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Quantization of the Hitchin moduli spaces, Liouville theory and the geometric Langlands correspondence I

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Abstract

We discuss the relation between Liouville theory and the Hitchin integrable system, which can be seen in two ways as a two step process involving quantization and hyperkähler rotation. The modular duality of Liouville theory and the relation between Liouville theory and the SL(2)-WZNW-model give a new perspective on the geometric Langlands correspondence and on its relation to conformal field theory.

1 Introduction

1.1 Motivation

It is known for a while that the low-energy theory of $N = 2$ supersymmetric gauge theories in four dimensions can be described in terms of the

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data characterizing an algebraically integrable system, which is canonically associated to a given gauge theory [12, 17, 30]. More recently it was found that studying the gauge theories in finite volume [56], or in the presence of certain deformations like the so-called Omega-deformation [51] is a useful tool to extract some highly non-trivial non-perturbative information about such gauge theories. It was in particular recently argued in [53] that the gauge theory in the presence of a certain one-parameter deformation can at low energies effectively be described in terms the quantization of the above-mentioned algebraically integrable system. An amazing correspondence was furthermore observed in [1, 56] between the partition functions of a certain class of gauge theories on S^4 and the correlation functions in Liouville theory [74, 78, 80]. Knowing the modular transformation properties of the Liouville conformal blocks [57, 74, 75, 78] now allows us to investigate and test the S-duality conjectures in these gauge theories, as illustrated in [2,11].

It seems, however, that the deeper reasons for this relationship between a two-dimensional (2D) and a 4D theory remain to be understood. A clue in this direction may be seen in the fact that the instanton partition functions which represent the building blocks of the partition functions studied in [1,56] are obtained by specializing the two-parameter family $\mathcal{Z}(a, \epsilon_1, \epsilon_2; q)$ of instanton partition functions introduced in [47, 49, 51]. The functions $\mathcal{Z}(a, \epsilon_1, \epsilon_2; q)$ not only allow one to obtain the Seiberg–Witten prepotential of the gauge theory on $\mathbb{R}^4$ in the limit where both ϵ_1 and ϵ_2 tend to zero [51, 52], but also the Yang's potential determining the spectrum of the quantized integrable model mentioned above in the limit where only one of the two parameters ϵ_1 or ϵ_2 vanishes. This was observed in [53] for a certain class of examples, and is expected to hold much more generally. The functions $\mathcal{Z}(a, \epsilon_1, \epsilon_2; q)$ were identified with the conformal blocks of Liouville theory in [1].

This indicates that the relationship between certain gauge theories and Liouville theory involves in particular a two-parametric deformation of the algebraically integrable model associated to the gauge theories on $\mathbb{R}^4$, which ultimately produces Liouville theory as a result. One of my intentions in this paper is to clarify in which sense this point of view is correct. Such a study may be seen as being complementary to the recent work of Nekrasov and Witten [55], where certain aspects of the correspondence between Liouville theory and gauge theory were understood by studying a certain two-parameter generalization of the setup from [53]. We will make some comments on this relation in the conclusions.

Another piece of motivation comes from the relations discussed in [39] between 4D gauge theories and the geometric Langlands correspondence. A puzzling aspect of the resulting picture is the fact that the geometric

Langlands correspondence is also related to conformal field theory as shown in the works of Beilinson, Drinfeld, Feigin, Frenkel and others, see [24] for a nice review and further references. However, the relation between the gauge theory approach to the geometric Langlands correspondence of [39] and the conformal field theory approach has remained mostly unclear up to now. The author feels that the above-mentioned relations between the gauge theory and conformal field theory offer new clues in this regard. It is therefore my second main aim to clarify the relations between the quantization of the Hitchin system, the geometric Langlands correspondence and the Liouville conformal field theory.

1.2 From the Hitchin integrable system to Liouville theory

One of my aims is to explain that it is possible to understand the relation between the Liouville and the Hitchin systems in two ways as the result of a two-step process which is a combination of a one-parameter deformation and quantization, schematically:

$$\begin{array}{ccccc} & & \text{Hitchin system} & & \\ & (\mathrm{A})_{\epsilon_2} \swarrow & & \searrow (\mathrm{B})_{\epsilon_1} & \\ \begin{array}{c}\text{Fuchsian}\\ \text{isomonodromic}\\ \text{deformations}\end{array} & & & & \begin{array}{c}\text{quantized}\\ \text{Hitchin}\\ \text{systems}\end{array} \\ & (\mathrm{C})_{\epsilon_1} \searrow & & \swarrow (\mathrm{D})_{\epsilon_2} & \\ & & \text{Liouville theory} & & \end{array} \qquad (1.1)$$

where the arrows may be schematically characterized as follows:

(A) Hyperkähler rotation for the Hitchin moduli space $\mathcal{M}_{\mathrm{H}}(C)$. This is explained in Section 3.
(B) Quantization of the Hitchin system in the sense discussed in [53] and [55] with quantization conditions determined by Yang's potential (Section 4).
(C) Quantization of the Hitchin moduli spaces $\mathcal{M}_{\mathrm{H}}(C)$. This is explained in Section 6.
(D) This arrow will be referred to as *quantum* hyperkähler rotation. The motivation for this terminology come from the closure of the diagram together with the observation that the quantized Hitchin system can be recovered from Liouville theory in suitable limits, as discussed in Section 5.

The parameters ϵ_1, ϵ_2 that govern the different relations will also be parameterized as

$$\epsilon_1 = \hbar b, \quad \epsilon_2 = \hbar/b, \qquad (1.2)$$

with b being the parameter that is often used in the literature on Liouville theory.

Liouville theory is known to be related to the quantum theory of Teichmüller spaces [76, 77]. The Teichmüller spaces can be identified with one of the connected components of the moduli spaces of flat $\mathrm{SL}(2,\mathbb{R})$-connections on Riemann surfaces. We will show that the relations above can be understood as natural generalizations of the relation between Liouville theory and the quantization of Teichmüller spaces to the other components of the moduli spaces of flat $\mathrm{SL}(2,\mathbb{R})$-connections.

