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Null asymptotics of solutions of the Einstein–Maxwell equations in general relativity and gravitational radiation

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Abstract

We prove that for spacetimes solving the Einstein–Maxwell (EM) equations, the electromagnetic field contributes at highest order to the nonlinear memory effect of gravitational waves. In [5] Christodoulou showed that gravitational waves have a nonlinear memory. He discussed how this effect can be measured as a permanent displacement of test masses in a laser interferometer gravitational-wave detector. Christodoulou derived a precise formula for this permanent displacement in the Einstein vacuum (EV) case. We prove in Theorem 2.6 that for the EM equations this permanent displacement exhibits a term coming from the electromagnetic field. This term is at the same highest order as the purely gravitational

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term that governs the EV situation. On the other hand, in Section 3, we show that to leading order, the presence of the electromagnetic field does not change the instantaneous displacement of the test masses. Following the method introduced by Christodoulou in [5] and asymptotics derived by Zipser in [8, 9], we investigate gravitational radiation at null infinity in spacetimes solving the EM equations. We study the Bondi mass loss formula at null infinity derived in [9]. We show that the mass loss formula from [9] is compatible with the one in Bondi coordinates obtained in [4]. And we observe that the presence of the electromagnetic field increases the total energy radiated to infinity up to leading order. Moreover, we compute the limit of the area radius at null infinity in Theorem 2.7.

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1 Introduction and main results

In this paper we investigate the null asymptotics for spacetimes solving the Einstein–Maxwell (EM) equations, compute the radiated energy, derive limits at null infinity and compare them with the Einstein vacuum (EV) case. We show that the presence of the electromagnetic field does not affect the instantaneous displacement of the test masses of a laser interferometer detector at leading order, as it only comes in at lower order. But the electromagnetic field does contribute to the nonlinear effect of the displacement of the test masses. The EM case gives us a wonderful opportunity to observe mass loss and also to measure gravitational radiation. It is crucial to understand fully the behavior of the gravitational field also when other fields are present and to investigate their interplay. The only way to achieve this is to compute the null asymptotics of the spacetimes.

A major goal of mathematical General Relativity (GR) and astrophysics is to precisely describe and finally observe gravitational radiation, one of the predictions of GR. In order to do so, one has to study the null asymptotical limits of the spacetimes for typical sources. Among the latter we find binary neutron stars and binary black hole mergers. In these processes typically mass and momenta are radiated away in the form of gravitational waves. Bondi, van der Burg and Metzner studied these in [3]. Christodoulou showed in his paper [5] that every gravitational-wave burst has a nonlinear memory. The insights of this work are based on the precise description of null infinity obtained by Christodoulou and Klainerman in [7]. Among the many pioneering results they derived the Bondi mass loss formula. This is all in the regime of the EV equations. Then Zipser studied the Einstein–Maxwell equations in [8,9] and computed limits along the lines of [7] for this case. She derived a Bondi mass loss formula, where in addition to the one obtained by Christodoulou and Klainerman, a component of the electromagnetic field comes in. Thus the mass radiated away goes into the gravitational and the electromagnetic field. Here, we rely on the methods introduced in [7], used in [8,9] and by one of the present authors in [1,2]. There is a large literature about gravitational radiation. However, in the present paper, we only give the references that are relevant to our investigations.

The main results of this paper are the following. We first recall the Bondi mass loss formula obtained in [9] for spacetimes solving the EM equations:

$$\frac{\partial}{\partial u} M\left(u\right) = \frac{1}{8\pi} \int_{S^2} \left(\left|\Xi\right|^2 + \frac{1}{2} \left|A_F\right|^2 \right) d\mu_{\overset{\circ}{\gamma}}.$$

Compared to the formula obtained in [7] for spacetimes solving the EV equations, we have an additional term, $|A_F|^2$, from the electromagnetic field. Furthermore, we compare the above mass loss formula to the corresponding formula in Bondi coordinates [4]:

$$\frac{\partial}{\partial w} M(w) = - \int_{S^2} \left((\partial_w c)^2 + (\partial_w d)^2 + \frac{1}{2}(X^2 + Y^2) \right) d\mu_{\overset{\circ}{\gamma}}$$

and show that the two formulae agree.

As shown in the work of Christodoulou [5], $\Sigma^+ - \Sigma^-$ is the term that governs the permanent displacement of test particles. Using this fact, Christodoulou shows that the gravitational field has a non-linear "memory" which can be detected by a gravitational-wave experiment in a spacetime solving the EV equations. We will describe this experiment in the last section as well. In Section 2.3 of our paper, we study the permanent displacement formula for uncharged test particles of the same gravitational-wave experiment in a spacetime solving the EM equations and show that the electromagnetic field contributes to the nonlinear effect. We first obtain Theorem 2.6 which determines $\Sigma^+ - \Sigma^-$ in the EM case. From Theorem 2.6, we observe that the electromagnetic field changes the leading order term of the permanent displacement of test particles. Then in the last section, we study in details a gravitational-wave experiment for our findings. We observe that the electromagnetic field does not enter the leading order term of the Jacobi equation. As a result, to leading order, it does not change the instantaneous displacement of test particles. But the electromagnetic field does contribute at highest order to the nonlinear effect of the permanent displacement of test masses. Furthermore, in Theorem 2.7 we compute the limit of the area radius r on any null hypersurface C_u as t goes to ∞ and show that the result coincides with the one obtained in [7] for EV.

We follow the method introduced by Christodoulou in [5] to study the effect of gravitational waves. The treatment is based on the asymptotic behavior of the gravitational field obtained at null and spatial infinity. These rigorous asymptotics allow us to study the structure of the spacetimes at null infinity. The spacetime is foliated by a time function t and by an optical function u. The corresponding lapse functions are denoted by ϕ and a. Each level set of u, C_u, is an outgoing null hypersurface and each level set of t, H_t, is a maximal space-like hypersurface. We pick a suitable pair of normal vectors along the null hypersurface. The flow along these vector fields generates a family of diffeomorphisms ϕ_u of S^2. We use ϕ_u to pull back tensor fields in our spacetime. This allows us to study their limit at null infinity along the null hypersurface C_u. Then we study the effect of gravitational waves by taking the limit as u goes to $\pm\infty$. Christodoulou in [5] gives a complete explanation of the structure at null infinity.

The methods introduced in [7], used in [8, 9], reveal the structure of the null asymptotics of our spacetimes. In these works, to prove the stability result, the data are assumed to be small. However, as far as the study of the null asymptotics is concerned, the data can be large. We give a brief outline in the last part of this introduction of the methods of [7].

In GR the fundamental equations are the Einstein equations linking the curvature of the spacetime to its matter content:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, R = 8\pi T_{\mu\nu} \tag{1.1}$$

(setting $G = c = 1$), $\mu, \nu = 0, 1, 2, 3$. $G_{\mu\nu}$ denotes the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, R the scalar curvature tensor, g the metric tensor and $T_{\mu\nu}$ is the energy-momentum tensor.

Here, we are discussing the EM equations. This means that $T_{\mu\nu}$ on the right-hand side of (1.1) is the stress-energy tensor of the electromagnetic field. The twice contracted Bianchi identities imply that

$$D^\nu G_{\mu\nu} = 0. \tag{1.2}$$

This is equivalent to the following equation, namely, that the divergence of the stress-energy tensor of the electromagnetic field vanishes:

$$D^\nu T_{\mu\nu} = 0 \tag{1.3}$$

with

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right), \tag{1.4}$$

where F denotes the electromagnetic field. Note that F is an antisymmetric covariant 2-tensor. As $T_{\mu\nu}$ is trace free, the Einstein equations (1.1) take the form

$$R_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.5}$$

We find that the scalar curvature is identically zero. We write the EM equations as

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.6}$$

$$D^\alpha F_{\alpha\beta} = 0, \tag{1.7}$$

$$D^\alpha \, {}^*F_{\alpha\beta} = 0. \tag{1.8}$$

As a consequence of the Maxwell equations, we have

$$\Box F = 0, \tag{1.9}$$

where $\Box$ is the de Rham Laplacian with respect to the metric g.

We split the Riemannian curvature $R_{\alpha\beta\gamma\delta}$ into its traceless part, namely the Weyl tensor $W_{\alpha\beta\gamma\delta}$, and a part including the spacetime Ricci curvature

$R_{\alpha\beta}$ and spacetime scalar curvature R:

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}) - \frac{1}{6}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) R. \tag{1.10}$$

One can then define the Bel–Robinson tensor

$$Q_{\alpha\beta\gamma\delta} = W_{\alpha\rho\gamma\sigma}W_{\beta\ \delta}^{\ \rho\ \sigma} + {}^*W_{\alpha\rho\gamma\sigma}{}^*W_{\beta\ \delta}^{\ \rho\ \sigma}. \tag{1.11}$$

The Bianchi equations for the Weyl tensor in the presence of an electromagnetic field read

$$D^{\alpha}W_{\alpha\beta\gamma\delta} = \frac{1}{2}(D_{\gamma}R_{\beta\delta} - D_{\delta}R_{\beta\gamma}). \tag{1.12}$$

In [7] Christodoulou and Klainerman derived the asymptotic behavior in the case of strongly asymptotically flat initial data of the following type.

