by $(\mathcal{F}(K), S, \varphi)$ is stated in the next

Lemma 4.3. *Let K be a causal poset with a symmetry group S. Then there exists a bijective correspondence between covariant representations of the* C^**-dynamical system $(\mathcal{F}(K), S, \varphi)$ and causal and covariant connections of K.*

Proof. ($\Rightarrow$) Let (π, Γ) be a covariant representation of $(\mathcal{F}(K), S, \varphi)$ and (π^*, Γ) the induced representation of $(\widehat{\mathrm{F}}(K), S)$ (see Remark 2.3). Finally let $(\check{\pi}^*, \Gamma)$ be the covariant representation of $(\mathrm{F}(K), S)$ defined by $\check{\pi}^* := \pi^* \circ Q^F$. By restricting $\check{\pi}^*$ to $\mathrm{T}_1(K)$ one can easily see that the pair $(\check{\pi}^*, \Gamma)$ is a causal and covariant connection of K.

($\Leftarrow$) Let (u, Γ) be a causal and covariant connection of K. Since $\mathrm{T}_1(K)$ generates $\mathrm{F}(K)$ we can extend u over all $\mathrm{F}(K)$,

$$u(w) := u(b_n) \cdots u(b_1) \quad w = b_n \cdots b_1 \in \mathrm{F}(K), \;\; b_k \in \mathrm{T}_1(K), \;\; k = 1, \ldots, n.$$

This yields a covariant representation (u, Γ) of $(\mathrm{F}(K), S)$. By property (3) of Definition 4.2, u annihilates on the subgroup of causal commutators of $\mathrm{F}(K)$, so it admits an extension to a representation $(\hat{u}, \Gamma)$ of $(\widehat{\mathrm{F}}(K), S)$. Finally, this induces a covariant representation $(\hat{u}_*, \Gamma)$ of $(\mathcal{F}(K), S, \varphi)$, (see Remark 2.3).

It is easily seen that these two constructions are each other's inverses. □

Some observations are in order. First, this lemma enhances the results obtained in [32], since it proves the existence of non-trivial causal and covariant connections. Secondly, the above correspondence preserves the strongly continuity of the representation of the symmetry group. So we have a 1–1 correspondence between continuously covariant representations of the net of causal loops induced by $(\mathcal{F}(K), S, \varphi)$ and causal and continuously covariant connections of K.

4.3 Connection systems

The representations of the net of causal loops induced by $(\mathcal{A}(K), S, \alpha)$ generalize those induced by $(\mathcal{F}(K), S, \varphi)$. But, as in general there does not exist a canonical extension of covariant representations between an inclusion of C^*-dynamical systems, we cannot follow the route of the previous section to get a correspondence with the cohomology of posets. So, following the scheme (4.1), we first associate a representation of the group of loops $\mathrm{L}(K)$ to any representation of $(\mathcal{A}(K), S, \alpha)$ and, afterwards, associate 1-cochains of K to representations of $\mathrm{L}(K)$.

To begin, we give the following

Definition 4.4. Let K be a causal poset with a symmetry group S. A **causal and covariant representation** of $\mathrm{L}(K)$ is a pair (w, Γ), where w is a unitary representation of $\mathrm{L}(K)$ on a Hilbert space $\mathcal{H}$ and Γ is a unitary representation of S on $\mathcal{H}$ satisfying the relations

(1) $\Gamma_s \, w(p) = w(s(p)) \, \Gamma_s$ for any $s \in S$ and loop p;
(2) $[w(p), w(q)] = 0$ for any pair p, q of causally disjoint loops.

We say that (w, Γ) is **equivalent** to another such a representation (w', Γ') on a Hilbert space $\mathcal{H}'$ if there is a field $t : \Sigma_0(K) \ni o \to t(o) \in \mathcal{U}(\mathcal{H}, \mathcal{H}')$ of unitary operators such that

$$t(o) \, w(p) = w'(p) \, t(o), \quad t(s(o)), \quad \Gamma_s = \Gamma'_s \, t(o), \quad p : o \to o, \ s \in S.$$

If Γ is strongly continuous, then we say that (w, Γ) is **continuously covariant**.

Now, following the same reasoning of the proof of Lemma 4.3, it follows that: *(continuously) covariant representations of the dynamical system* $(\mathcal{A}(K), S, \alpha)$ *are in a 1–1 correspondence with causal and (continuously) covariant representations of* $\mathrm{L}(K)$.

What remains to be understood is how to relate such representations of $\mathrm{L}(K)$ to connection 1-cochains of K. To this end, we recall a standard procedure to get 1-cochains from functions defined over loops. This procedure, borrowed from algebraic topology, has been used to establish a correspondence between representations of the fundamental group of K and 1-cocycles in [35]. Basically it makes use of the notion of a path-frame introduced in [11]. A *path-frame* P_o *over a pole* o is a choice of a path $p_{(o,a)} : o \to a$ for any element a of the poset K, subjected to the condition that $p_{(o,o)} = 1$. Clearly, since any causal poset is pathwise connected, path frames formed by elements of $\mathrm{T}_1(K)$ exist.[7]

So, following [11], given a path frame P_o and a causal and covariant representation (w, Γ) of $\mathrm{L}(K)$ define

$$u(b) := w(p_{(o,\partial_0 b)} \, b \, \overline{p_{(o,\partial_1 b)}}), \quad b \in \mathrm{T}_1(K).$$

[7] This holds true for the causal posets we are considering, namely those of Section 2.1.1. According to the definition of $\mathrm{T}_1(K)$, it is enough to observe that if $b = (a; a, o) \in \mathrm{N}_1(K)$, i.e., is a 1-simplex of the nerve, we can always find $\tilde{a} \Supset a$ (see Section 2.1.1). Then the 1-simplex $(\tilde{a}; a, o) \in \mathrm{T}_1(K)$ joins the faces a and o.

This is a connection 1-cochain since $u(\bar{b}) = u(b)^*$ but, in general, u is neither covariant nor causal in the sense of the Definition 4.2.

To get causality and covariance we need a *covariant path-frame system*, that is a collection of path frames $\mathcal{P} := \{P_o \,|\, o \in K\}$ satisfying

$$s(P_a) = P_{s(a)}, \quad a \in K,\, s \in S, \tag{4.5}$$

which amounts to saying that $s(p_{(a,x)}) = p_{(s(a),s(x))}$, where $p_{(a,x)} \in P_a$ whilst $p_{(s(a),s(x))} \in P_{s(a)}$. Note that, according to this definition, if s is in the stabilizer S_a of a, then $s(P_a) = P_a$.

We now give a necessary and sufficient condition for the existence of covariant path-frame systems, afterwards we apply this result to two remarkable examples.