1.3 Separation of variables

It is known for a while that the Hitchin system is related to the conformal field theory by a similar-looking two-step procedure of deformations and formal quantization.

$$
\begin{array}{ccccc}
 & & \text{Hitchin system} & & \\
 & (a)_\epsilon \swarrow & & \searrow (b)_\hbar & \\
\begin{array}{c}\text{Isomonodromic}\\ \text{deformations}\end{array} & & & & \begin{array}{c}\text{Beilinson–Drinfeld}\\ \text{system}\end{array} \\
 & (c)_{\check{k}} \searrow & & \swarrow (d)_k & \\
 & & \text{KZB-equations} & &
\end{array}
\tag{1.3}
$$

$(a)_\epsilon$ The Hitchin system can be obtained as a limit of the isomonodromic deformation system as shown in [45, 46].

$(b)_\hbar$ The quantization of the Hitchin Hamiltonians constructed by Beilinson–Drinfeld [3].

$(c)_k$ The Knizhnik–Zamolodchikov–Bernard (KZB) equations of the Wess-Zumino-Novikov-Witten (WZNW) conformal field theory can be obtained as a formal quantization of the ismonodromic deformation system, as was observed for $g = 0$ in [32, 58] and shown for $g > 0$ in [5].

$(d)_{\check{k}}$ The eigenvalue equation for the Gaudin Hamiltonians, which are the $g = 0$ cases of the quantized Hitchin Hamiltonians arise in the critical level limit of the Knizhnik–Zamolodchikov (KZ) equations as shown for $g = 0$ in [61] and for $g > 0$ in [5].

The whole diamond of relations was discussed in [5, 46].

The diagram (1.3) is of course not unrelated to the previous one in (1.1). On the classical level there are two natural representations for the Hitchin system, one coming from the representation of an open dense subset of the Hitchin moduli space $\mathcal{M}_{\mathrm{H}}(C)$ as $T^*\mathrm{Bun}_G$, the other is related to a natural map Sov from $\mathcal{M}_{\rm H}(C)$ to a Hilbert scheme $(T^*C)^{[h]}$ of points on T^*C which can also be used to introduce a set of coordinates for $\mathcal{M}_{\rm H}(C)$. The change of variables Sov is closely related to what is called the separation of variables (SOV) in the integrable systems literature [64].

Moreover indeed, we are going to explain that the full set of relations between the diagrams (1.1) and (1.3) originates from the change of variables underlying the SOV method. For the quantized Hitchin system the relation between the diagrams was found in [64] for $g = 0$ and for $g = 1$ in [14]. It is related to the quantum version of the SOV method. At the bottom of (1.3) and (1.1) one finds on the level of systems of differential equations a correspondence between the null-vector decoupling equations of Belavin–Polyakov–Zamolodchikov (BPZ) and the KZ equations discovered in [66]. The correspondence between the respective systems of differential equations can be extended to a correspondence between Liouville theory and SL(2)-WZNW-model on the level of the full correlation functions as was established in [59] for $g = 0$ and extended to higher genus in [36]. We will finally show that the relation between the Fuchsian isomonodromic deformation equations and the theory of isomonodromic deformations of flat holomorphic connections can also be seen to follow from a variant of the change of variables as used in the SOV method.

The relations in (1.3) were so far only discussed on the level of system of differential equations. The connection with Liouville theory allows us to go much further: It enables us to construct and parameterize interesting spaces of solutions to the KZB equations which are complete in the sense that all the monodromies can be represented as linear transformations.

1.4 Geometric Langlands correspondence

Our second main aim in this article is to point out relations to the geometric Langlands correspondence and a certain generalization thereof. The geometric Langlands correspondence (see [24] for a nice review and further references) is often schematically presented as a correspondence between

$$\boxed{{}^{\rm L}G - \text{local systems}} \quad \longrightarrow \quad \boxed{\mathcal{D} - \text{modules on } \mathrm{Bun}_G} \tag{1.4}$$

It is connected to the quantization of the Hitchin Hamiltonians [3] by noting that an important part of the $\mathcal{D}$-module structure on the right-hand side of (1.4) can be represented as the system of eigenvalue equations

$$\mathsf{H}_r \Psi = E_r \Psi, \tag{1.5}$$

for the quantized Hitchin Hamiltonians H_r, with H_r being certain second-order differential operators on a line bundle on Bun_G.

We are going to propose that important aspects of the geometric Langlands correspondence can be understood as arising in a suitable limit from a correspondence between the conformal blocks of Liouville theory and those of the SL(2)-WZNW model that will be described below. This correspondence is based on the relations observed in [66] between the BPZ and KZ systems of differential equations. We are going to show that this correspondence opens the way to *construct* the conformal blocks of the SL(2)-WZNW model from those of Liouville theory. The possibility to reconstruct the SL(2)-WZNW model from the Liouville theory,

$$\boxed{\mathsf{Liou}_b} \quad \longrightarrow \quad \boxed{\mathsf{WZNW}_k(\mathfrak{sl}_2)} \tag{1.6}$$

may be seen as a kind of inversion of the Drinfeld–Sokolov reduction. The correspondence (1.6) will be shown to reproduce important aspects of the geometric Langlands correspondence in the limit $k \to -2$, which is called the critical level limit. The KZ equations yield the eigenvalue equations for the Hitchin Hamiltonians representing the right-hand side of (1.4). This limit is related to the limit $b \to \infty$ in the Liouville theory. Liouville theory has the profound property to be self-dual under inversion of the parameter b, which means that almost[1] all characteristic quantities of Liouville theory like in particular the conformal blocks are unchanged if one replaces b by $1/b$. This phenomenon will be referred to as the modular duality of Liouville theory. The modular duality of Liouville theory implies that the critical level limit is equivalent to the classical limit in Liouville theory. Fuchsian differential equations of the second order arise naturally in this limit. The monodromies of the solutions to these Fuchsian differential equations are the local systems on the left-hand side of (1.4).