Definition 1.1. We define a strongly asymptotically flat initial data set in the sense of [7] (studied by Christodoulou and Klainerman) to be an initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighborhood of infinity such that, as $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, $\bar{g}_{ij}$ and k_{ij} are:

$$\bar{g}_{ij} = \left(1 + \frac{2M}{r}\right)\delta_{ij} + o_4\left(r^{-3/2}\right), \tag{1.13}$$

$$k_{ij} = o_3\left(r^{-5/2}\right) \tag{1.14}$$

with M denoting the mass.

Under a smallness condition on the initial data, Christodoulou and Klainerman proved in [7] that this can be extended uniquely to a smooth, globally hyperbolic and geodesically complete spacetime solving the EV equations. The resulting spacetime is globally asymptotically flat. Together with the existence and uniqueness theorem comes a precise description of the asymptotic behavior of the spacetime. While the smallness condition was imposed in order to ensure completeness, the results about the behavior at null infinity are largely independent of the smallness. The decay behavior of the components of the Weyl tensor are given below. And the limits at null infinity of the relevant quantities are given in Section 2.

In [7] as well as in [8,9] and [1,2] the Weyl tensor W in (M, g) is decomposed with respect to the null frame e_4, e_3, e_2, e_1. That is, e_4 and e_3 form a null pair which is supplemented by e_A, $A = 1, 2$, a local frame field for

$S_{t,u} = H_t \cap C_u$. Given this null pair, e_3 and e_4, we can define the tensor of projection from the tangent space of M to that of $S_{t,u}$:

$$\Pi^{\mu\nu} = g^{\mu\nu} + \frac{1}{2}(e_4^\nu e_3^\mu + e_3^\nu e_4^\mu).$$

We decompose the second fundamental form k_{ij} of H_t into

$$k_{NN} = \delta, \tag{1.15}$$
$$k_{AN} = \epsilon_A, \tag{1.16}$$
$$k_{AB} = \eta_{AB}, \tag{1.17}$$

where N is the unit normal vector of $S_{t,u}$ in H_t. Let T be the future-directed unit normal to H_t. We define

$$\theta_{AB} = \langle \nabla_A N, e_B \rangle.$$

The Ricci coefficients of the null standard frame $T - N, T + N, e_2, e_1$ are given by the following:

$$\chi'_{AB} = \theta_{AB} - \eta_{AB}, \tag{1.18}$$
$$\underline{\chi}'_{AB} = -\theta_{AB} - \eta_{AB}, \tag{1.19}$$
$$\underline{\xi}'_A = \phi^{-1} \nabla\!\!\!\!/_A \phi - a^{-1} \nabla\!\!\!\!/_A a, \tag{1.20}$$
$$\underline{\zeta}'_A = \phi^{-1} \nabla\!\!\!\!/_A \phi - \epsilon_A, \tag{1.21}$$
$$\zeta'_A = \phi^{-1} \nabla\!\!\!\!/_A \phi + \epsilon_A, \tag{1.22}$$
$$\nu' = -\phi^{-1} \nabla\!\!\!\!/_N \phi + \delta, \tag{1.23}$$
$$\underline{\nu}' = \phi^{-1} \nabla\!\!\!\!/_N \phi + \delta, \tag{1.24}$$
$$\omega' = \delta - a^{-1} \nabla\!\!\!\!/_N a. \tag{1.25}$$

We use χ, $\underline{\chi}$, etc. for the Ricci coefficients of the null frame $a^{-1}(T - N)$, $a(T + N), e_2, e_1$.

Definition 1.2. We define the null components of W as follows:

$$\underline{\alpha}_{\mu\nu}(W) = \Pi_\mu{}^\rho \, \Pi_\nu{}^\sigma \, W_{\rho\gamma\sigma\delta} \, e_3^\gamma \, e_3^\delta, \tag{1.26}$$
$$\underline{\beta}_\mu(W) = \frac{1}{2} \Pi_\mu{}^\rho \, W_{\rho\sigma\gamma\delta} \, e_3^\sigma \, e_3^\gamma \, e_4^\delta, \tag{1.27}$$
$$\rho(W) = \frac{1}{4} W_{\alpha\beta\gamma\delta} \, e_3^\alpha \, e_4^\beta \, e_3^\gamma \, e_4^\delta, \tag{1.28}$$
$$\sigma(W) = \frac{1}{4} {}^*W_{\alpha\beta\gamma\delta} \, e_3^\alpha \, e_4^\beta \, e_3^\gamma \, e_4^\delta, \tag{1.29}$$
$$\beta_\mu(W) = \frac{1}{2} \Pi_\mu{}^\rho \, W_{\rho\sigma\gamma\delta} \, e_4^\sigma \, e_3^\gamma \, e_4^\delta, \tag{1.30}$$
$$\alpha_{\mu\nu}(W) = \Pi_\mu{}^\rho \, \Pi_\nu{}^\sigma \, W_{\rho\gamma\sigma\delta} \, e_4^\gamma \, e_4^\delta. \tag{1.31}$$

The estimates in [7] yield the decay behavior:

$$
\begin{aligned}
\underline{\alpha}(W) &= O(r^{-1}\tau_-^{-5/2}),\\
\underline{\beta}(W) &= O(r^{-2}\tau_-^{-3/2}),\\
\rho(W) &= O(r^{-3}),\\
\sigma(W) &= O(r^{-3}\tau_-^{-1/2}),\\
\alpha(W), \beta(W) &= o(r^{-7/2}),
\end{aligned}
$$

where $\tau_-^2 = 1 + u^2$ and $r(t, u)$ is the area radius of the surface $S_{t,u}$.

In [8, 9], Zipser works with the same conditions on the metric, second fundamental form and curvature, in addition she imposes a decay condition on the electromagnetic field F, namely

$$
F|_H = o_3\left(r^{-5/2}\right). \tag{1.32}
$$

The null components of the electromagnetic field are written as

$$
\begin{aligned}
F_{A3} &= \underline{\alpha}(F)_A, \quad F_{A4} = \alpha\,(F)_A\,,\\
F_{34} &= 2\rho\,(F)\,, \quad F_{12} = \sigma\,(F)\,.
\end{aligned} \tag{1.33}
$$

The corresponding null decomposition $\{\underline{\alpha}\,({}^*F)\,, \alpha\,({}^*F)\,, \rho\,({}^*F)\,, \sigma\,({}^*F)\}$ of ${}^*F$ is given by

$$
\begin{aligned}
\underline{\alpha}\,({}^*F)_A &= -\underline{\alpha}\,(F)^B\,\epsilon_{BA}, \quad \alpha\,({}^*F)_A = \alpha\,(F)^B\,\epsilon_{BA},\\
\rho\,({}^*F) &= \sigma\,(F)\,, \quad \sigma\,({}^*F) = -\rho\,(F)\,,
\end{aligned} \tag{1.34}
$$

where the Hodge dual of a tensor u tangent to $S_{t,u}$ is defined by

$$
{}^*u_A = \epsilon_A{}^B u_{B.}
$$

The estimates in [8, 9] yield the decay behavior:

$$
\begin{aligned}
\underline{\alpha}(F) &= O(r^{-1}\tau_-^{-3/2}),\\
\rho(F), \sigma(F) &= O(r^{-2}\;\tau_-^{-1/2}),\\
\alpha(F) &= o(r^{-5/2}).
\end{aligned}
$$

One of the main difficulties in [7] is that a general spacetime has no symmetries and thus does not have suitable vectorfields to construct integral conserved quantities. To overcome this difficulty, Christodoulou and Klainerman use the “closeness” of their spacetimes to the Minkowski spacetime and construct quasi-conformal vector fields. The main step is carried out within a bootstrap argument in the “last slice,” namely in a space-like hypersurface which is a level set of the time function t. First, the authors foliate the spacetime by functions t and u near the initial slice. From the foliations, one constructs vectorfields that are almost Killing. Combining these vectorfields with the Bel–Robinson tensor, one obtains local estimates for the Weyl curvature tensor W and the electromagnetic field F. With these estimates, one constructs a new optical function which is defined on a larger domain in the spacetime. Then, following a continuity argument, one obtains a smooth, globally hyperbolic and geodesically complete spacetime solving the Einstein equations. The resulting spacetime is globally asymptotically flat, satisfying the above decay properties.