Proposition 4.5. *Let K be a causal poset with a symmetry group S. K has a covariant path-frame system if, and only if, for any pair $o, a \in K$ there is a path joining a to o whose generators are invariant under $S_a \cap S_o$.*

Proof. ($\Rightarrow$) Let $\mathcal{P}$ be a covariant path-frame system. According to (4.5) if s is an element of the stabilizer S_a of a, then $s(P_a) = P_a$. Let y be such that $S_y \cap S_a \neq 1$. Then for any $s \in S_y \cap S_a$ we have that $s(p_{(a,y)}) = p_{(a,y)}$. This, according to (3.5) and (3.6), amounts to saying that all the 1-simplices generating the path $p_{(a,y)}$ are invariant under $S_a \cap S_y$.

($\Leftarrow$) For any element in the orbit space K/S choose a representative a. We first construct a suited path frame over a. To this end consider the orbit space K/S_a. For any orbit in this space choose a representative y and define $p_{(a,y)}$ as a path whose generating 1-simplices are invariant under $S_a \cap S_y$. Clearly if $S_a \cap S_y = 1$ no restriction is imposed on the choice of the path. Afterwards if $z = sy$ for some $s \in S_a$ define

$$p_{(a,z)} := s(p_{(a,y)})$$

This defines a path frame P_a over a. Note that if $r \in S_a$, then

$$r(p_{(a,z)}) = rs(p_{(a,y)}) = p_{(a,r(z))}. \tag{4.6}$$

So P_a agrees with the above observation. We now construct a path frame $P_{a'}$ for any element a' in the orbit of a. If $a' = s(a)$ for $s \in S$, then we set

$$p_{(a',y)} := s(p_{(a,s^{-1}(y))}), \quad y \in K.$$

This definition is well posed. In fact if $r(a) = a'$ then $s^{-1}r$ belongs to the stabilizer of a. Hence by equation (4.6) we have

$$r(p_{(a,r^{-1}(y))}) = s(s^{-1}r)(p_{(a,r^{-1}y)}) = s(p_{(a,(s^{-1}rr^{-1})(y))}) = s(p_{(a,s^{-1}y)}).$$

The collection P_a, with $a \in K$, is manifestly covariant. □

On this ground, the cases of Minkowski spacetime and S^1 can be completely understood.

Corollary 4.6. *The following assertions hold.*

(i) *The set of double cones of the Minkowski spacetime has a covariant path-frame system with respect to the Poincaré group.*
(ii) *The set of non-empty, open, connected intervals of S^1, having a proper closure, does not have a covariant path-frame system with respect to the Möbius group.*

Proof. (i) Fix a reference frame of the Minkowski spacetime. According to the definition of a double cone given in Section 2.1.1, it is enough to study the following situation: o_R is a double cone based on the subspace C_0 at time $t = 0$ whose base is centred at the origin of the reference frame; o is any other double cone. Recall that the stabilizer of o_R is $SO(3)$, the subgroup of spatial rotations. Now, let $S = S_{o_R} \cap S_o$ be the intersection of the stabilizers of o_R and o. Take $R' > 0$ large enough that $o_R, o \subseteq o_{R'}$. Then $S = S_{o_R} \cap S_o \cap S_{o_{R'}}$ because o_R and $o_{R'}$ have the same stabilizer. Hence the 1-simplex $(o_{R'}; o, o_R)$ satisfies the condition of Proposition 4.5.

(ii) We show that the condition of Proposition 4.5 is not verified. As observed in Section 2.1.1, the stabilizer of an interval a, which is isomorphic to the modular group, is also the stabilizer of the open interval $S^1 \setminus \{cl(a)\}$ of a. Since the modular group acts transitively within these intervals, there is no other interval having the same stabilizer. □

Just a comment about this result. Actually the above proofs seem to point out a topological obstruction to the existence of a path-frame system. The reason why the poset of double cones of the Minkowski spacetime has path-frame systems relies on the fact that this poset is upward directed. This implies that the poset is contractible, (see [35]). Clearly this does not happen for the intervals of the circle, whose associated poset has homotopy group $\mathbb{Z}$.

Given a covariant path-frame system $\mathcal{P}$, we associate with any $o \in K$ the connection 1-cochain

$$\mathrm{u}_{\mathcal{P}_o}(b) := w(p_{(o,\partial_0 b)}\, b\, \overline{p_{(o,\partial_1 b)}}), \quad b \in \mathrm{T}_1(K). \tag{4.7}$$

Clearly, as before, if we look at a single connection $\mathrm{u}_{\mathcal{P}_o}$ we have neither covariance nor causality, however these two properties arise if we look at the collection $\mathrm{u}_{\mathcal{P}} := \{\mathrm{u}_{\mathcal{P}_o}, o \in K\}$. In fact since we are using a covariant path-frame system we have

$$\begin{aligned}\mathrm{Ad}_{\Gamma_s}(\mathrm{u}_{\mathcal{P}_o}(b)) &= \mathrm{Ad}_{\Gamma_s}(w(p_{(o,\partial_0 b)}\, b\, \overline{p_{o,\partial_1 b}})) \\ &= w(s(p_{(o,\partial_0 b)})\, s(b)\, \overline{s(p_{o,\partial_1 b})}) = \mathrm{u}_{\mathcal{P}_{s(o)}}(s(b)).\end{aligned}$$

Moreover, w is recovered from the system $\mathrm{u}_{\mathcal{P}}$ by observing that $w(p) = \mathrm{u}_{\mathcal{P}_a}(p)$, for any loop $p : a \to a$, because $p_{(a,a)} = 1$. Causality follows from this relation, in the sense that $[\mathrm{u}_{\mathcal{P}_a}(p), \mathrm{u}_{\mathcal{P}_o}(q)] = 0$ for any $p : a \to a$, $q : o \to o$ with $p \perp q$.

Hence we may give the following

Definition 4.7. Let K be a causal poset with a symmetry group S. A **causal and covariant connection system** is a pair (u, Γ) where u is a family $\{\mathrm{u}_o, o \in K\}$ of connection 1-cochains of K in a Hilbert space $\mathcal{H}$ and Γ is a unitary representation of S on $\mathcal{H}$ such that

(1) $[\mathrm{u}_o(p), \mathrm{u}_a(q)] = 0$, if $p \perp q$ and $p : o \to o$, $q : a \to a$.
(2) $\mathrm{Ad}_{\Gamma_s} \circ \mathrm{u}_o = \mathrm{u}_{s(o)} \circ s$, for any $s \in S$ and $o \in K$.