On the level of the representation theory of chiral algebras a related way to explain the local geometric Langlands correspondence was developed in [18], see also [23] and in particular [24, Section 8.6] for a nice discussion. Relations between the geometric Langlands correspondence and the SOV method have first been discussed in [22], which was an important source of inspiration for this work.

There are two elements that the relationship with Liouville theory adds to the story. First, it allows one to lift certain aspects of the geometric

[1] The only exception being the dependence on the cosmological constant, the parameter in front of the interaction term $e^{2b\varphi}$ in the Liouville action.

Langlands correspondence from the local level (opers on a disc versus representations of the current algebra at the critical level) to the global level where both sides of (1.4) are associated to Riemann surfaces. Even more interesting appears to be the possibility to extend the geometric Langlands correspondence from the level of $\mathcal{D}$-modules to the level of the multivalued holomorphic solutions of the differential equations coming from the $\mathcal{D}$-module structure.

1.5 Modular duality versus Langlands duality

The modular duality of Liouville theory offers another way to construct an SL(2)-WZNW model from Liouville theory [27], obtained from the first by the exchange $b \to b^{-1}$, schematically

$$\boxed{\mathsf{WZNW}_{\check{k}}(\mathfrak{sl}_2)} \quad\longleftarrow\quad \boxed{\mathsf{Liou}_b} \quad\longrightarrow\quad \boxed{\mathsf{WZNW}_{k}(\mathfrak{sl}_2)} \tag{1.7}$$

The level $\check{k}$ of the SL(2)-WZNW model on the left is determined by

$$\check{k} + 2 = (k+2)^{-1} = -b^2. \tag{1.8}$$

We are going to show that the corresponding relations between spaces of conformal blocks lead to another approach to the geometric Langlands correspondence in which both sides of (1.4) are obtained in the limit $b \to \infty$. The same limit that reduces the KZB equations to the eigenvalue equations of the quantized Hitchin Hamiltonians is now observed to be the classical limit $\check{k} \to \infty$ for the dual WZNW model $\mathsf{WZNW}_{\check{k}}(\mathfrak{sl}_2)$. Local systems will be found to arise very naturally in the classical limit $\check{k} \to \infty$ of the WZNW model. This means that the somewhat asymmetric looking geometric Langlands correspondence (1.4) is obtained in the limit $b \to \infty$ from a much more symmetric looking duality between two WZNW-models at different levels,

$$\begin{array}{ccccc}
\boxed{\text{PSL(2)} - \text{local systems}} & & \leftrightarrow & & \boxed{\mathcal{D}\text{-modules on } \text{Bun}_{\text{SL(2)}}} \\
\uparrow & & & & \uparrow \\
\boxed{\mathsf{WZNW}_{\check{k}}(\mathfrak{sl}_2)} & \longleftarrow & \boxed{\mathsf{Liou}_b} & \longrightarrow & \boxed{\mathsf{WZNW}_{k}(\mathfrak{sl}_2)}
\end{array} \tag{1.9}$$

It seems natural to call the relations schematically represented at the bottom of (1.9) a *quantum geometric Langlands correspondence.* Other approaches to defining "quantum" versions of the geometric Langlands correspondence have been discussed in [24, 26, 40, 67].

The author views this paper as a first look on a huge iceberg, most of which remains invisible. It is hoped that this look stimulates further investigations of this subject.

In the first part of our present paper we will mostly illustrate the picture proposed above by examples related to Riemann surfaces of genus 0. The forthcoming second part of the paper [79] will discuss the cases of higher genus in more detail. Nevertheless, whenever easily possible we will present the relevant background and the main claims in full generality already in this paper.

2 The classical Hitchin system

The following is a (rather incomplete) reminder of some basic definitions and results about the Hitchin system.

2.1 Self-duality equations versus Higgs pairs

The Hitchin moduli space $\mathcal{M}_{\rm H}(C)$ on a Riemann surface C is the space of solutions (A,θ) of the SU(2) self-duality equations

$$F_A + R^2[\theta,\bar\theta] = 0, \qquad \begin{aligned} \bar\partial_A\theta + \theta\,\bar\partial_A = 0,\\ \partial_A\bar\theta + \bar\theta\,\partial_A = 0, \end{aligned} \tag{2.1}$$

where $d_A = d + A$ is an SU(2)-connection on a vector bundle V, and θ is a holomorphic one-form with values in $\mathrm{End}(V)$, modulo SU(2) gauge transformations. $\mathcal{M}_{\rm H}(C)$ is a space of complex dimension $6g-6+2n$ if $C = C_{g,n}$ is a Riemann surface of genus g with n marked points.

Decomposing d_A into the $(1,0)$ and $(0,1)$ parts ∂_A and $\bar\partial_A$, respectively, we may associate to each solution a holomorphic vector bundle $\mathcal{E}$ with holomorphic structure being defined by $\bar\partial_A = \bar\partial + A^{0,1}$. Equations (2.1) imply in particular that θ is holomorphic with respect to the holomorphic structure defined by $\bar\partial_A$. This means that each solution of the self-duality equations (2.1) defines a Higgs pair $(\mathcal{E},\theta)$, which is a pair $(\mathcal{E},\theta)$ of objects, with $\mathcal{E}$ being a holomorphic vector bundle, and $\theta \in H^0(C, \mathrm{End}(\mathcal{E})\otimes\Omega^1_C)$. Conversely, Higgs pairs come from solutions of the self-duality equations iff they are stable, which means that any θ-invariant sub-bundle of V must have a degree that is smaller than half of the degree of V [34].