2 Null asymptotics

2.1 Asymptotic behavior and Bondi mass

We need precise data at null infinity. In particular, we have to know the Bondi mass and the asymptotic behavior of the components of the curvature and the electromagnetic field. Zipser described them in [8,9] following the discussion in Chapter 17 of [7], making changes as necessary due to the presence of the electromagnetic field. The parameters of the foliations and the components of the Weyl tensor behave exactly as in [7]. That is, the following holds: along the null hypersurfaces C_u as $t \to \infty$, it is

$$\lim_{C_u, t\to\infty} \phi = 1, \qquad \lim_{C_u, t\to\infty} a = 1 \tag{2.1}$$

and

$$\lim_{C_u, t\to\infty} (r\mathrm{tr}\chi) = 2, \qquad \lim_{C_u, t\to\infty} \left(r\mathrm{tr}\underline{\chi}\right) = -2. \tag{2.2}$$

Furthermore, we let

$$H = \lim_{C_u, t\to\infty} \left(r^2 \left(\mathrm{tr}\,\chi' - \frac{2}{r} \right) \right). \tag{2.3}$$

From the existence theorem of [8,9], Zipser makes the following conclusions, which are generalizations of conclusions 17.0.1–17.0.4 in [7].

Following the convention in [7,9], the pointwise norms $|\quad|$ of the tensors on S^2 relate to the metric $\overset{\circ}{\gamma}$, which is the limit of the induced metrics on $S_{t,u}$ rescaled by r^{-2} for each u as $t \to \infty$.

Theorem 2.1. *On any null hypersurface C_u, the normalized curvature components $r\underline{\alpha}(W)$, $r^2\underline{\beta}(W)$, $r^3\rho(W)$, $r^3\sigma(W)$, $r\underline{\alpha}(F)$, $r^2\rho(F)$, $r^2\sigma(F)$ have limits as $t \to \infty$, in particular*

$$\begin{aligned}
&\lim_{C_u, t\to\infty} r\underline{\alpha}(W) = A_W(u,\cdot), \quad \lim_{C_u, t\to\infty} r^2\underline{\beta}(W) = B_W(u,\cdot),\\
&\lim_{C_u, t\to\infty} r^3\rho(W) = P_W(u,\cdot), \quad \lim_{C_u, t\to\infty} r^3\sigma(W) = Q_W(u,\cdot),\\
&\lim_{C_u, t\to\infty} r\underline{\alpha}(F) = A_F(u,\cdot),\\
&\lim_{C_u, t\to\infty} r^2\rho(F) = P_F(u,\cdot), \quad \lim_{C_u, t\to\infty} r^2\sigma(F) = Q_F(u,\cdot)
\end{aligned}$$

with A_W being a symmetric traceless covariant 2-tensor, B_W and A_F are 1-forms and P_W, Q_W, P_F, Q_F functions on S^2 depending on u. The following decay properties hold:

$$\begin{aligned}
&|A_W(u,\cdot)| \le C(1+|u|)^{-5/2}, \quad |B_W(u,\cdot)| \le C(1+|u|)^{-3/2},\\
&\left|P_W(u,\cdot) - \overline{P}_W(u)\right| \le (1+|u|)^{-1/2}, \quad \left|Q_W(u,\cdot) - \overline{Q}_W(u)\right| \le (1+|u|)^{-1/2},\\
&|A_F(u,\cdot)| \le C(1+|u|)^{-3/2},\\
&|P_F(u,\cdot)| \le (1+|u|)^{-1/2}, \quad |Q_F(u,\cdot)| \le (1+|u|)^{-1/2}
\end{aligned}$$

and

$$\lim_{u\to-\infty} \overline{P}_W(u) = 0, \quad \lim_{u\to-\infty} \overline{Q}_W(u) = 0.$$

The existence of the limits in the conclusion follows from the estimates in the existence theorem of [9] (i.e., [8]).

Theorem 2.2. *On the null hypersurface C_u, the normalized shear $r^2\widehat{\chi}'$ has limit as $t \to \infty$:*

$$\lim_{C_u, t\to\infty} r^2\widehat{\chi}' = \Sigma(u,\cdot)$$

with Σ being a symmetric traceless covariant 2-tensor on S^2 depending on u.

The proof is the same as in [7] because the propagation equation stays unaltered:

$$\frac{d\widehat{\chi}_{AB}}{ds} = -\mathrm{tr}\,\chi\widehat{\chi}_{AB} - \alpha(W)_{AB}.$$

Theorem 2.3. *On any null hypersurface C_u, the limit of $r\widehat{\eta}$ exists as $t \to \infty$, that is*

$$\lim_{C_u, t \to \infty} r\widehat{\eta} = \Xi(u, \cdot)$$

with Ξ being a symmetric traceless 2-covariant tensor on S^2 depending on u and having the decay property

$$|\Xi(u, \cdot)|_{\overset{\circ}{\gamma}} \leq C(1 + |u|)^{-3/2}.$$

Further, it is

$$\lim_{C_u, t \to \infty} r\widehat{\theta} = -\frac{1}{2} \lim_{C_u, t \to \infty} r\underline{\widehat{\chi}}' = \Xi$$

as well as

$$\frac{\partial \Sigma}{\partial u} = -\Xi, \tag{2.4}$$

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{4} A_W. \tag{2.5}$$

Zipser proves this result as conclusion 3 in [9]. The argument is along the lines of the proof of conclusion 17.0.3 in [7].

Zipser follows [7] to derive the Bondi mass formula by calculating a propagation equation for the Hawking mass enclosed by a 2-surface $S_{t,u}$. The Hawking mass is defined as

$$m(t, u) = \frac{r}{2}\left(1 + \frac{1}{16\pi}\int_{S_{t,u}} \operatorname{tr}\chi \operatorname{tr}\underline{\chi}\right). \tag{2.6}$$

Let

$$\underline{\mu} = -di\!\!/v\, \underline{\zeta} + \frac{1}{2}\widehat{\chi} \cdot \underline{\widehat{\chi}} - \rho(W) - \frac{1}{2}\left(\rho^2(F) + \sigma^2(F)\right). \tag{2.7}$$

With respect to the l-pair, one has the null structure equations

$$\frac{d\operatorname{tr}\underline{\chi}}{ds} + \frac{1}{2}\operatorname{tr}\chi \operatorname{tr}\underline{\chi} = -2\underline{\mu} + 2|\zeta|^2,$$

$$\frac{d\operatorname{tr}\chi}{ds} + \frac{1}{2}(\operatorname{tr}\chi)^2 = -|\widehat{\chi}|^2 - |\alpha(F)|^2.$$

One computes

$$\frac{d}{ds}\operatorname{tr}\chi \operatorname{tr}\underline{\chi} + \operatorname{tr}\chi\left(\operatorname{tr}\chi \operatorname{tr}\underline{\chi}\right) = -2\underline{\mu}\operatorname{tr}\chi + 2\operatorname{tr}\chi|\zeta|^2 - \operatorname{tr}\underline{\chi}|\widehat{\chi}|^2 - \operatorname{tr}\underline{\chi}|\alpha(F)|^2,$$

thus

$$\frac{\partial}{\partial t}\int_{S_{t,u}} \operatorname{tr}\chi \operatorname{tr}\underline{\chi} = -2\int_{S_{t,u}} a\phi\underline{\mu} \operatorname{tr}\chi + \int_{S_{t,u}} a\phi\left(-\operatorname{tr}\underline{\chi}\,|\widehat{\chi}|^2 - \operatorname{tr}\underline{\chi}\,|\alpha(F)|^2 + 2\operatorname{tr}\chi\,|\zeta|^2\right). \quad (2.8)$$

Using the Gauss equation

$$K = -\frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} - \rho(W) - \frac{1}{2}\left(\rho^2(F) + \sigma^2(F)\right),$$

one derives

$$\underline{\mu} = -\mathrm{d}\!\!/\mathrm{iv}\,\underline{\zeta} + K + \frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi}. \quad (2.9)$$

By the Gauss–Bonnet formula and formulas (2.7), (2.9) conclude that

$$\int_{S_{t,u}} \underline{\mu} = \int_{S_{t,u}} \left(\frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} - \rho(W) - \frac{1}{2}\left(\rho^2(F) + \sigma^2(F)\right)\right) = 4\pi\left(1 + \frac{1}{16\pi}\int_{S_{t,u}} \operatorname{tr}\chi\operatorname{tr}\underline{\chi}\right) = \frac{8\pi}{r}m. \quad (2.10)$$

Moreover, by

$$\frac{d}{dt}r = \frac{r}{2}\overline{\phi a \operatorname{tr}\chi},$$

and (2.8), (2.10), it is

$$\frac{\partial}{\partial t}m(t,u) = -\frac{r}{16\pi}\int_{S_{t,u}} \left(a\phi\operatorname{tr}\chi - \overline{\phi a\operatorname{tr}\chi}\right)\underline{\mu} + \frac{r}{8\pi}\int_{S_{t,u}} a\phi\left(\frac{1}{2}\operatorname{tr}\chi\,|\zeta|^2 - \frac{1}{4}\operatorname{tr}\underline{\chi}\,|\widehat{\chi}|^2 - \frac{1}{4}\operatorname{tr}\underline{\chi}\,|\alpha(F)|^2\right). \quad (2.11)$$

Note that $K + \frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi} = O\left(r^{-3}\right)$, $\underline{\mu} = O\left(r^{-3}\right)$. From the asymptotic behavior of the right-hand side of (2.11), it follows that

$$\frac{\partial}{\partial t}m(t,u) = O\left(r^{-2}\right).$$

This means that $m(t,u)$ has a limit for any fixed u as $t \to \infty$, namely the Bondi mass of the null hypersurface C_u. As in [7], it is denoted by $M(u)$.