We say that (u, Γ) is **equivalent** to (u', Γ') if there is a family $\mathrm{t} := \{\mathrm{t}_o,\ o \in K\}$ of fields $\mathrm{t}_o : \Sigma_0(K) \ni a \to \mathrm{t}_o(a) \in \mathcal{U}(\mathcal{H}, \mathcal{H}')$ of unitary operators satisfying the relations

$$\mathrm{t}_o(\partial_0 b)\mathrm{u}_o(b) = \mathrm{u}'_o(b)\mathrm{t}_o(\partial_1 b), \quad \mathrm{t}_{s(o)} \circ s = \Gamma'_s \mathrm{t}_o \Gamma_{s^{-1}}, \quad b \in \mathrm{T}_1(K), \quad s \in S.$$

If Γ is strongly continuous, then we say that (u, Γ) is **continuously covariant**.

Note that if (u, Γ) is a connection as in Definition 4.2 then setting $\mathrm{u}_o := u$ for any o yields a causal and covariant connection system (u, Γ). Thus the notion of a connection system generalizes that of a connection.

Now, it is easy to see that any causal and covariant connection system (u, Γ) defines a causal representation of the dynamical system $(\mathcal{A}(K), S, \alpha)$.

To this end, define

$$w(p) := \mathrm{u}_o(p), \quad p : o \to o. \tag{4.8}$$

Clearly $w(p)^* = w(\overline{p})$ for any loop p; causality and covariance of w follow from properties 1 and 2 of Definition 4.7. In particular, concerning covariance, if $p : o \to o$ then

$$\mathrm{Ad}_{\Gamma_s}(w(p)) = \mathrm{Ad}_{\Gamma_s}(\mathrm{u}_o(p)) = \mathrm{u}_{s(o)}(s(p)) = w(s(p)).$$

The observation below Definition 4.4 and the (4.7) and (4.8) establish a correspondence between causal and covariant connection systems of K and representations of the net of causal loops induced by $(\mathcal{A}(K), S, \alpha)$. One can easily check that this correspondence is, up to equivalence, bijective.

Proposition 4.8. *There is, up to equivalence, a 1–1 correspondence between causal and (continuously) covariant connection systems of K and (continuously) covariant representations of the net of causal loops induced by $(\mathcal{A}(K), S, \alpha)$.*

4.4 Gauge transformations

Intuitively a gauge transformation of a causal and covariant connection system is a transformation preserving covariance and causality. More precisely,

Definition 4.9. A **gauge transformation** of a causal and covariant connection system (u, Γ) is a collection $\mathrm{g} := \{\mathrm{g}_a,\ a \in K\}$ of fields $\mathrm{g}_a : \Sigma_0(K) \ni o \to \mathrm{g}_a(o) \in \mathcal{UH}$ of unitary operators satisfying the following properties

(1) $\mathrm{g}_{s(a)}(s(o))\,\Gamma_s = \Gamma_s\,\mathrm{g}_a(o)$, for any $a, o \in K$ and any $s \in S$;
(2) $\mathrm{Ad}_{\mathrm{g}_a(a)}(\mathrm{u}(\mathcal{A}_o)) = \mathrm{u}(\mathcal{A}_o)$ for any inclusion $a \le o$,

where by $\mathrm{u}(\mathcal{A}_o)$ we mean the fibre $\mathcal{A}_o$ of the net of causal loops in the representation induced by u. We denote the set of gauge transformations by $\mathcal{G}$.

We now draw on some consequences of this definition. First, given a gauge transformation g of (u, Γ), we define

$$\mathrm{u}^{\mathrm{g}}_o(b) := \mathrm{g}_o(\partial_0 b)\,\mathrm{u}_o(b)\,\mathrm{g}^*_o(\partial_1 b), \quad b \in \mathrm{T}_1(K), \tag{4.9}$$

for any $o \in K$, and observe that the pair $(\mathrm{u}^{\mathrm{g}}, \Gamma)$ is a causal and covariant connection system. Covariance easily follows from the definition of gauge transformation. Concerning causality, let $p : o \to o$ and $q : a \to a$ be two causally disjoint loops. This, according to (3.4), amounts to saying that there are $\tilde{o}, \tilde{a} \in K$ such that $|p| \leq \tilde{o}$ and $|q| \leq \tilde{a}$ and $\tilde{o} \perp \tilde{a}$. Property 2 of Definition 4.9 implies that

$$\mathrm{u}^{\mathrm{g}}_{o}(p) \in \mathrm{u}(\mathcal{A}_{\tilde{o}}), \quad \mathrm{u}^{\mathrm{g}}_{a}(q) \in \mathrm{u}(\mathcal{A}_{\tilde{a}}).$$

Hence, $\mathrm{u}^{\mathrm{g}}_{o}(p)$ and $\mathrm{u}^{\mathrm{g}}_{a}(q)$ commute because so do the algebras $\mathcal{A}_{\tilde{o}}$ and $\mathcal{A}_{\tilde{a}}$. Secondly, by (4.9) the causal and covariant representation (w, Γ) of $\mathrm{L}(K)$ associated with (u, Γ) by (4.8), transforms, under a gauge transformation g, as

$$w^{\mathrm{g}}(p) = \mathrm{g}_{o}(o)\, w(p)\, \mathrm{g}^{*}_{o}(o), \quad p : o \to o. \tag{4.10}$$

Thirdly, as one can easily deduce by Definition 4.9, gauge transformations form a group under the composition

$$(\mathrm{g} \cdot \mathrm{h})_{a}(o) := \mathrm{g}_{a}(o)\, \mathrm{h}_{a}(o), \quad a, o \in K.$$

So we shall refer to $\mathcal{G}$ as the group of *gauge transformations.* Fourthly, for any $o \in K$, let G_o the subgroup of $\mathcal{UH}$ generated by

$$\{\mathrm{g}_{a}(a) \in \mathcal{UH} \,|\, \mathrm{g} \in \mathcal{G},\ a \leq o\}. \tag{4.11}$$

The group G_o acts by automorphisms upon the algebra $\mathrm{u}(\mathcal{A}_o)$. We refer to G_o as *the gauge group over o.* Note that according to Definition 4.9, we have that $G_o \subseteq G_a$ for any inclusion $o \leq a$ and that G_o is spatially isomorphic to $G_{s(o)}$ for any $s \in S$. So we have a *covariant net of gauge groups* $(G, \mathrm{id}, \mathrm{Ad}_{\Gamma})_K$ whose fibres are the gauge groups G_o.