We will allow for a finite number of regular singularities on C. Introducing a local coordinate y_r near the singular point z_r, $r = 1, \dots, n$, we will require that the singular behavior is of the form

$$A = \frac{1}{2i} A_r \left(\frac{dy_r}{y_r} - \frac{d\bar{y}_r}{\bar{y}_r} \right) + \text{regular}, \quad \theta = \frac{1}{2} \theta_r \frac{dy_r}{y_r} + \text{regular}, \tag{2.2}$$

with θ_r and A_r being simultaneously diagonalizable matrices, and A_r skew-Hermitian.

There is a natural slice within $\mathcal{M}_{\mathrm{H}}(C)$ defined by the condition $\theta = 0$. It is clearly isomorphic to $\mathrm{Bun}_G(C)$, the moduli space of holomorphic bundles on C. Sections θ of $H^0(C, \mathrm{End}(\mathcal{E}) \otimes K_C)$, where K_C is the canonical line bundle, naturally represent vectors in the cotangent space of $\mathrm{Bun}_G(C)$. It follows that an open dense subset of $\mathcal{M}_{\mathrm{H}}(C)$ is naturally isomorphic to the cotangent bundle $T^*\mathrm{Bun}_G(C)$.

2.2 The Hitchin integrable system

To begin with, let us consider an SL(2) Higgs pair $(\mathcal{E}, \theta)$. Associate to it the quadratic differential

$$\vartheta = \mathrm{tr}(\theta^2). \tag{2.3}$$

Expanding ϑ with respect to a basis $\{\vartheta_1, \dots, \vartheta_{3g-3+n}\}$ of the $3g-3+n$-dimensional space of quadratic differentials,

$$\vartheta = \sum_{r=1}^{3g-3+n} H_r \, \vartheta_r, \tag{2.4}$$

defines functions H_r, $r = 1, \dots, 3g-3+n$ on $\mathcal{M}_{\mathrm{H}}(C)$, which are called Hitchin's Hamiltonians. The subspaces $\Theta_E \subset \mathcal{M}_{\mathrm{H}}(C)$ defined by the equations $H_r = E_r$ for $E = (E_1, \dots, E_{3g-3+n})$ are abelian varieties (complex tori) for generic E. This means that $\mathcal{M}_{\mathrm{H}}(C)$ can be described as a torus fibration with base $\mathcal{B}$, which can be identified with the space $\mathcal{Q}(C)$ of quadratic differentials on the underlying Riemann surface C.

There is a complex structure I on $\mathcal{M}_{\mathrm{H}}(C)$ for which both E and complex analytic coordinates for the fibers Θ_E are holomorphic. Associated with the complex structure I is the holomorphic symplectic structure Ω_I, which can be defined as

$$\Omega_I = 2iR \int_C \mathrm{tr}(\delta\theta \wedge \delta A^{0,1}), \tag{2.5}$$

where $\bar{\partial}_A = \bar{\partial} + A^{0,1}$. The functions H_r are Poisson-commuting with respect to the Poisson structure coming from the symplectic structure Ω_I.

The assertions above can be summarized in the statement that $\mathcal{M}_{\mathrm{H}}(C)$ is an algebraically completely integrable system in complex structure I. It is useful to encode the values of E into the definition of the spectral curve

$$\Sigma = \{(v, y) | \det(v - \theta(y)) = 0\}, \tag{2.6}$$

which defines a double cover Σ of the surface C.

Certain generalizations of this set-up will become relevant for us later. Instead of considering holomorphic $G = \mathrm{SL}(2)$-bundles one may consider bundles in $G = \mathrm{GL}(2)$. One may furthermore consider Higgs fields θ in $H^0(C, \mathrm{End}(\mathcal{E}) \otimes L)$, with L being a line bundle different from the canonical line bundle K_c. In this case one gets additional degrees of freedom and additional Hamiltonians from $\mathrm{tr}(\theta)$. This will be discussed in more detail in Part II of this paper.

2.3 SOV

In the SOV method [64, 65], one maps the dynamics of an integrable system to the motion of a divisor on the spectral curve. It furnishes a set of canonically conjugate variables which can be used as a starting point for the quantization of the model.

Let Bun_G be the moduli space of holomorphic vector bundles $\mathcal{E}$ on V. In the case of SL(2)-bundles on $C_{g,n}$, for example, we have

$$d := \dim_{\mathbb{C}}(\mathcal{M}_k) = 3g - 3 + n. \tag{2.7}$$

The SOV amounts to the existence of a birational map

$$\mathsf{Sov} : T^*\mathcal{M} \to (T^*C)^{[d]},$$

from $T^*\mathcal{M}$ to the Hilbert scheme of points on T^*C, which is a symplectomorphism on open dense subsets. The open dense subset of $(T^*C)^{[d]}$, which is relevant here is the set

$$\mathcal{Y} \equiv ((T^*C)^d - \Delta)/S_d,$$

with Δ being union of all diagonals and S_d is the symmetric group. On this subset one may choose coordinates $(y, v) \equiv [(y_1, v_1), \dots, (y_d, v_d)]$ such that the symplectic form Ω_I becomes

$$\omega = \sum_{r=1}^{d} dv_r \wedge dy_r. \tag{2.8}$$

The main idea behind the definition of the coordinates (y, v) can be described most easily in the case of $g = 0$ with n marked points corresponding to the Gaudin model. Choosing a gauge where $A^{0,1} = 0$, the Higgs pair (V, θ) is characterized by Higgs fields of the form

$$\theta = \begin{pmatrix} \theta^0 & \theta^+ \\ \theta^- & -\theta^0 \end{pmatrix}, \quad \theta^a = \sum_{r=1}^{n} \frac{\theta_r^a}{y - z_r}, \tag{2.9}$$

subject to the global $\mathfrak{sl}_2$-invariance constraints $\sum_{r=1}^{n} \theta_r^a = 0$ for $a = -, 0, +$. $\vartheta(y)$ is the form

$$\vartheta(y) = \sum_{r=1}^{n} \left(\frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r} \right), \tag{2.10}$$

where δ_r are central elements, and the H_r are the Hitchin Hamiltonians. In the following we will mostly consider a slightly simpler version of this model obtained by sending $z_n \to \infty$, $\theta_n^- \to 0$ and imposing $\sum_{r=1}^{n-1} \theta_r^a = \delta_{a,0}\sqrt{\delta_n}$ for $a = -, 0$. The difference is in the treatment of the global $\mathfrak{sl}_2$-invariance, and will turn out to be inessential even on the quantum level.