The terms appearing due to the presence of the electromagnetic field are shown to decay fast enough so that the mass decays at the same rate as in [7]. In particular,

$$m(t,u) = M(u) + O\left(r^{-1}\right)$$

as $t \to \infty$ on C_u.

Following [7], Zipser calculates a Bondi mass loss formula by considering

$$\frac{\partial}{\partial u} m(t,u)$$

with

$$\frac{\partial}{\partial u} m(t,u) = \frac{1}{2}\overline{a \operatorname{tr}\theta} m + \frac{r}{32\pi}\int_{S_{t,u}} a\left(\nabla_N \underline{\mu} + \operatorname{tr}\theta \underline{\mu}\right).$$

As $l = a^{-1}(T+N)$ and $\underline{l} = a(T-N)$,

$$\begin{aligned} a(\nabla_N \underline{\mu} + \operatorname{tr}\theta \underline{\mu}) &= \frac{1}{2}a^2\left(\mathbf{D}_4 \underline{\mu} + \operatorname{tr}\chi\, \underline{\mu}\right) \\ &\quad - \frac{1}{2}\left(\mathbf{D}_3 \underline{\mu} + \operatorname{tr}\underline{\chi}\, \underline{\mu}\right) \end{aligned}$$

and

$$\begin{aligned} &\mathbf{D}_4 \underline{\mu} + \operatorname{tr}\chi\, \underline{\mu} = O(r^{-4}), \\ &\mathbf{D}_3 \underline{\mu} + \operatorname{tr}\underline{\chi}\, \underline{\mu} = -\frac{1}{4}\operatorname{tr}\chi \left|\widehat{\underline{\chi}}\right|^2 - \frac{1}{2}\operatorname{tr}\chi \left|\underline{\alpha}(F)\right|^2 + O(r^{-4}). \end{aligned}$$

Thus,

$$\frac{\partial}{\partial u} m(t,u) = \frac{r}{64\pi}\int_{S_{t,u}} \operatorname{tr}\chi\left(\left|\widehat{\underline{\chi}}\right|^2 + \frac{1}{2}\left|\underline{\alpha}(F)\right|^2\right) + O\left(r^{-1}\right).$$

Following [7], Zipser uses the following facts to derive the Bondi mass loss formula: the metric $\widetilde{\gamma} = \phi_{t,u}^*\left(r^{-2}\gamma\right)$ converges to the standard metric $\overset{\circ}{\gamma}$ of the unit sphere S^2 as $t \to \infty$ for each u ($\phi_{t,u}^*$ is a diffeomorphism from S^2 to $S_{t,u}$), moreover $\frac{r}{2}\operatorname{tr}\chi$ converges to 1, and $r\widehat{\underline{\chi}}$ converges to -2Ξ. This yields

$$\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi}\int_{S^2}\left(|\Xi|^2 + \frac{1}{2}|A_F|^2\right) d\mu_{\overset{\circ}{\gamma}}.$$

The right-hand side of this expression is positive and integrable in u. Thus, $M(u)$ is a non-decreasing function of u and has finite limits $M(-\infty)$ for

$u \to -\infty$ and $M(\infty)$ for $u \to \infty$. Further, Zipser concludes from (2.10) that $M(-\infty) = 0$, and $M(\infty)$ is the total mass.

Theorem 2.4. *The Hawking mass $m(t,u)$ tends to the Bondi mass $M(u)$ as $t \to \infty$ on any null hypersurface C_u. That is,*

$$m(t,u) = M(u) + O(r^{-1})$$

and $M(u)$ verifies the Bondi mass loss formula

$$\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi} \int_{S^2} \left(|\Xi|^2 + \frac{1}{2} |A_F|^2 \right) d\mu_{\overset{\circ}{\gamma}}$$

with $d\mu_{\overset{\circ}{\gamma}}$ being the area element of the standard unit sphere S^2.

We see that in the Bondi mass loss formula the limiting term A_F of the electromagnetic field comes in. At this point, let us compare this with the Bondi mass loss formula obtained in [7, p. 499]: $\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi} \int_{S^2} |\Xi|^2 \, d\mu_{\overset{\circ}{\gamma}}$. In fact, the electromagnetic field contributes to the change of the Bondi mass by $\frac{1}{16\pi} \int_{S^2} |A_F|^2 \, d\mu_{\overset{\circ}{\gamma}}$.

The decay behavior of A_F is the same as for Ξ. See Theorems 2.1 and 2.3. Similarly as in [5, 7] for the EV case, we can define now the new function

$$F = \frac{1}{8} \int_{-\infty}^{+\infty} \left(| \, \Xi \, |^2 + \frac{1}{2} \, | \, A_F \, |^2 \right) du. \tag{2.12}$$

Then $\frac{F}{4\pi}$ is the total energy radiated to infinity in a given direction per unit solid angle. Thus the integrand in (2.12) is proportional to the power radiated to infinity at a given retarded time u, in a given direction, per unit area on S^2 (per unit solid angle). Already in [5] Christodoulou tells us how to adapt the formula for F when matter radiation is present, that is also in the EM case.

In the next two subsections, we also need the following theorem for H.

Theorem 2.5. *The function H satisfies*

$$\frac{\partial H}{\partial u} = 0, \tag{2.13}$$

$$\bar{H} = 0. \tag{2.14}$$

Proof. In the EV case, equation (2.13) is proved in conclusion 17.0.5 of [7] where one uses the fact that

$$\nabla_N \mathrm{tr}\chi' + \frac{1}{2}\chi' = O(r^{-3}).$$

In the EM case, it is easy to see that the additional terms involving the electromagnetic field are also $O(r^{-3})$. Thus equation (2.13) is still true in the EM case.

In the EV case, equation (2.14) is proved in Lemma 17.0.1 in [7]. In the proof, we need to show that $r^2\bar{\delta}$ converges to $2M(u)$. From Proposition 4.4.4 in [7], we have

$$4\pi r^3\bar{\delta} = \int_{u_0}^{u} du' \left(\int_{S_{t,u}} ar\hat{\theta}\cdot\hat{\eta} - \frac{1}{2}\kappa(\delta - \bar{\delta}) - ra^{-1}\nabla\!\!\!/\, a\cdot\epsilon + r(\mathrm{div}\, k)_N \right).$$

Following the proof of Lemma 17.0.1 in [7], we see that

$$\int_{S_{t,u}} ar\hat{\theta}\cdot\hat{\eta} - \frac{1}{2}\kappa(\delta - \bar{\delta}) - ra^{-1}\nabla\!\!\!/\, a\cdot\epsilon = r\int_{S^2} |\Xi|^2 d\mu_{\overset{\circ}{\gamma}} + O(1).$$

Moreover, in the EM case, due to the constraint equation, we have

$$(\mathrm{div}\, k)_N = R_{0N} = 2F_0{}^{\rho}F_{N\rho}. \tag{2.15}$$

Using equation (2.15), we see that

$$\int_{S_{t,u}} r(\mathrm{div}\, k)_N = \frac{r}{2}\int_{S^2} |A_F|^2 d\mu_{\overset{\circ}{\gamma}} + O(1),$$

since the leading terms of F_{0A} and F_{NA} are both $\frac{1}{2}\underline{\alpha}(F)_A$. As a result, we can still conclude that

$$r\bar{\delta} = \frac{2}{r^2}\int_{u_0}^{u} r\frac{\partial}{\partial u}m(t,u) + O(r^{-1}).$$

The rest of the proof follows easily from the proof of Lemma 17.0.1 in [7]. □

2.2 Compare result with mass loss in Bondi coordinates

In this subsection, we compare the mass loss formula obtained in [8,9], cited above in Theorem 2.4, with the mass loss formula in Bondi coordinates.

Bondi coordinates are first defined by Bondi *et al.* in [3] for axially symmetric vacuum spacetimes. The main motivation for such coordinates is to study gravitational radiation at null infinity. The form of the metric is chosen such that many computations are simplified at null infinity. As a result, one can derive many useful theorems and formulae assuming the existence of such coordinates. In particular, the Bondi mass loss formula in Bondi coordinates is first derived for axially symmetric vacuum spacetimes in [3] and is generalized to the EM case in [4]. However, for a given spacetime, it is hard to tell whether such coordinates exist. On the other hand, spacetimes studied in [8] are obtained by evolving small initial data on a space-like hypersurface by EM equations. In particular, it is not clear whether all spacetimes studied in [8] admit such coordinates. Here we show that for the leading term in the mass loss formula, the two different coordinate systems give the same result.