As an interesting case, let us analyze how the connection system associated with a causal and covariant representation (w, Γ) of $\mathrm{L}(K)$ depends on the choice of the path-frame system. Consider two path-frame systems $\mathcal{P}$ and $\mathcal{Q}$ and the corresponding connection systems $\mathrm{u}_{\mathcal{P}}$ and $\mathrm{u}_{\mathcal{Q}}$. To be precise by equation (4.7), we have

$$\mathrm{u}_{\mathcal{P}_o}(b) = w(p_{o,\partial_0 b}\, b\, \overline{p_{(o,\partial_1 b)}}), \quad \mathrm{u}_{\mathcal{Q}_o}(b) = w(q_{o,\partial_0 b}\, b\, \overline{q_{(o,\partial_1 b)}}),$$

for any $b \in \mathrm{T}_1(K)$ and $o \in K$. Then observe that if we set

$$\mathrm{g}_{o}(a) := w(q_{(o,a)} \overline{p_{(o,a)}}), \quad a \in K,$$

since $g_o(o) = 1$ for any o, then we get a gauge transformation g of $(\mathrm{u}_{\mathcal{P}}, \Gamma)$ such that $\mathrm{u}^{\mathrm{g}}_{\mathcal{P}} = \mathrm{u}_{\mathcal{Q}}$. So a changing of a path-frame system leads to a gauge transformation of the connection system. Moreover, w is invariant under this gauge transformation, that is $w^{\mathrm{g}} = w$.

4.5 Existence of continuously covariant representations

In this section we focus on the Minkowski spacetime $\mathbb{M}^4$ and prove the existence of continuously Poincaré-covariant (in symbols $\mathscr{P}_+^\uparrow$-covariant) representations of the net of causal loops over the set of double cones of $\mathbb{M}^4$. We also point out that this result applies to S^1, when the symmetry is given by the Möbius group, up to some modifications.

The approach we shall use points out the universality of the net of causal loops. In fact, we shall see that any Hermitian scalar quantum field over $\mathbb{M}^4$ gives a causal and continuously $\mathscr{P}_+^\uparrow$-covariant connection, hence a continuously $\mathscr{P}_+^\uparrow$-covariant representation of net of causal loops. We deserve a further investigation to analyze the explicit physical interest of this correspondence.

Concretely, we first define a *translator* that allows us to pass from quantum fields to connections. This is a 1-cochain of the poset taking values in the set of real, compactly supported smooth functions of the spacetime, which is invariant under the action of the symmetry group.

4.5.1 From simplices to test functions

In this section we define the above cited translator and prove its existence in the case of the posets associated with the Minkowski spacetime and with S^1.

Consider a spacetime X and denote the associated causal disjointness relation by $\perp$. Let S be a symmetry group of X, i.e., a subgroup of conformal diffeomorphisms of X. Concretely, if X is the Minkowski spacetime then S is the Poincaré group; if X is the circle then S is the Möbius group. Take a base K of X of open and relatively compact subsets of X, such that K is globally invariant under the action of S.

We denote the space of compactly supported, real valued, smooth functions on X by $\mathcal{D}(X)$. Similarly to the definition of net of C*-algebras given in Section 2.3, we have a net of vector spaces $(\mathcal{D}, \mathrm{id})_K$ assigning to each element a of K the space $\mathcal{D}_a$ of those functions of $\mathcal{D}(X)$ vanishing outside the closure $\mathrm{cl}(a)$ in X of a. For any $n \in \mathbb{N}$, we consider the *n-cochain* vector space

$$\mathrm{C}^n(K, \mathcal{D}) := \{f : \Sigma_n(K) \ni x \mapsto f_x \in \mathcal{D}_{|x|}\}. \tag{4.12}$$

There is a coboundary operator $d : \mathrm{C}^n(K,\mathcal{D}) \to \mathrm{C}^{n+1}(K,\mathcal{D})$ defined, as usual, as

$$(df)_x = \sum_{k=0}^{n} (-1)^k f_{\partial_k x}, \quad x \in \Sigma_{n+1}(K), \quad f \in \mathrm{C}^n(K,\mathcal{D}), \tag{4.13}$$

satisfying the equation $d \circ d = 0$. This allows us to define cohomology groups, but this is out of the aims of the present paper. There is a left action of the symmetry group S on $\mathrm{C}^n(K,\mathcal{D})$ defined by

$$(sf)_x := f_{s(x)} \circ s, \quad x \in \Sigma_n(K), \ s \in S. \tag{4.14}$$

Now, considering the fixed-point subspace of $\mathrm{C}^n(K,\mathcal{D})$ under the action of S, i.e.,

$$\mathrm{C}^n(K,\mathcal{D})^S := \{f \in \mathrm{C}^n(K,\mathcal{D}) \,|\, (sf)_x = f_x, \quad x \in \Sigma_n(K), \ s \in S\}, \tag{4.15}$$

we note that

$$\mathrm{C}^n(K,\mathcal{D})^S \ni f \quad \iff \quad f_{s(x)} = f_x \circ s^{-1}, \tag{4.16}$$

for any $x \in \mathrm{T}_n(K)$ and $s \in S$. We also observe that if $f \in \mathrm{C}^n(K,\mathcal{D})^S$, then $df \in \mathrm{C}^{n+1}(K,\mathcal{D})^S$ because the coboundary operator d commutes with the action of the symmetry group. In this regards one should note that df might be equal to 0 even if f is not. This happens when f is an n-cocycle.

For our aims it is of importance to understand when $\mathrm{C}^n(K,\mathcal{D})^S$ is not trivial, i.e., different from 0. The next result clarifies this point.

Proposition 4.10. *The space $\mathrm{C}^n(K,\mathcal{D})^S$ is not trivial if, and only if, there is $y \in \Sigma_n(K)$ and a non-zero function $f_0 \in \mathcal{D}_{|y|}$ such that $f_0 \circ s^{-1} = f_0$ for any s in the stabilizer S_y of y.*

Proof. ($\Rightarrow$) Let $f \in \mathrm{C}^n(K,\mathcal{D})^S$ be a non-zero function. So $f_y \neq 0$ for some n-simplex y. By (4.16), $f_y = f_y \circ s^{-1}$ for any $s \in S_y$.

($\Leftarrow$) Let f_0 and y be as above. For any $x \in \Sigma_n(K)$, define

$$f_x := \begin{cases} f_0 \circ s^{-1}, & \text{if } x = s(y), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f_y = f_0$ and that f is different from zero only on the orbit of y. This definition is well posed because if $x = r(y)$ then $s^{-1}r \in S_y$, so that

$f_0 \circ s^{-1} = f_0 \circ (s^{-1}r)r^{-1} = f_0 \circ r^{-1}$. Thus for any x in the orbit of y, according to the definition of the action of S, we have

$$(s'f)_x = f_{s'(x)} \circ s' = f_{s's(y)} \circ s' = f_0 \circ (s's)^{-1}s' = f_0 \circ s^{-1} = f_x,$$

for any $s' \in S$, completing the proof. $\square$

Remark 4.11. Using the same idea of the above proof we can prove the following stronger property: there exists $f \in \mathrm{C}^n(K, \mathcal{D})^S$ which does not vanish on *any* n-simplex y for which there is a non-zero function $f_0 \in \mathcal{D}_{|y|}$ satisfying $f_0 \circ s^{-1} = f_0$ for any $s \in S_y$. So, in particular, f does not vanish on the n-simplices over which S acts freely.