The coordinates y_r are then found as the zeros of $\theta^-(y)$,

$$\theta^-(y) = u \frac{\prod_{j=1}^{n-3}(y - y_j)}{\prod_{i=1}^{n-1}(y - z_i)}, \quad u = \sum_{i=1}^{n-1} \mu_i z_i, \tag{2.11}$$

where $\mu_r = \mathrm{Res}_{y=z_r} \theta^-(y)$. The conjugate variables v_r can be found from the condition that the point (y_r, v_r) of T^*C lies on the curve Σ,

$$v_r^2 = \vartheta(y_r) = \mathrm{tr}(\theta^2(y_r)). \tag{2.12}$$

Given the tuple (y, v) one recovers the spectral curve Σ as the curve that goes through all points (y_r, v_r), while for fixed values of the conserved quantities one may view the equations $v_r^2 = \vartheta(y_r)$ as equations determining the "momenta" v_r in terms of the variables y_r and the values of the conserved Hamiltonians.

The SOV for $g > 0$ was discussed in [31, 44]. It can be recast in a form more similar to the $g = 0$ case as will be discussed in [79].

2.4 Special geometry of the base of the Hitchin fibration

It is known that the base of any algebraically completely integrable system canonically has special geometry [17]. In the case at hand it can be described as follows. The spectral curve Σ is a double covering of the surface C. On Σ let us introduce the differential

$$dS = vdy. \tag{2.13}$$

We then get the special coordinates a_r, a_s^D as the periods of S along the homology cycles α_r, β_s, $r, s = 1, \dots, h$, respectively,

$$a_r = \int_{\alpha_r} dS, \quad a_r^D = \int_{\beta_r} dS. \tag{2.14}$$

Both $a = (a_1, \dots, a_h)$ and $a^D = (a_1^D, \dots, a_h^D)$ represent systems of coordinates for the base $\mathcal{B}$. The change of coordinates can be described in terms of a holomorphic function $\mathcal{F}(a)$ called prepotential such that

$$a_r^D = \frac{\partial \mathcal{F}}{\partial a_r}. \tag{2.15}$$

There are coordinates $\tau = (\tau_1, \dots, \tau_h)$ on the torus fibers $\Theta_{E(a)}$ which are Poisson-conjugate to the variables a. The coordinates (a, τ) are action-angle variables for the Hitchin system.

3 Isomonodromic deformations as a deformation of the Hitchin system

3.1 Hitchin moduli space as space flat connections

There is a useful description of the Hitchin moduli space $\mathcal{M}_{\mathrm{H}}$ as a moduli space of flat complex connections. To each solution (A, θ) to the self-duality equations (2.1), we may associate the connection

$$\nabla = \nabla' + \nabla'', \qquad \begin{aligned} \nabla' &= \partial_A + R\,\theta, \\ \nabla'' &= \bar{\partial}_A + R\,\bar{\theta}. \end{aligned} \tag{3.1}$$

The connection is flat thanks to equations (2.1). In (3.1) we have introduced a parameter R, which can be eliminated by a rescaling of θ, $\bar{\theta}$, but which is sometimes useful.

Conversely, given a flat connection ∇ on a vector bundle V on C, there is a canonical way to associate to it a solution to the self-duality equations. For given connection ∇, let $\rho : \pi(X) \to \mathrm{PSL}(2,\mathbb{C})$ be its monodromy representation. The key result [9,13,62] to be used is the existence of a *canonical* Hermitian metric h on the fibers of V, which may be represented as a smooth ρ-equivariant *harmonic* map from the universal cover $\tilde{C}$ of C to $H = G/K$, with K being the maximal compact subgroup of $G = \mathrm{PSL}(2,\mathbb{C})$. The metric h allows us to decompose the connection ∇ into the component $\nabla_K = d + A$ preserving the subgroup K, and the component Θ orthogonal to Lie algebra of K. Decomposing further into the $(1,0)$ and $(0,1)$ parts $\nabla' = \partial_A + R\theta$ and $\nabla'' = \bar{\partial}_A + R\bar{\theta}$ yields a solution to the self-duality equations, as is reviewed in [63, Section 2].

3.2 Flat connection versus local systems

Using the complex structure of the underlying surface, it is possible to represent the connections ∇ in holomorphic terms. To this aim one may note that $\nabla'' = \bar{\partial}_A + R\bar{\theta}$ is an integrable holomorphic structure and ∇' is an integrable holomorphic connection on $\mathcal{E} = (V, \nabla'')$. We may introduce local trivializations such that $\nabla'' = \bar{\partial}$. The connection ∇ is then locally described by holomorphic differential operators of the form

$$\nabla' = (\partial_y + M(y))\, dy. \tag{3.2}$$

One may furthermore trivialize the bundle by means of a basis of local solutions of $\nabla' s = 0$. The transition functions between the patches of such a trivialization must then be *constant*. This means that a flat connection ∇ on a surface C canonically defines a *local system*, a vector bundle defined by a local trivialization with *constant* transition functions between the patches.

Let $\mathrm{Loc}_G(C)$ be the moduli space of G-local systems for a complex group G. The space $\mathrm{Loc}_{\mathrm{PSL}(2,\mathbb{C})}(C)$ is also known as the space of projective structures on C.