The mass loss formula from [8, 9] is

$$\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi} \int_{S^2} \left(|\Xi|^2 + \frac{1}{2} |A_F|^2 \right) d\mu_{\overset{\circ}{\gamma}}.$$

Let us recall the mass loss formula and asymptotic expansion for solutions to the EM equations in Bondi coordinates [4]. The line segment is

$$-UV dw^2 - 2U dw\, dr + \sigma_{ab}(dx^a + W^a\, dw)(dx^b + W^b\, dw), \quad a, b = 2, 3$$

with the electromagnetic field given by a skew-symmetric two tensor $F_{\mu\nu}$. We have the following asymptotics for the line segment:

$$V = 1 - \frac{2m}{r} + O(r^{-2}), U = 1 + O(r^{-2}) \text{ and } W^a = O(r^{-2}),$$
$$\sigma_{ab} = \begin{pmatrix} r^2 + 2cr + \cdots & -2dr \sin\theta + \cdots \\ -2dr \sin\theta + \cdots & \sin^2\theta(r^2 - 2cr) + \cdots \end{pmatrix}.$$

We also need the following asymptotics for $F_{\mu\nu}$:

$$F_{w\theta} = X + O(r^{-1}) \text{ and } F_{w\phi} = Y \sin\theta + O(r^{-1})$$

as well as

$$F_{ra} = O(r^{-2}),\ F_{ab} = O(1) \text{ and } F_{wr} = O(r^{-2}).$$

Level sets of w, C_w, are outgoing null hypersurfaces. Each C_w is then foliated by level sets of r, $S_{w,r}$. Hence, it is natural to consider the following pair of null vectors normal to $S_{w,r}$:

$$e_3 = \frac{\partial}{\partial w} - W^c \frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r} \quad \text{and} \quad e_4 = \frac{\partial}{\partial r},$$

since e_4 is a natural choice of null vector on C_w and

$$\lim_{r\to\infty} \langle e_3, e_4 \rangle = -1.$$

Let $M(w)$ be the Bondi mass. It is given by

$$M(w) = \frac{1}{8\pi} \int_{S^2} m \, d\mu_{\overset{\circ}{\gamma}}.$$

The mass loss formula reads

$$\frac{\partial}{\partial w} M(w) = - \int_{S^2} \left((\partial_w c)^2 + (\partial_w d)^2 + \frac{1}{2}(X^2 + Y^2) \right) d\mu_{\overset{\circ}{\gamma}}.$$

To show that the two mass loss formulae agree, we prove that

$$|\Xi|^2 = (\partial_w c)^2 + (\partial_w d)^2 \text{ and } |A_F|^2 = X^2 + Y^2.$$

First we compute

$$\begin{aligned}
-\underline{\chi}'\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) &= \left\langle \frac{\partial}{\partial w} - W^c \frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} \right\rangle \\
&= \left\langle \frac{\partial}{\partial w} - W^c \frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r}, \Gamma^r_{ab}\frac{\partial}{\partial r} + \Gamma^w_{ab}\frac{\partial}{\partial w} + \Gamma^c_{ab}\frac{\partial}{\partial x^c} \right\rangle \\
&= \left\langle \frac{\partial}{\partial w} - W^c \frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r}, \Gamma^r_{ab}\frac{\partial}{\partial r} + \Gamma^w_{ab}\frac{\partial}{\partial w} \right\rangle.
\end{aligned}$$

The Christoffel symbols are

$$\begin{aligned}
\Gamma^w_{ab} &= -\frac{1}{2} g^{wr} \partial_r g_{ab} = \frac{1}{2}\partial_r g_{ab} + O(1) \\
\Gamma^r_{ab} &= \frac{1}{2} g^{wr}(\partial_b g_{wa} + \partial_a g_{wb} - \partial_w g_{ab}) + \frac{1}{2} g^{rr}(-\partial_r g_{ab}) \\
&\quad + \frac{1}{2} g^{rc}(\partial_b g_{ca} + \partial_a g_{cb} - \partial_c g_{ab}) \\
&= \frac{1}{2}\partial_w g_{ab} - \frac{1}{2}\partial_r g_{ab} + O(1).
\end{aligned}$$

As a result, one finds

$$\left\langle \frac{\partial}{\partial w} - W^c \frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} \right\rangle = \frac{1}{4}\partial_r \sigma_{ab} - \frac{1}{2}\partial_w \sigma_{ab} + O(1).$$

One can easily see that up to $O(1)$ terms, $\partial_w \sigma_{ab}$ is traceless and $\partial_r \sigma_{ab}$ has zero traceless part. As a result,

$$|\Xi|^2 = (\partial_w c)^2 + (\partial_w d)^2.$$

For the second equality, we use the expression for e_3 and the asymptotics for $F_{\mu\nu}$. A direct computation shows that

$$|A_F|^2 = X^2 + Y^2.$$

2.3 Permanent displacement formula

The permanent displacement of the test masses of a laser interferometer gravitational-wave detector is governed by $\Sigma^+ - \Sigma^-$. Christodoulou showed in [5] how this works. We discuss the corresponding wave experiment in the EM case in Section 3. In the following, we are going to state and prove a theorem for $\Sigma^+ - \Sigma^-$ in the EM case. We point out that the final formula — even though in its form identical to the one obtained by Christodoulou and Klainerman in [7] — differs from the EV case by a contribution from the electromagnetic field. The form of the formula is not altered due to the fact that the corresponding extra electromagnetic terms cancel. However, the limiting term A_F enters the new formula nonlinearly.

Theorem 2.6. *Let $\Sigma^+(\cdot) = \lim_{u\to\infty} \Sigma(u, \cdot)$ and $\Sigma^-(\cdot) = \lim_{u\to -\infty} \Sigma(u, \cdot)$. Let*

$$F(\cdot) = \int_{-\infty}^{\infty} \left(| \, \Xi(u, \cdot) \, |^2 \; + \frac{1}{2} \, | \, A_F(u, \cdot) \, |^2 \right) du. \tag{2.16}$$

Moreover, let Φ be the solution with $\bar{\Phi} = 0$ on S^2 of the equation

$$\overset{\circ}{\not{\Delta}} \, \Phi = F - \bar{F}.$$

Then $\Sigma^+ - \Sigma^-$ is given by the following equation on S^2:

$$\overset{\circ}{\not{div}} \, (\Sigma^+ - \Sigma^-) = \overset{\circ}{\not{\nabla}} \, \Phi. \tag{2.17}$$

Proof. Equation (2.4) in Theorem 2.3 yields that Σ tends to limits Σ^+ as $u \to \infty$ and Σ^- as $u \to -\infty$. Also, it is

$$\Sigma(u) = \Sigma^- - \int_{-\infty}^{u} \Xi(u')du'$$

and

$$\Sigma^+ - \Sigma^- = - \int_{-\infty}^{\infty} \Xi(u')du'.$$

When taking the limits for the Hodge system ((2.23), (2.24)) on C_u as $t \to \infty$, we will compute the corresponding limits for and involving Ψ, Ψ'. From Zipser's work [9, Chapter 9.1, (9.13) and Lemma 9], we know

$$\triangle\Psi = r|\hat{\eta}\,|^2 - \frac{r}{4} \mid \underline{\alpha}(F) \mid^2, \tag{2.18}$$

$$\triangle\Psi' = -ra^{-1}\lambda\big(\mid \hat{\eta} \mid^2 - \overline{\mid \hat{\eta} \mid^2}\big) + \frac{r^2a^{-1}}{4}\big(a\not{D}_4 \mid \underline{\alpha}(F) \mid^2 - \overline{a\not{D}_4 \mid \underline{\alpha}(F) \mid^2}\big), \tag{2.19}$$

whereas in the work of Christodoulou and Klainerman [7, Chapter 11.2, (11.2.2b) and (11.2.7b)], it is

$$\triangle\Psi = r \mid \hat{\eta} \mid^2, \tag{2.20}$$

$$\triangle\Psi' = -ra^{-1}\lambda\big(\mid \hat{\eta} \mid^2 - \overline{\mid \hat{\eta} \mid^2}\big). \tag{2.21}$$

We compute the following limits in the new case:

$$\begin{aligned} &\lim_{C_u, t\to\infty} \Psi = \boldsymbol{\Psi}, \qquad \lim_{C_u, t\to\infty} \Psi' = \boldsymbol{\Psi}', \\ &\lim_{C_u, t\to\infty} r\nabla_N\Psi = \Omega(u, \cdot), \qquad \lim_{C_u, t\to\infty} r\nabla_N\Psi' = \Omega'(u, \cdot). \end{aligned} \tag{2.22}$$

We proceed by investigating the Hodge system for ϵ. The Hodge system for ϵ reads, see also [9, Chapter 9]:

$$\begin{aligned} \not{d}\!\not{i}\!\not{v}\, \epsilon = &-\nabla_N \delta - \frac{3}{2}\mathrm{tr}\,\theta\delta + \hat{\eta}\cdot\hat{\theta} \\ &- 2(a^{-1}\not{\nabla} a)\cdot\epsilon + \frac{1}{4} \mid \alpha(F) \mid^2 - \frac{1}{4} \mid \underline{\alpha}(F) \mid^2, \end{aligned} \tag{2.23}$$

$$\not{c}\!\not{u}\!\not{r}\!\not{l}\, \epsilon = \sigma(W) + \hat{\theta} \wedge \hat{\eta}. \tag{2.24}$$