We shall use this result later to show the existence of invariant cochains for the Minkowski spacetime and for the circle. We now make a step toward the main result of this section and specialize to the set of 1-cochains. Note that $\mathrm{C}^1(K, \mathcal{D})$ is a $\mathbb{Z}_2$-graded vector space under the action $\mathrm{C}^1(K, \mathcal{D}) \ni f \to \overline{f} \in \mathrm{C}^1(K, \mathcal{D})$ defined as

$$\overline{f}_b := f_{\overline{b}}, \quad b \in \Sigma_1(K). \tag{4.17}$$

This yields the direct sum decomposition in the even/odd spectral subspaces

$$\mathrm{C}^1(K, \mathcal{D}) \to \mathrm{C}^{1,\mathrm{ev}}(K, \mathcal{D}) \oplus \mathrm{C}^{1,\mathrm{odd}}(K, \mathcal{D}), \quad f \mapsto f^{\mathrm{ev}} \oplus f^{\mathrm{odd}}, \tag{4.18}$$

where

$$f^{\mathrm{ev}} := \frac{1}{2}(f + \overline{f}), \quad f^{\mathrm{odd}} := \frac{1}{2}(f - \overline{f}) \tag{4.19}$$

are such that

$$\overline{f^{\mathrm{ev}}} = f^{\mathrm{ev}}, \quad \overline{f^{\mathrm{odd}}} = -f^{\mathrm{odd}}. \tag{4.20}$$

Note that $(df)^{\mathrm{ev}} = 0$ while $(df)^{\mathrm{odd}} = df$ for any $f \in \mathrm{C}^0(K, \mathrm{D})$. Now, since the $\mathbb{Z}_2$-grading commutes with the induced S-action, the direct sum decompositions (4.18) and (4.19) apply also to the subspace $\mathrm{C}^1(K, \mathcal{D})^S$:

$$\mathrm{C}^1(K, \mathcal{D})^S \to \mathrm{C}^{1,\mathrm{ev}}(K, \mathcal{D})^S \oplus \mathrm{C}^{1,\mathrm{odd}}(K, \mathcal{D})^S, \quad f \mapsto f^{\mathrm{ev}} \oplus f^{\mathrm{odd}}. \tag{4.21}$$

Let us consider the mapping $\delta : \mathrm{C}^0(K, \mathcal{D}) \to \mathrm{C}^1(K, \mathcal{D})$ defined as

$$(\delta f)_b := f_{\partial_0 b} - f_{\partial_1 b} + f_{|b|} = (df)_b + f_{|b|}, \quad b \in \Sigma_1(K), \tag{4.22}$$

which is twist of the coboundary operator d at the level of 0-cochains. Observe that δ, like d, commutes with the S-action; so $\delta : \mathrm{C}^0(K, \mathcal{D})^S \to$

$\mathrm{C}^1(K,\mathcal{D})^S$. Concerning the grading (4.18) we have that

$$(\delta f)_b^{\mathrm{ev}} = f_{|b|}, \quad (\delta f)_b^{\mathrm{odd}} = (df)_b, \quad b \in \Sigma_1(K). \tag{4.23}$$

We now apply these results to posets associated with the Minkowski spacetime and to the circle.

Proposition 4.12. *Let K be the set of double cones of the Minkowski spacetime $\mathbb{M}^4$. Then there exists $f \in \mathrm{C}^0(K,\mathcal{D})^{\mathscr{P}_+^\uparrow}$ such that $f_a \neq 0$ and is non-negative for any 0-simplex a. Furthermore, the 1-cochain $\delta f \in \mathrm{C}^1(K,\mathcal{D})^{\mathscr{P}_+^\uparrow}$ satisfies the following properties*

(i) $(\delta f)_b^{\mathrm{ev}} = f_{|b|} \neq 0$ *is non-negative for any 1-simplex b;*
(ii) $(\delta f)_b^{\mathrm{odd}} = f_{\partial_0 b} - f_{\partial_1 b} \neq 0$ *for any 1-simplex b such that $\partial_0 b \neq \partial_1 b$.*

Proof. As observed in Section 2.1.1, any double cone lays in the orbit of a double cone o_R whose base belongs to the subspace of the Minkowski spacetime at $t = 0$, is centred at the origin of the reference frame and has radius equal to $R > 0$. The stabilizer of any such double cone is the subgroup of spatial rotations $SO(3)$. So, consider a double cone o_R and let $h : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function supported in $[-1, 1]$. Define

$$f_R(t,x,y,z) := h\left(2 \cdot \frac{t}{R}\right) \cdot h\left(4 \cdot \frac{x^2+y^2+z^2}{R^2}\right). \tag{4.24}$$

Here, f_R is a non-negative smooth function supported in o_R that satisfies $f_R \circ s^{-1} = f_R$ for any $s \in SO(3)$. So, applying Proposition 4.10 and the Remark 4.11, f_R defines an invariant 0-cochain f such that $f_a \neq 0$ for any $a \in \Sigma_0(K)$.

As observed above $\delta f \in \mathrm{C}^1(K,\mathcal{D})^{\mathscr{P}_+^\uparrow}$. Using the equation (4.23) (i) is obvious, while (ii) easily follows from the definition of f. □

The argument used in the proof of the previous proposition cannot be applied to the circle with the Möbius group as a symmetry group. The stabilizer of an interval acts transitively within the interval (see Section 2.1.1). So there does not exist any invariant non-zero smooth function supported within the interval. Hence, according to Proposition 4.10, there does not exist any non-trivial, Möbius invariant 0-cochain. However this is not the case for 1-cochains.