Two alternative realizations of local systems will be used. First, each local system canonically defines a representation

$$\rho : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C}). \tag{3.3}$$

Conversely, each such representation ρ canonically defines a local system, see e.g., [24, Section 3.1] for the three-line proof. The space of local systems is therefore isomorphic to the space $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2, \mathbb{C}))$ of representations of the fundamental group $\pi_1(C)$ in $\mathrm{PSL}(2, \mathbb{C})$. In the following, we will often identify the representations ρ of $\pi_1(C)$ with the corresponding local systems.

Alternatively, one may associate to each local system a pair of objects $(\mathcal{E}, \nabla')$, where $\mathcal{E}$ is a holomorphic vector bundle on C, and ∇' is a holomorphic connection, which may be locally represented in the form

$$\nabla' = \frac{\partial}{\partial y} + M(y), \tag{3.4}$$

where $M(y)$ is a matrix-valued holomorphic function. The correspondence between local systems and pairs $(\mathcal{E}, \nabla')$ is called the Riemann–Hilbert correspondence.

It may also be useful to consider holomorphic vector bundles $\tilde{\mathcal{E}}$ with *meromorphic* connections $\tilde{\nabla}'$. As illustrated later, we may then have pairs $(\tilde{\mathcal{E}}, \tilde{\nabla}')$ which have the same monodromy representation $\rho : \pi(X) \to \mathrm{PSL}(2, \mathbb{C})$ as a given local system $(\mathcal{E}, \nabla')$.

3.3 Hyperkähler structure

For a given Higgs bundle $(\mathcal{E}, \theta)$ one may introduce, generalizing the decomposition (3.1), a one-parameter family of flat connections as

$$\nabla_\zeta = \nabla'_\zeta + \nabla''_\zeta, \qquad \begin{aligned} \nabla'_\zeta &= \partial_A + \frac{1}{\zeta} R\,\theta, \\ \nabla''_\zeta &= \bar{\partial}_A + \zeta R\,\bar{\theta}. \end{aligned} \tag{3.5}$$

Associated to this one-parameter family of flat connections are a one-parameter family of natural complex structures $J^{(\zeta)}$ and holomorphic symplectic forms ϖ_ζ on the Hitchin moduli space $\mathcal{M}_{\mathrm{H}}(C)$ [34]. The complex structures $J^{(\zeta)}$ can be characterized by the property that holomorphic functions of the flat connection ∇_ζ like the traces of monodromies of ∇_ζ are holomorphic in complex structure I_ζ. The holomorphic symplectic forms ϖ_ζ

can be defined as

$$\varpi_\zeta = \frac{1}{2}\int_C \operatorname{tr}(\delta\mathcal{A}_\zeta \wedge \delta\mathcal{A}_\zeta), \tag{3.6}$$

where $\mathcal{A}$ is defined by $\nabla_\zeta = d + \mathcal{A}_\zeta$. The form ϖ_ζ can be expanded as

$$\varpi_\zeta = -\frac{i}{2\zeta}\,\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-, \tag{3.7}$$

where, in particular, $\omega_+ \equiv \Omega_I$, the natural holomorphic symplectic form associated to the Higgs bundle picture for $\mathcal{M}_{\mathrm{H}}(C)$ defined in (2.5).

In order to describe the situation in purely holomorphic terms, let $\mathcal{E}_{\zeta R}$ be the holomorphic structure on the vector bundle V defined by $\nabla''_\zeta = \bar\partial_A + \zeta R\theta$. On $\mathcal{E}_{\zeta R}$ let us, following [63, Section 4], consider the holomorphic ϵ-connection, which locally is obtained from ∇'_ζ by

$$\partial_\epsilon \equiv \epsilon\nabla'_\zeta = \epsilon\partial + I(y), \quad \epsilon = \frac{\zeta}{R}. \tag{3.8}$$

$I(y)$ transforms under gauge transformations as $I \to g^{-1}Ig + \epsilon g^{-1}\partial g$.

3.4 Drinfeld–Sokolov reduction

Important for us will be a special class of local systems called *opers* [4], which in the case $\mathfrak{g} = \mathfrak{sl}_2$ may be described as bundles admitting a connection that locally looks as in (3.4) with

$$M(y) = \begin{pmatrix} 0 & t(y) \\ 1 & 0 \end{pmatrix}. \tag{3.9}$$

The equation $(\partial_y + M(y))\phi = 0$ now implies that the component χ of $\phi = (\eta, \chi)$ solves a second-order differential equation of the form

$$(\partial_y^2 + t(y))\chi = 0. \tag{3.10}$$

Under holomorphic changes of the local coordinates on C, the differential operator $\partial_y^2 + t(y)$ transforms as

$$t(y) \mapsto (y'(w))^2 t(y(w)) - \frac{1}{2}\{y, w\}, \quad \{y, w\} \equiv \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2, \tag{3.11}$$

which is the transformation law characteristic for a *projective* connection. The transformation law (3.11) follows from the transformation law for a connection if one takes into account that a compensating gauge transformation is generically needed in order to recover the form (3.9) of the connection after having changed the local coordinate.