We observe that the $\not{c}\!\not{u}\!\not{r}\!\not{l}$ equation coincides with the one obtained by Christodoulou and Klainerman in [7], whereas the $\not{d}\!\not{i}\!\not{v}$ equation contains the extra terms $\mid \alpha(F) \mid^2$ and $\mid \underline{\alpha}(F) \mid^2$ from the electromagnetic field. (See [7, Chapter 17, ((17.0.12a), (17.0.12b))].) According to [9, Chapter 9], one has

$$\begin{aligned}
&\nabla_N \delta - \hat{\theta} \cdot \hat{\eta} + \frac{1}{4} \mid \underline{\alpha}(F) \mid^2 \\
&= -2r^{-3}(\nabla_N r)p + r^{-2}\nabla_N p - r^{-2}(\nabla_N r)\nabla_N \Psi + r^{-1}\nabla_N^2 \Psi \\
&= -\hat{\chi} \cdot \hat{\eta} - r^{-1} \not{\triangle} \Psi - r^{-2}\big(r \operatorname{tr} \theta + a^{-1}\lambda\big)\nabla_N \Psi \\
&\quad - r^{-1}a^{-1} \not{\nabla} a \cdot \not{\nabla} \Psi + r^{-2}\nabla_N p - 2r^{-3}a^{-1}\lambda p \qquad (2.25)
\end{aligned}$$

with

$$p = r\nabla_N q + q' + \Psi'.$$

This differs from [7, Chapter 17, (17.0.12c)] by the extra curvature term from the electromagnetic field. Zipser derived in [9, Chapter 9, Lemma 9],

$$\triangle q = r(\mu - \overline{\mu}) + I, \qquad (2.26)$$

where

$$\begin{aligned}
I &= \frac{1}{2} {}^{(rN)}\hat{\pi}_{ij} k_{ij} + \frac{r}{4} \mid \alpha(F) \mid^2 - \frac{r}{4} \mid \underline{\alpha}(F) \mid^2 - \triangle \Psi \\
&= r\hat{\chi} \cdot \hat{\eta} - \kappa\delta - 2ra^{-1} \not{\nabla} a \cdot \epsilon + \frac{r}{4} \mid \alpha(F) \mid^2
\end{aligned}$$

and μ is the mass aspect function given by

$$\mu = -\rho(W) - \widehat{\chi} \cdot \hat{\eta}.$$

Recall the radial decomposition of $\triangle$ to be $\nabla_N^2 = \triangle - \operatorname{tr} \theta \nabla_N - \not{\triangle} - a^{-1} \not{\nabla} a \cdot \not{\nabla}$. Now, we obtain from the last equations that

$$\begin{aligned}
\triangle q &= \nabla_N^2 q + \operatorname{tr} \theta \nabla_N q + \not{\triangle} q + a^{-1} \not{\nabla} a \cdot \not{\nabla} q \\
&= -r(\rho - \bar{\rho}) - r\overline{\hat{\chi} \cdot \hat{\eta}} - \kappa\delta - 2ra^{-1} \not{\nabla} a \cdot \epsilon + \frac{r}{4} \mid \alpha(F) \mid^2 . \qquad (2.27)
\end{aligned}$$

We proceed as follows: Substituting first for $\nabla_N p$ from (2.27) in (2.25) and then the resulting terms from (2.25) in (2.23) yields

$$\begin{aligned}
\not{div}\, \epsilon &= \rho - \bar{\rho} + \hat{\chi} \cdot \hat{\eta} - \overline{\hat{\chi} \cdot \hat{\eta}} + r^{-1} \not{\triangle} \Psi - r^{-2}\nabla_N \Psi' \\
&\quad - r^{-3}a^{-1}\lambda \Psi' + l.o.t., \qquad (2.28) \\
\not{curl}\, \epsilon &= \sigma(W) + \hat{\theta} \wedge \hat{\eta}. \qquad (2.29)
\end{aligned}$$

Let

$$E = \lim_{C_u, t \to \infty} \left(r^2 \epsilon \right).$$

We multiply equations (2.28) and (2.29) by r^3 and take the limits on C_u as $t \to \infty$. This yields

$$\overset{\circ}{c\not{u}rl}\, E = Q + \Sigma \wedge \Xi, \tag{2.30}$$

$$\overset{\circ}{d\not{i}v}\; E = P - \bar{P} + \Sigma \cdot \Xi - \overline{\Sigma \cdot \Xi} + \overset{\circ}{\not{\triangle}}\; \Psi - \Psi' - \Omega'. \tag{2.31}$$

Then we investigate the limits as $u \to +\infty$ and $u \to -\infty$. Considering the last equations for ϵ, respectively E, and using Theorems 2.3 and 2.1 one finds that E tends to a limit E^+ as $u \to +\infty$ and to E^- as $u \to -\infty$.

By conclusions along the lines of [7, Chapter 17], we obtain

$$\overset{\circ}{c\not{u}rl}\; (E^+ - E^-) = 0.$$

In order to compute $\overset{\circ}{d\not{i}v}\; (E^+ - E^-)$, we have to consider especially the corresponding limits for the terms involving $\mathbf{\Psi}$ and $\mathbf{\Psi'}$, that is also Ω'.

Much like Christodoulou and Klainerman computed the formulas in Lemma 17.0.2, on p. 504 of [7], we derive the new results in which the electromagnetic field term $\underline{\alpha}(F)$, respectively, its limit A_F, is present. To do that, we use the fact that

$$\not{D}_4 \underline{\alpha}(F)_A = -\frac{1}{2} \mathrm{tr}\, \chi\, \underline{\alpha}(F)_A + \mathrm{l.o.t.} \tag{2.32}$$

The derivation of (2.32) can be found in Zipser's work [9] on p. 351 formula (4.15). Now, considering (2.19) and using (2.32), we find that

$$\not{D}_4 \mid \underline{\alpha}(F) \mid^2 = -\mathrm{tr}\, \chi \mid \underline{\alpha}(F) \mid^2 \; + \mathrm{l.o.t.} \tag{2.33}$$

Using (2.33), (2.20) and (2.21), we deduce formulas for $\mathbf{\Psi}$, $\mathbf{\Psi'}$, Ω, Ω' by computing the limits (2.22). We give the formulas:

$$\mathbf{\Psi} = -\frac{1}{2^{\frac{1}{2}} 4\pi} \int_{-\infty}^{+\infty} \left\{ \int_{S^2} \frac{\mid \Xi \mid^2 (u', \omega')}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' + \frac{1}{2} \int_{S^2} \frac{\mid A_F \mid^2 (u', \omega')}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' \right\} du',$$

$$\mathbf{\Psi'} = \frac{1}{2^{\frac{1}{2}} 4\pi} \int_{-\infty}^{+\infty} \left\{ \int_{S^2} \frac{\mid \Xi \mid^2 (u', \omega') - \overline{\mid \Xi \mid^2 (u')}}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' \right.$$
$$\left. + \frac{1}{2} \int_{S^2} \frac{\mid A_F \mid^2 (u', \omega') - \overline{\mid A_F \mid^2 (u')}}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' \right\} du',$$

$$\Omega = \frac{1}{2^{\frac{3}{2}} 4\pi} \int_{-\infty}^{+\infty} \left\{ \int_{S^2} \frac{\mid \Xi \mid^2 (u', \omega')}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' + \frac{1}{2} \int_{S^2} \frac{\mid A_F \mid^2 (u', \omega')}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' \right\} du'$$
$$+ \frac{1}{2} \int_{-\infty}^{+\infty} \left\{ \mathrm{sgn}(u - u') \Big(\mid \Xi \mid^2 (u', \omega') + \frac{1}{2} \mid A_F \mid^2 (u', \omega') \Big) \right\} du',$$
$$\Omega' = -\frac{1}{2^{\frac{3}{2}} 4\pi} \int_{-\infty}^{+\infty} \left\{ \int_{S^2} \frac{\mid \Xi \mid^2 (u', \omega') - \overline{\mid \Xi \mid^2 (u')}}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' \right.$$
$$\left. + \frac{1}{2} \int_{S^2} \frac{\mid A_F \mid^2 (u', \omega') - \overline{\mid A_F \mid^2 (u')}}{(1 - \omega\omega')^{\frac{1}{2}}} d\omega' \right\} du'$$
$$- \frac{1}{2} \int_{-\infty}^{+\infty} \left\{ \mathrm{sgn}(u - u') \Big(\Big(\mid \Xi \mid^2 (u', \omega') - \overline{\mid \Xi \mid^2 (u')} \Big) \right.$$
$$\left. + \frac{1}{2} \Big(\mid A_F \mid^2 (u', \omega') - \overline{\mid A_F \mid^2 (u')} \Big) \Big) \right\} du'.$$