Proposition 4.13. *Let K be the set of open, connected intervals of S^1 having a proper closure. Then there exists $f \in \mathrm{C}^1(K,\mathcal{D})^{M\ddot{o}b}$ such that $f_b \neq 0$ and non-negative for any non-degenerate 1-simplex.*

Proof. The Möbius group acts freely on non-degenerate 1-simplices (see Section 3.1 for the definition of a degenerate 1-simplex). Therefore, we set $f_b = 0$ if b is a degenerate 1-simplex. For non-degenerate 1-simplices, we choose a representative element $y \in \Sigma_1(K)$ for any orbit and take a non-zero and non-negative smooth function f_0 supported within $|y|$. Then proof follows from Proposition 4.10 and Remark 4.11. □

4.5.2 From quantum fields to causal and continuously covariant connections

We now prove the existence of continuously covariant representations of the net of causal loops in the case of the Minkowski spacetime. This will be done by showing that any scalar quantum field satisfying the Wightman axioms, and other two standard properties, induces a causal and continuously covariant connection. For brevity, we shall outline the features of a quantum field strictly necessary for the aims of the present paper and refer the reader to the standard textbooks in axiomatic QFT [1, 8, 22, 25, 34, 41] for a complete description of the Wightman axioms.

Let $(\Phi, \Gamma, \mathcal{H})$ be an Hermitian scalar quantum field in the Minkowski spacetime $\mathbb{M}^4$ satisfying the Wightman axioms. The field Φ is an operator valued distribution defined on a dense domain D of a separable Hilbert space $\mathcal{H}$: a linear mapping $f \mapsto \Phi(f)$ associating an unbounded operator $\Phi(f)$ to any Schwartz function $f \in \mathcal{S}(\mathbb{R}^4)$ such that $\Phi(f)$ and its adjoint $\Phi(f)^*$ map D into D. Γ is a strongly continuous unitary representation of the Poincaré group $\mathscr{P}_+^\uparrow$ on $\mathcal{H}$ implementing the covariance of the field:

$$\Gamma_s : D \to D, \quad \mathrm{Ad}_{\Gamma_s}(\Phi(f))D = \Phi(f \circ s^{-1})D, \quad s \in \mathscr{P}_+^\uparrow, \quad f \in \mathcal{S}(\mathbb{R}^4). \tag{4.25}$$

In what follows we focus on two properties that are a natural strengthening of the axioms and that are relevant for our aims. First, we assume that the field is *self-adjoint*, namely

P1. $\Phi(f)$ is an essentially self-adjoint operator on the domain D, for any real Schwartz function $f \in \mathcal{S}_\mathrm{r}(\mathbb{R}^4)$.

From now on, we shall work with the closure of $\Phi(f)$ that, with a little abuse of notation, we shall denote by the same symbol. Using **P1** we can define the exponential $\exp(\mathrm{i}\Phi(f))$ for any $f \in \mathcal{S}_\mathrm{r}(\mathbb{R}^4)$. It turns out that $\exp(\mathrm{i}\Phi(f))$ is a unitary operator of $\mathcal{H}$ satisfying the relations

$$\exp(\mathrm{i}\Phi(-f)) = \exp(\mathrm{i}\Phi(f))^* \quad \text{and} \quad \mathrm{Ad}_{\Gamma_s}(\exp(\mathrm{i}\Phi(f))) = \exp(\mathrm{i}\Phi(f \circ s^{-1})), \tag{4.26}$$

for any $f \in \mathcal{S}_{\mathrm{r}}(\mathbb{R}^4)$ and for any $s \in \mathscr{P}_+^{\uparrow}$; the latter relation, in particular, derives from (4.25). Secondly, we assume the property of *strong causality* of the field, namely

P2. For any pair $f, g \in \mathcal{S}_r(\mathbb{R}^4)$

$$\mathrm{supp}(f) \perp \mathrm{supp}(g) \quad \Rightarrow \quad [\exp(\mathrm{i}\Phi(f)), \exp(\mathrm{i}\Phi(g))] = 0. \tag{4.27}$$

We stress that **P1** and **P2** are not required in the usual scheme of the Wightman axioms. However, these two properties are verified by any scalar field satisfying the Osterwalder–Schrader axioms [22] and, in particular, by any known example of a Hermitian scalar quantum field.

We now are ready to prove the main results of this section.

Theorem 4.14. *Let* $(\Phi, \Gamma, \mathcal{H})$ *be a Hermitian scalar quantum field in the Minkowski spacetime* $\mathbb{M}^4$ *satisfying the Wightman axioms and the properties* $\mathbf{P_1}$ *and* $\mathbf{P_2}$*. Let* K *be the set of double cones in the Minkowski spacetime. Given a non-zero* $f \in \mathrm{C}^1(K, \mathcal{D})^{\mathscr{P}_+^{\uparrow}}$*, define*

$$[\Phi f](b) := \exp\left\{ \mathrm{i} \int_{b_{\circledcirc}} |f_b^{\mathrm{ev}}(x)| d^4x \cdot \Phi(f_b^{\mathrm{odd}}) \right\}, \quad b \in \Sigma_1(K), \tag{4.28}$$

where d^4x *is the Lebesgue measure of* $\mathbb{R}^4$ *and* $b_{\circledcirc} := |b| - (\partial_1 b \cup \partial_0 b) \subseteq X$ *shall be called* ***the corona of*** b*. Then the pair* $([\Phi f], \Gamma)$ *is a causal and continuously* $\mathscr{P}_+^{\uparrow}$*-covariant connection of* K*. In particular* $[\Phi f](b) = 1$ *for any* $b \in \mathrm{N}_1(K)$*.*

Proof. Clearly $[\Phi f](b) = 1$ when $b \in \mathrm{N}_1(K)$, because in this case $b_{\circledcirc} = \emptyset$. Since $f_{\bar{b}}^{\mathrm{odd}} = -f_b^{\mathrm{odd}}$, $f_{\bar{b}}^{\mathrm{ev}} = f_b^{\mathrm{ev}}$ and $\bar{b}_{\circledcirc} = b_{\circledcirc}$, by (4.26) we have that $[\Phi f]$ $(\bar{b}) = [\Phi f](b)^*$. This proves that $[\Phi f]$ is a connection 1-cochain. Causality follows by property $\mathbf{P_2}$. About covariance, since $f^{\mathrm{ev}}, f^{\mathrm{odd}} \in \mathrm{C}^1(K, \mathcal{D})^{\mathscr{P}_+^{\uparrow}}$, relation (4.16) implies that $f_{s(b)}^{\mathrm{ev}} = f_b^{\mathrm{ev}} \circ s^{-1}$ and $f_{s(b)}^{\mathrm{odd}} = f_b^{\mathrm{odd}} \circ s^{-1}$ for any 1-simplex b and any Poincaré transformation s. These relations, the (4.26) and the invariance of the Lebesgue measure under transformations of the Poincaré group, imply that the pair $([\Phi f], \Gamma)$ is a causal and continuously covariant connection of K. In fact, since $s(b_{\circledcirc}) = s(b)_{\circledcirc}$ for any 1-simplex b and any Poincaré transformation s, we have that $\int_{b_{\circledcirc}} |f_b^{\mathrm{ev}}(x)| d^4x = \int_{s(b)_{\circledcirc}} |f_b^{\mathrm{ev}}$