It is useful to note that any local system can be represented in the form (3.9) away from finitely many points on C, as discussed in [24, Section 9.6]. In order to see this for $G = \mathrm{SL}(2)$ in a simple way, let us represent the elements of the connection matrices $M(y)$ as

$$M(y) = \begin{pmatrix} \alpha(y) & \beta(y) \\ \gamma(y) & -\alpha(y) \end{pmatrix}. \tag{3.12}$$

$\gamma(y)$ may be set to one by a singular gauge transformation

$$\partial_y + M' \equiv g \cdot (\partial_y + M) \cdot g^{-1}, \quad g = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, \tag{3.13}$$

where $h(y) = \sqrt{\gamma(y)}$. The gauge transformation g is singular at the zeros $w_1, \dots, w_d$ of $\gamma(y)$. This is where $M'(y)$ has additional singularities. By means of a further gauge transformation one may set the diagonal elements of $M'(y)$ to zero, leading to

$$M'(y) = \begin{pmatrix} 0 & t(y) \\ 1 & 0 \end{pmatrix}. \tag{3.14}$$

The corresponding equation $(\partial_y^2 + t(y))\chi = 0$ has regular singular points $z_1, \dots, z_n$ *and* $w_1, \dots, w_d$. The behavior near the singular points is of the form

$$\begin{aligned} t(y) &\sim \frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r}, \quad \text{near } y = z_r, \\ t(y) &\sim \frac{-3}{4(y - w_k)^2} + \frac{\kappa_k}{y - w_k}, \quad \text{near } y = w_k. \end{aligned} \tag{3.15}$$

However, the additional singularities as $w_1, \dots, w_d$ are gauge artefacts, and the monodromy of $\partial_y + M'(y)$ is the same as the one of $\partial_y + M(y)$. The singular points $w_1, \dots, w_d$ of $t(y)$ are called apparent singularities which expresses the fact that the monodromy around these singular points is trivial in $\mathrm{PSL}(2, \mathbb{C})$. It can be shown [22, Section 3.9] that this implies the equations

$$t_{k,2} + t_{k,1}^2 = 0, \quad \text{where } t(y) = \sum_{i=0} t_{k,i}(y - w_k)^{i-2}. \tag{3.16}$$

These equations give relations between the parameters w_k, $\kappa_{k'}$ and H_r of the projective connection $\partial_y^2 + t(y)$.

3.5 Space of opers

Of particular importance for us will be the cases where $d = 0$, where there are no apparent singularities. Let $\mathrm{Op}_{\mathfrak{sl}_2}(C)$ the space of $\mathfrak{sl}_2$-opers on a Riemann surface C. Two opers P and P' differ by a holomorphic quadratic differential $\vartheta = P - P'$. This implies that the space $\mathrm{Op}_{\mathfrak{sl}_2}(C_{g,n})$ of $\mathfrak{sl}_2$-opers on a fixed surface $C_{g,n}$ of genus g with n marked points is $3g-3+n$-dimensional. Complex analytic coordinates for $\mathrm{Op}_{\mathfrak{sl}_2}(C_{g,n})$ are obtained by picking a reference projective connection P_0, a basis $\vartheta_1, \dots, \vartheta_{3g-3+n}$ for the vector space of quadratic differentials, and writing any other projective connection P as

$$P = P_0 + \sum_{r=1}^{3g-3+n} H_r \, \vartheta_r. \tag{3.17}$$

The parameters H_r are sometimes called accessory parameters.

The monodromy representations $\rho_P : \pi_1(C_{g,n}) \to \mathrm{PSL}(2, \mathbb{C})$ of the differential operators P will generate a $3g-3+n$-dimensional subspace of the space $\mathrm{Loc}_{\mathrm{PSL}(2,\mathbb{C})}(C_{g,n})$ of local systems. Varying the complex structure of the underlying surface C, too, we get a subspace of $\mathrm{Loc}_{\mathrm{PSL}(2,\mathbb{C})}(C)$ of complex dimension $6g-6+2n$. The space of opers forms an affine bundle $\mathcal{P}$ over the Teichmüller space of deformations of the complex structure of C. Standard Teichmüller theory identifies the space of quadratic differentials with the holomorphic cotangent space of the Teichmüller space of deformations of the complex structure of C. It follows that $\mathcal{P}$ is canonically isomorphic to the cotangent bundle $T^*\mathcal{T}(C)$ over the Teichmüller space $\mathcal{T}(C)$. It is important that the mapping $\mathcal{P} \to \mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2, \mathbb{C}))$ defined by associating to the projective connection P its monodromy representation ρ_P is locally biholomorphic, and that the corresponding mapping $T^*\mathcal{T} \to \mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2, \mathbb{C}))$ is symplectic in the sense that the canonical cotangent bundle symplectic structure is mapped to the natural symplectic structure $\Omega_J \equiv \varpi_\zeta|_{\zeta=1}$ on the space of flat complex connections, see [42] and references therein. We may, therefore, choose a set of local coordinates $q = (q_1, \dots, q_{3g-3+n})$ on $\mathcal{T}(C_{g,n})$ which are conjugate to the coordinates H_r defined above in the sense that the Poisson brackets coming from this symplectic structure are

$$\{q_r, q_s\} = 0, \quad \{H_r, q_s\} = \delta_{r,s}, \quad \{H_r, H_s\} = 0. \tag{3.18}$$

Other useful sets of coordinates for the space of opers can be defined in terms of the monodromy map $\mathbf{M} : \mathrm{Op}_{\mathfrak{sl}_2}(C_{g,n}) \to \mathrm{Hom}(\pi_1(C_{g,n}), \mathrm{PSL}(2, \mathbb{C}))$ as follows. Let $\mathcal{C}$ be a pants decomposition of $C_{g,n}$ defined by a collection $\{\gamma_1, \dots, \gamma_{3g-3+n}\}$ of simple mutually non-intersecting closed curves. To each curve γ_r there corresponds a unique generator γ_r of the fundamental group $\pi_1(C_{g,n})$. For given oper, let

$$L_r := 2\cosh\frac{l_r}{2} := \mathrm{tr}(\rho_P(\gamma_r)). \tag{3.19}$$

The tuple $l_P = (l_1, \dots, l_{3g-3+n})$ can be used to parameterize $\mathrm{Op}_{\mathfrak{sl}_2}(C_{g,n})$ at least locally. For given $l = (l_1, \dots, l_{3g-3+n})$ one may generically find accessory parameters $H_r = H_r(l, q)$ such that (3.19) is satisfied (Riemann–Hilbert correspondence).