Straightforward calculation shows that when evaluating the difference of the limits as $u \to +\infty$ and $u \to -\infty$ in (2.31), the contribution of $\overset{\circ}{\not\Delta} \Psi$, Ψ' and Ω' comes only from terms in Ω'. We find that Ω' tends to limits $\Omega'^{+}(\cdot)$ and $\Omega'^{-}(\cdot)$ as $t \to \infty$ and $t \to -\infty$, respectively. Thus, we conclude

$$\Omega'^{+}(\cdot) - \Omega'^{-}(\cdot) = \int_{-\infty}^{+\infty} \Big(\mid \Xi(u, \cdot) \mid^2 - \overline{\mid \Xi(u, \cdot) \mid^2} + \frac{1}{2} \mid A_F(u, \cdot) \mid^2$$
$$- \frac{1}{2} \overline{\mid A_F(u, \cdot) \mid^2} \Big) du. \tag{2.34}$$

Finally, we obtain

$$\overset{\circ}{\not{d}i\not{v}} \, (E^{+} - E^{-}) = -\Omega'^{+} + \Omega'^{-}$$
$$= \int_{-\infty}^{+\infty} \Big(- \mid \Xi(u, \cdot) \mid^2 + \overline{\mid \Xi(u, \cdot) \mid^2} - \frac{1}{2} \mid A_F(u, \cdot) \mid^2$$
$$+ \frac{1}{2} \overline{\mid A_F(u, \cdot) \mid^2} \Big) du. \tag{2.35}$$

Proceeding along the lines of [7, Chapter 17], it is

$$(E^{+} - E^{-}) = \overset{\circ}{\not\nabla} \Phi \tag{2.36}$$

with Φ being the solution of vanishing mean of

$$\overset{\circ}{\not\Delta} \Phi = -\Omega'^{+} + \Omega'^{-} \quad \text{on } S^2.$$

Accordingly, also by conclusions along the lines of [7, Chapter 17], we derive (2.38). To see this, we consider the normalized null Codazzi equation

$$\begin{aligned}(\not{div}\,\hat{\chi})_A - \frac{1}{2}\not\nabla_A \mathrm{tr}\chi + \epsilon_B \hat{\chi}_{AB} - \frac{1}{2}\epsilon_A \mathrm{tr}\chi \\ = -\beta(W)_A - \rho(F)\alpha(F)_A - \epsilon_{AB}\sigma(F)\alpha(F)_B. \end{aligned} \tag{2.37}$$

Multiply equation (2.37) by r^3 and take the limit as $t \to \infty$ on C_u. We obtain

$$\overset{\circ}{\not{div}}\ \Sigma = \overset{\circ}{\not\nabla}\ H + E$$

as in [7, Conclusion 17.0.8, p. 510] since the extra terms from the electromagnetic field in (2.37) decay fast enough. Due to equation (2.13), we conclude

$$\overset{\circ}{\not{div}}\ (\Sigma^+ - \Sigma^-) = E^+ - E^-. \tag{2.38}$$

Thus, the theorem is proved. □

2.4 Limit for r as $t \to \infty$ on null hypersurface C_u

We shall use the fact that the constraint on the space-like scalar curvature, which is given by

$$R = |k|^2 + R_{00},$$

differs from the constraint in the vacuum case only by the term R_{00}, which is a quadratic in F.

Building on the results of Christodoulou and Klainerman in [7] as well as the results of Zipser in [9] (i.e., [8]), we can now prove the following results.

Theorem 2.7. *As $t \to \infty$ we obtain on any null hypersurface C_u*

$$r = t - 2M(\infty)\log t + O(1).$$

Proof. We recall from [7, p. 503], with $\phi' = \phi - 1$,

$$\begin{aligned}\frac{dr}{dt} &= \frac{r}{2}\overline{\phi \mathrm{tr}\chi'} \\ &= \frac{r}{2}\overline{(1+\phi')\left(\frac{2}{r} + \left(\mathrm{tr}\chi' - \frac{2}{r}\right)\right)} \\ &= 1 + \overline{\phi'} + O(r^{-2}).\end{aligned}$$

In the last equality, we use equation (2.14).

From [9, p. 465]

$$R_{00} = \frac{1}{2}(| \underline{\alpha}(F) |^2 + | \alpha(F) |^2) + \rho(F)^2 + \sigma(F)^2 \tag{2.39}$$

with $\underline{\alpha}(F), \alpha(F), \rho(F), \sigma(F)$ the components of the electromagnetic field. Moreover, the lapse equation in our situation is given by

$$\triangle\phi = (| k |^2 + R_{00})\phi. \tag{2.40}$$

We integrate the lapse equation (2.40) on H_t in the interior of $S_{t,u'}$ to obtain

$$\int_{S_{t,u}} \nabla_N \phi' = \int_{u_0}^{u} du' \int_{S_{t,u'}} a\phi(| k |^2 + R_{00}).$$

In view of (2.39) and the fact that all the terms on the right-hand side of (2.39) except $\underline{\alpha}(F)$ are of lower order, we estimate

$$\int_{S_{t,u}} \nabla_N \phi' = \int_{u_0}^{u} du' \int_{S_{t,u'}} a\phi(| k |^2 + \frac{1}{2} | \underline{\alpha}(F) |^2) + \text{l.o.t.}$$

We see that

$$\int_{S_{t,u'}} a\phi(| k |^2 + \frac{1}{2} | \underline{\alpha}(F) |^2) \to \int_{S^2} | \Xi |^2 + \frac{1}{2} | A_F |^2 .$$

Consider the Bondi mass loss formula in Theorem 2.4. Then, as $t \to \infty$ we conclude

$$\int_{S_{t,u}} \nabla_N \phi' - 8\pi M(u) = O(r^{-1}) \tag{2.41}$$

on each C_u. In view of ϕ' we compute

$$\begin{aligned}\overline{\phi'} &= \frac{1}{4\pi r^2} \int_{S_{t,u}} \phi' = -\frac{1}{4\pi} \int_B \text{div}(r^{-2}\phi' N) \\ &= \frac{1}{4\pi} \int_B \Bigg(-\frac{1}{a(r(t,u'))^2} \overline{a \operatorname{tr} \theta} N\phi' + \frac{1}{(r(t,u'))^2} \phi' \underbrace{(\text{div} N)}_{= tr\theta} \\ &\quad + \frac{1}{(r(t,u'))^2} \nabla_N \phi' \Bigg)\end{aligned}$$

$$= -\frac{1}{4\pi}\int_u^\infty \frac{1}{(r(t,u'))^2}du'\left(\int_{S_{t,u'}} a\nabla_N\phi' + (a\mathrm{tr}\,\theta - \overline{a\,\mathrm{tr}\,\theta})\phi'\right)$$
$$= -\frac{1}{4\pi}\int_u^\infty \frac{1}{(r(t,u'))^2}du'\left(\int_{S_{t,u'}} \nabla_N\phi'\right) + O(r^{-2}),$$

where B denotes the exterior of $S_{t,u}$. Therefore, from (2.41) it follows on C_u as $t \to \infty$,

$$\overline{\phi'}(t,u) = -2\int_u^\infty \frac{1}{(r(t,u'))^2}M(u')du' + O(r^{-2}) = -\frac{2}{r}M(\infty) + O(r^{-2}).$$

Thus, we obtain on any cone C_u for $t \to \infty$,

$$\frac{dr}{dt} = 1 - \frac{2}{r}M(\infty) + O(r^{-2}). \tag{2.42}$$

Thus, the statement of our theorem follows, which closes the proof. □

3 Wave experiments

We are now going to show how the results above relate to experiment. In [5] Christodoulou established his breaking result on the nonlinear memory effect. The idea of the gravitational-wave experiment and setup is given in [5], discussing a laser interferometer gravitational-wave detector. There Christodoulou explained how the theoretical result on $\Sigma^+ - \Sigma^-$ leads to an effect measurable by such detectors. This effect manifests itself in a permanent displacement of the test masses of the detector after a wave train has passed. In the present EM case, we find a result on the displacement of test masses which is two-fold. Considering the Jacobi equation (see (3.10)), the highest order term remains unchanged. However, there is an extra term at highest order from the electromagnetic field in the formula for $\Sigma^+ - \Sigma^-$, the permanent displacement of test masses, as we have shown in the proof of Theorem 2.6. In the present section, we shall show how the electromagnetic field enters the experiment and we will derive results for this case.

We will follow the lines of argumentation by Christodoulou in [5, 6].

Let us briefly review the setup of a laser interferometer experiment: Three test masses are suspended by pendulums of equal length. Denote by m_0 the reference mass, which is also the location of the beam splitter. For time-like scales much shorter than the period of the pendulums, the motion of the masses in the horizontal plane can be considered free. By laser interferometry the distance of the masses m_1 and m_2 from the reference mass m_0 is measured. Whenever the light travel times between m_0 and m_1 and m_2, respectively, differ, this shows in a difference of phase of the laser light at m_0.

The masses m_0, m_1, m_2 move along geodesics γ_0, γ_1, γ_2 in spacetime. Let T be the unit future-directed tangent vectorfield of γ_0 and t the arc length along γ_0. For each t denote by H_t the space-like, geodesic hyperplane through $\gamma_0(t)$ orthogonal to T.