$(s^{-1}(x))|d^4x = \int_{s(b)_\circledcirc} |f^{\mathrm{ev}}_{s(b)}(x)|d^4x$. Then

$$\begin{aligned}\mathrm{Ad}_s([\Phi f](b)) &= \exp\left\{\mathrm{i}\int_{b_\circledcirc} |f^{\mathrm{ev}}_b(x)|d^4x \cdot \Phi(f^{\mathrm{odd}}_b \circ s^{-1})\right\} \\ &= \exp\left\{\mathrm{i}\int_{s(b)_\circledcirc} |f^{\mathrm{ev}}_{s(b)}(x)|d^4x \cdot \Phi(f^{\mathrm{odd}}_{s(b)}\cdot)\right\} = [\Phi f](s(b)),\end{aligned}$$

and this completes the proof. □

Hence, any Hermitian scalar quantum field defines a causal and continuously $\mathscr{P}^{\uparrow}_{+}$-covariant connection and this defines, in turn, a continuously $\mathscr{P}^{\uparrow}_{+}$-covariant representation of the net of causal loops. However, the general hypothesis of the previous theorem cannot exclude the triviality of this construction, i.e., Φ might annihilate on the 1-cochain f^{odd} even if f^{odd} is not trivial like, for instance, that constructed in Proposition 4.12.

We avoid this eventuality in the case of the free Hermitian scalar field $(\Phi_m, \Gamma_m, \mathcal{H}_m)$, of mass $m \geq 0$, in $\mathbb{M}^4$. This field satisfies all the properties outlined above, and we briefly recall its definition following closely the reference [34]. The Hilbert space $\mathcal{H}_m$ is the symmetric Fock space associated to the 1-particle Hilbert space $L^2(\mathrm{H}_m, d\Omega_m)$, where H_m is the hyperboloid of mass $m \geq 0$ of $\mathbb{R}^4$ and $d\Omega_m$ is the $\mathscr{L}^{\uparrow}_{+}$-invariant measure on H_m. Γ_m is a strongly continuous unitary representation of the Poincaré group $\mathscr{P}^{\uparrow}_{+}$ on $\mathcal{H}_m$, and the generators of the translation subgroup have joint spectrum on the closure of the forward lightcone $\overline{V}_+$. The field is defined as $\Phi_m(f) := \Phi_S(E_m(\Re(f))) + i\Phi_S(E_m(\Im(f)))$ for any test function$f \in \mathcal{S}(\mathbb{R}^4)$, where Φ_S is the Segal quantization of $L^2(\mathrm{H}_m, d\Omega_m)$ and $E_m : \mathcal{S}(\mathbb{R}^4) \to L^2(\mathrm{H}_m, d\Omega_m)$ is defined as

$$E_m(f) := (2\pi)^{-2} \int \mathrm{e}^{\mathrm{i}\, p\cdot x} f(x) d^4x \upharpoonright \mathrm{H}_m, \tag{4.29}$$

where $p \cdot x = p_0 t - \mathbf{p} \cdot \mathbf{x}$. So $E_m(f)$ is nothing but the restriction to the hyperboloid of mass m of the Fourier transformation of f.

We now are ready to prove the existence of non-trivial, causal and continuously $\mathscr{P}^{\uparrow}_{+}$-covariant connections.

Theorem 4.15. *Let K be the set of double cones in the Minkowski spacetime. Then:*

(i) *there exists a non-flat, causal and continuously $\mathscr{P}^{\uparrow}_{+}$-covariant connection of K;*

(ii) *the net of causal loops* $(\mathcal{A}, \mathrm{id}, \alpha)_K$ *has a non-trivial and continuously* $\mathscr{P}_+^\uparrow$*-covariant representation.*

Proof. (ii) follows from (i) and Lemma 4.3. So let us prove (i). Consider the free Hermitian scalar field $(\Phi_m, \Gamma_m, \mathcal{H}_m)$. Let f be the invariant 0-cochain constructed in Proposition 4.12. As observed in that proposition the twisted cobord δf is an element of $\mathrm{C}^1(K, \mathcal{D})^{\mathscr{P}_+^\uparrow}$, $\delta f_b^{\mathrm{odd}} = f_{\partial_0 b} - f_{\partial_1 b}$ is different from zero on 1-simplices having different faces, and $\delta f_b^{\mathrm{ev}} = f_{|b|}$ is different from zero for any 1-simplex. Using the formula (4.28), define

$$[\Phi_m \delta f](b) := \exp\left\{ \mathrm{i} \int_{b_\circledcirc} |(\delta f)_b^{\mathrm{ev}}(x)| d^4x \cdot \Phi_m((\delta f)_b^{\mathrm{odd}}) \right\}, \quad b \in \Sigma_1(K).$$

By Theorem 4.14, the pair $([\Phi_m \delta f], \Gamma_m)$ is a causal and continuously $\mathscr{P}_+^\uparrow$-covariant connection. We now want to show that this connection is not trivial. This amounts to showing that $\Phi_m((\delta f)_b^{\mathrm{odd}}) \neq 0$ or, equivalently, that $E_m((\delta f^{\mathrm{odd}})_b) \neq 0$ for some 1-simplex b which is does not belong to the nerve of K.

We start by showing that $E_m(f_a) \neq 0$ for any 0-simplex a. Afterwards we shall prove that there are 1-simplices b such that $E_m((\delta f^{\mathrm{odd}})_b) \neq 0$. Recall that the 0-cochain f is obtained by letting act the Poincaré group on the set of functions f_R, $R > 0$ which are associated to double cones whose base lays on the subspace at $t = 0$ of the reference frame of the Minkowski spacetime. Recalling the definition of these functions (4.24) we observe that

$$\hat{f}_R(p) := \int \mathrm{e}^{\mathrm{i}\, p \cdot x} f_R(x) d^4x = f'_R(p_0) \cdot f''_R(\mathbf{p}),$$

where

$$f'_R(p_0) := \int \mathrm{e}^{\mathrm{i} p_0 t} h(2t/R) dt, \quad f''_R(\mathbf{p}) := \int \mathrm{e}^{-\mathrm{i}\mathbf{p} \cdot \mathbf{x}} h(4|\mathbf{x}|^2/R^2) d^3x.$$