Take (E_1, E_2, E_3) to be an orthonormal frame for H_0 at $\gamma_0(0)$, and parallelly propagate it along γ_0 to obtain the orthonormal frame field (T, E_1, E_2, E_3) along γ_0. It follows that (E_1, E_2, E_3) at each t is an orthonormal frame for H_t at $\gamma_0(t)$. Then one assigns to a point p in spacetime, lying in a neighborhood of γ_0, the cylindrical normal coordinates (t, x^1, x^2, x^3), based on γ_0, if $p \in H_t$ and $p = \exp X$ with $X = \sum_i x^i E_i \in T_{\gamma_0(t)} H_t$. Denote by d the distance of p from the center $\gamma_0(t)$ on H_t, that is, $d = \mid X \mid = \sqrt{\sum_i (x^i)^2}$. The difference between the metric and the Minkowski metric $\eta_{\mu\nu}$ in these coordinates is

$$g_{\mu\nu} - \eta_{\mu\nu} = O(R\ d^2). \tag{3.1}$$

As usual, we put $c = 1$. Now, let τ be the time scale in which the curvature varies significantly. Then, the displacements of the masses from their initial positions will be $O(R\tau^2)$. Assume that

$$\frac{d}{\tau} << 1. \tag{3.2}$$

Then we can read off from the differences in phase of the laser light differences in distance of m_1 and m_2 from m_0. Further, in view of (3.2) one can replace the geodesic equation for γ_1 and γ_2 by the Jacobi equation (geodesic deviation from γ_0):

$$\frac{d^2 x^k}{dt^2} = -R_{kTlT}\ x^l \tag{3.3}$$

with $R_{kTlT} = R\ (E_k, T, E_l, T)$. One is free to assume that the source is in the E_3-direction. This was derived by Christodoulou for the EV case in [5], and can also be found in his [6].

We now investigate the formula (3.3) for the EM situation. If we assume the test masses not to be charged, then formula (3.3) stays the same, but through the EM equations and in view of (1.10) the electromagnetic field comes in. We shall see that it enters at lower order though. From (1.10) one can write

$$R_{k0l0} = W_{k0l0} + \frac{1}{2}(g_{kl}R_{00} + g_{00}R_{kl} - g_{0l}R_{k0} - g_{k0}R_{0l}). \tag{3.4}$$

As there is from the EM equations:

$$R_{00} = 8\pi T_{00},$$

and in particular, we have

$$R_{00} = \frac{1}{2}(\mid \underline{\alpha}(F) \mid^2 + \mid \alpha(F) \mid^2) + \rho(F)^2 + \sigma(F)^2, \tag{3.5}$$

we can investigate the components of the Ricci curvature on the right-hand side of (3.4). The component R_{00} includes the term $\mid \underline{\alpha}(F) \mid^2$. Recall that $\underline{\alpha}(F)$ is the part of the electromagnetic field with worst decay behavior. However it enters as a quadratic the formula for R_{00}.

To proceed, we consider $L = T - E_3$, $\underline{L} = T + E_3$. The leading components of the curvature are

$$\underline{\alpha}_{AB}(W) = R(E_A,\ \underline{L},\ E_B,\ \underline{L}), \tag{3.6}$$

$$\underline{\alpha}_{AB}(W) = \frac{A_{AB}(W)}{r} + o\ (r^{-2}). \tag{3.7}$$

And the leading components of the electromagnetic field are

$$\underline{\alpha}_A(F) = F(E_A, \underline{L}), \tag{3.8}$$

$$\underline{\alpha}_A(F) = \frac{A_A(F)}{r} + o\ (r^{-2}). \tag{3.9}$$

In the following the kth Cartesian coordinate of the mass m_A for $A = 1, 2$ will be denoted by $x^k{}_{(A)}$. Then the Jacobi equation becomes

$$\frac{d^2\ x^k_{(A)}}{d\ t^2} = -\frac{1}{4}r^{-1}A_{AB}\ x^l_{(B)} - \frac{1}{8}r^{-2} \mid A_F \mid^2 x^l_{(B)} + O\ (r^{-2}), \tag{3.10}$$

that is

$$\frac{d^2\ x^3{}_{(C)}}{d\ t^2} = 0, \tag{3.11}$$

$$\frac{d^2\ x^A{}_{(C)}}{d\ t^2} = -\frac{1}{4}\ r^{-1}\ A_{AB}\ x^B{}_{(D)} - \frac{1}{8}r^{-2} \mid A_F \mid^2 x^B{}_{(D)} + O\ (r^{-2}). \tag{3.12}$$

From the Jacobi equation (3.10) we see that the electromagnetic field enters on the right-hand side at order (r^{-2}) only. Thus, we have shown that the electromagnetic field does not contribute at leading order to the deviation measured by the Jacobi equation. Therefore, at leading order, we can rely on the results for the EV case, derived by Christodoulou in [5]. Instead of (3.10) he obtained

$$\frac{d^2\ x^k_{(A)}}{d\ t^2} = -\frac{1}{4}\ r^{-1}\ A_{AB}\ x^l_{(B)} + O(r^{-2}). \tag{3.13}$$

As in [5] one obtains that in the vertical direction there is no acceleration to leading order (r^{-1}). Initially, m_1 and m_2 are at rest at equal distance d_0 and at right angles from m_0. This implies the following initial conditions, as $t \to -\infty$: ${x^3}_{(A)} = 0$, ${\dot{x}^3}_{(A)} = 0$, ${x^B}_{(A)} = d_0 \delta^B_A$, ${\dot{x}^B}_{(A)} = 0$. The right-hand side being very small, one can substitute the initial values on the right-hand side. Then the motion is confined to the horizontal plane. One has to leading order

$$\ddot{x}^A_{\ (B)} = -\frac{1}{4}\ r^{-1}\ d_0\ A_{AB}. \tag{3.14}$$

One obtains

$${\dot{x}^A}_{(B)}\ (t) = -\frac{1}{4}\ d_0\ r^{-1} \int_{-\infty}^{t} A_{AB}(u)\ du. \tag{3.15}$$

In the following, let us revisit the result (3.19) from [5]. In view of equation (2.5), i.e., $\frac{\partial \Xi}{\partial u} = -\frac{1}{4}\ A$ and $\lim_{|u| \to \infty} \Xi = 0$ we obtain

$$-\int_{-\infty}^{t} A_{AB}(u)\ du = \Xi(t) \tag{3.16}$$

and

$${\dot{x}^A}_{(B)}(t) = \frac{d_0}{r}\ \Xi_{AB}(t). \tag{3.17}$$

As $\Xi \to 0$ for $u \to \infty$, the test masses return to rest after the passage of the gravitational wave. Taking into account (2.4), i.e., $\frac{\partial \Sigma}{\partial u} = -\Xi$, and integrating again:

$${x^A}_{(B)}(t) = -\left(\frac{d_0}{r}\right)(\Sigma_{AB}(t) - \Sigma^-). \tag{3.18}$$

The limit $t \to \infty$ is taken and it follows that the test masses experience permanent displacements. Thus $\Sigma^+ - \Sigma^-$ is equivalent to an overall displacement of the test masses:

$$\triangle\ {x^A}_{(B)} = -\left(\frac{d_0}{r}\right)\ (\Sigma^+_{AB} - \Sigma^-_{AB}). \tag{3.19}$$

The right-hand side of (3.19) includes terms from the electromagnetic field at highest order as given in our Theorem 2.6. Even though the form of (3.19) is as in the EV case investigated by Christodoulou in [5, 6], the nonlinear contribution from the electromagnetic field is present in $\Sigma^+_{AB} - \Sigma^-_{AB}$.

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References

[1] L. Bieri, *An extension of the stability theorem of the Minkowski space in general relativity*, ETH Zurich, Ph.D. thesis, **17178**, Zurich, 2007.

[2] L. Bieri, *Extensions of the stability theorem of the Minkowski space in general relativity. Solutions of the Einstein vacuum equations*, AMS-IP, Studies in Advanced Mathematics, Cambridge, MA, 2009.

[3] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems*, Proc. Roy. Soc. **A.269** (1962), 21–52.

[4] M.G.J. van der Burg, *Gravitational waves in general relativity X. Asymptotic expansions for the Einstein–Maxwell field*, Proc. Roy. Soc. **A.310** (1969), 221–230.

[5] D. Christodoulou, *Nonlinear nature of gravitation and gravitational-wave experiments*, Phys. Rev. Lett. **67**(12) (1991), 1486–1489.

[6] D. Christodoulou, *Mathematical problems of general relativity theory I and II*, vol. 1, EMS Publishing House, ETH Zürich, 2008, vol. 2 to appear: EMS Publishing House, ETH Zürich.

[7] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Math. Series, **41**, Princeton University Press, Princeton, NJ, 1993.

[8] N. Zipser, *The global nonlinear stability of the trivial solution of the Einstein–Maxwell equations*, Ph.D. thesis, Harvard Univ. Cambridge MA, 2000.

[9] N. Zipser, *Extensions of the stability theorem of the Minkowski space in general relativity. Solutions of the Einstein–Maxwell equations*, AMS-IP, Studies in Advanced Mathematics, Cambridge, MA, 2009.