Let us study the zeroes of the function $\hat{f}_R$. Note that p is a zero of $\hat{f}_R$ if, and only if, either p_0 is a zero of f'_R or $\mathbf{p}$ is a zero of f''_R. Observe that both f'_R and f''_R are holomorphic entire functions, since h is a compactly supported smooth function. Therefore the zeroes of f' are at most a countable set of $\mathbb{R}$. Instead, since f''_R is, up to a factor, the Fourier transform of a spherically symmetric function, if $f''_R(\mathbf{q}) = 0$ then $f''_R(\mathbf{p}) = 0$ for any $\mathbf{p} \in \mathbb{R}^3$ such that $|\mathbf{p}| = |\mathbf{q}| = r$. So the zeroes are distributed on spherical surfaces and the collection Z of the radius $r > 0$ of such surfaces, cannot have an accumulation point. In fact, if it were so, being the function $z \mapsto f''_R(z, 0, 0)$

holomorphic entire, since $f''_R(r,0,0) = 0$ for any $r \in Z$, we should have that $f''_R(z,0,0) = 0$ for any $z \in \mathbb{C}$. This implies that any $r \geq 0$ belongs to Z, so $f''_R = 0$ and this leads to a contradiction because f''_R is the Fourier transform of a non-zero smooth function. In conclusion we have that $E_m(f_R) \neq 0$ for any $R > 0$, because $\hat{f}_R = 0$, at most on a countable collection of measure zero subsets of H_m, associated to the zeroes of f'_R and of f''_R. This also implies that $E_m(f_a) \neq 0$ in $L^2(\mathrm{H}_m, d\Omega_m)$ since the 0-cochain f is obtained by letting act the Poincaré group of the function f_R for any $R > 0$.

Consider now the following 1-simplex b: $\partial_1 b := o_R$; $\partial_0 b := o_R + y$ with $0 \neq y \in \mathbb{R}^4$; $|b|$ is a double cone greater than $\partial_1 b \cup \partial_0 b$. Since $f_{\partial_0 b}(x) = f_R(x - y)$, we have that

$$\hat{f}_{\partial_0 b}(p) = \int \mathrm{e}^{\mathrm{i}\, p\cdot x} f_R(x-y) d^4x = \mathrm{e}^{\mathrm{i}} p \cdot y \hat{f}_R(p) = \mathrm{e}^{\mathrm{i}p\cdot y} \hat{f}_{\partial_1 b}(p).$$

Hence

$$E_m((\delta f)^{\mathrm{odd}}_b) = E_m(f_{\partial_0 b} - f_{\partial_1 b}) = E_m(f_{\partial_1 b}) \cdot (\mathrm{e}^{\mathrm{i}p\cdot y} - 1) \upharpoonright \mathrm{H}_m$$

and this is not zero in $L^2(\mathrm{H}_m, d\Omega_m)$. This shows that the connection 1-cochain $[\Phi_m \delta f]$ is not trivial. In addition $[\Phi_m \delta f]$ results to be not flat, i.e., does not satisfy the 1-cocycle identity on all $\Sigma_1(K)$. In fact $[\Phi_m \delta f](b)$ depends in general on the support $|b|$ of the 1-simplex b through the factor $\int_{b_{\circledcirc}} |(\delta f)^{\mathrm{ev}}_b| d\mu$, according to the definition of $b_{\circledcirc}$. $\square$

Remark 4.16. The results of this section cannot be fully applied to the case of the circle S^1 with respect to Möbius group. In fact, even if Proposition 4.13 shows that the space $\mathrm{C}^1(K, \mathcal{D})^{\text{Möb}}$ is not zero, there does not exist a Möbius-invariant measure on S^1, so the formula (4.28) cannot be applied.

There are two ways to avoid this problem. The first one is to consider the rotation subgroup of Möb as a symmetry group, which clearly admits an invariant measure. The second approach consists in modifying (4.28): as a first step, we consider $f \in \mathrm{C}^1(K, \mathcal{D})^{\text{Möb}}$ and make the cutoff

$$\tilde{f}_b := \begin{cases} 0, & b \in \mathrm{N}_1(K) \\ f_b, & b \in \mathrm{T}_1(K). \end{cases}$$

The above definition is well-posed since $\mathrm{N}_1(K)$ and $\mathrm{T}_1(K)$ are globally stable under the Möb-action and have void intersection, so there is no risk to define

$\tilde{f}$ in different ways over some 1-simplex. This allows us to define

$$[\Phi f]_{\mathrm{odd}}(b) := \exp\left\{ \mathrm{i}\Phi(\tilde{f}_b^{\mathrm{odd}}) \right\}, \quad b \in \Sigma_1(K).$$

The above expression yields a causal and covariant connection which is expected to be non-trivial when f^{odd} is non-zero on $\mathrm{T}_1(K)$. The price to pay for the construction of $[\Phi f]_{\mathrm{odd}}$ is the loss of any explicit information on the support (corona) of b. In conclusion, non-triviality of the invariant subspace of $\mathrm{C}^1(K, \mathcal{D})$ under the symmetry group action seems to be the only strong hypothesis in Theorem 4.14, under which we expect that a version of the result holds in any spacetime where a causal, selfadjoint Wightman field can be defined.

5 Concluding remarks

In this paper we gave a model independent construction of a causal and covariant net of C*-algebras over a spacetime, called the net of causal loops. This is constructed using as input only the underlying spacetime; more precisely, a base K of the topology of the spacetime encoding the causal and the symmetry structure. Generators of the local algebras of the net are free groups of loops of the poset K (the base K ordered under inclusion).

We showed that representations of nets of causal loops appear associated with quantum fields, under limitations that seems to be imposed only by the geometry of the spacetime and the symmetry group. Two remarkable examples have been constructed: the case of Minkowski spacetime and the case of S^1. In the Minkowski spacetime, this also has led to the prove of the existence of covariant representations satisfying the spectrum condition. Other insights could come from the theory of subsystems [13,14,16] since the net of causal loops defined in such representations turns out to be a subnet of the net generated by the given quantum field. Furthermore, having in mind applications to gauge theories, refining the construction associating quantum fields to representations of the net of causal loops, we hope to produce more interesting representations at the physical level. In particular, the notion of connection system may be a good starting point for a definition of local gauge transformation in the setting of AQFT.

At the mathematical level, a related question is to classify the above mentioned representations in terms of the equivariant cohomology of the complex defined by the nets of test functions $C^*(K, \mathcal{D})^S$, $* = 1, 2, \ldots$ and, in particular, to give conditions for irreducibility in terms of properties of the translator. This could give some light on the "charge" structure exhibited by nets of causal loops.

A further interesting direction of research is, in the case the spacetime has a topological symmetry group, the definition of a topology on the free group of loops making continuous the action of the symmetry. This could allow a sort of *Mackey's induction* in the sense that, for a given representation of the symmetry group, a continuously covariant representation of the group of loops (i.e., a continuously covariant representation of the net of causal loops) should be induced. Up to now, the authors obtained partial results on this topic, that presents difficulties arising from its combinatorial free group aspect.

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