3.6 Isomonodromic deformations

The representation of the connection ∇ in terms of holomorphic data $(\mathcal{E}, \nabla')$ was using the complex structure on C. It is natural to ask how $(\mathcal{E}, \nabla')$ vary if we consider variations of the complex structure of C for fixed monodromy of the connection ∇. This defines families of compatible flows on the space of pairs $(\mathcal{E}, \nabla')$ [5, 45, 63]. The differential equations characterizing these flows are called the isomonodromic deformation equations. For $g = 0$ one gets well-known systems of partial differential equations, and more explicit forms of the resulting equations for $g > 0$ were obtained in [45].

3.6.1 Example: the Schlesinger system

In the case of $g = 0$ with n punctures we can describe $\mathrm{Loc}_{\mathrm{SL}(2,\mathbb{C})}$ as the space of all meromorphic connections of the form

$$\partial_y + M(y) = \partial_y + \sum_{r=1}^{n} \frac{M_r}{y - z_r}, \qquad M_r \in \mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}, \tag{3.20}$$

with fixed conjugacy class of M_r. The equations

$$\begin{aligned} \frac{\partial}{\partial z_s} M_r &= \frac{[M_r, M_s]}{z_r - z_s}, \quad r \neq s, \\ \frac{\partial}{\partial z_r} M_r &= \sum_{s \neq r} \frac{[M_r, M_s]}{z_r - z_s} \end{aligned} \tag{3.21}$$

ensure that the monodromy of $\partial_y + M(y)$ stays constant under variations of the complex structure of $\mathcal{C}$. Equation (3.21) are integrable, and define what is called the Schlesinger system.

3.6.2 Separation of variables for the Schlesinger system

For the case at hand ($g = 0$) it is particularly easy to see that the relation between the holomorphic connection (3.20) and the second-order differential operator $\partial_y^2 + t(y)$ is based on a change of variables very similar to the one that was giving the SOV for the Gaudin model in Section 2.3. Following the discussion in Section 3.4 leads to the differential equation $(\partial_y^2 + t(y))\chi = 0$ with $t(y)$ of the form

$$t(y) = \sum_{r=1}^{n} \left(\frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r} \right) - \sum_{k=1}^{l} \left(\frac{3}{4(y - w_k)^2} - \frac{\kappa_k}{y - w_k} \right). \quad (3.22)$$

In order to eliminate the constraints following from projective invariance, let us choose $z_n = 0,\ z_{n-1} = 1,\ z_{n-2} = \infty$. $t(y)$ may then be written in the form

$$t(y) = \frac{\delta_n}{y^2} + \frac{\delta_{n-1}}{(1-y)^2} + \frac{\upsilon}{y(y-1)} + \sum_{r=1}^{n-3} \left(\frac{\delta_r}{(y - z_r)^2} + \frac{z_r(z_r - 1)}{y(y-1)} \frac{H_r}{y - z_r} \right)$$
$$- \sum_{k=1}^{d} \left(\frac{3}{4(y - w_k)^2} - \frac{w_k(w_k - 1)}{y(y-1)} \frac{\kappa_k}{y - w_k} \right). \quad (3.23)$$

In the case where $d = n - 3$, equations (3.16) can be written explicitly as

$$\kappa_k^2 + \sum_{r=1}^{n} \left(\frac{\Delta_r}{(w_k - z_r)^2} + \frac{H_r}{w_k - z_r} \right) - \sum_{\substack{k=1 \\ k' \neq k}}^{d} \left(\frac{3}{4(w_k - w_{k'})^2} - \frac{\kappa_k}{w_k - w_{k'}} \right) = 0, \quad (3.24)$$

Equation (3.24) can be solved to express H_r in (3.23) in terms of variables w_k and κ_k. The resulting expression is a quadratic polynomial $H_r(\kappa, w)$ in the variables κ_k. This is precisely the form of an projective connection considered in the theory of the Garnier systems. The monodromy of the projective connection $\partial_y^2 + t(y)$ stays constant under a variation of the variables z_r provided that κ_k, w_k are varied according to

$$\frac{\partial w_k}{\partial z_r} = \frac{\partial H_r}{\partial \kappa_k}, \qquad \frac{\partial \kappa_k}{\partial z_r} = -\frac{\partial H_r}{\partial w_k}. \quad (3.25)$$

These equations are nothing but the rewriting of the Schlesinger equations (3.21) in terms of the separated variables w_k defined by the condition of $\gamma(w_k) = 0$, where $\gamma(y)$ is defined in (3.12).

3.6.3 Symplectic structure

The Hamiltonian form of the isomonodromic deformation equations (3.25) naturally suggests the Poisson structure

$$\{w_r, w_s\} = 0, \quad \{\kappa_r, w_s\} = \delta_{r,s}, \quad \{\kappa_r, \kappa_s\} = 0. \tag{3.26}$$

In the generalization to higher genus [37, 38] it is natural to set $d = 3g - 3 + n$. The positions w_k of the apparent singularities together with the residues κ_k introduced in (3.15) then form a local set of coordinates for the subset of $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ given by the monodromies of the Fuchsian differential operators $\partial_y^2 + t(y)$. This Poisson structure (3.26) coincides with the one coming from the holomorphic symplectic form Ω_J on Hitchin moduli space [38].

3.7 Real slices

A real slice in the space $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ is naturally defined by the requirement that the representation $\rho \in \mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. The space $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))$ has finitely many connected components, as will be described in the following.

3.7.1 Representation in terms of Higgs pairs

Let us first describe how these components are represented in terms of Higgs pairs $(\mathcal{E}, \theta)$. For a given effective divisor D of degree d, and chosen square-root $K_C^{\frac{1}{2}}$ of the canonical line bundle let us consider holomorphic bundles $\mathcal{E}$ of the form

$$\mathcal{E} = L_1 \oplus L_2, \quad \begin{aligned} L_2 &:= K_C^{+\frac{1}{2}}, \\ L_1 &:= K_C^{-\frac{1}{2}} \otimes D. \end{aligned} \tag{3.27}$$

Let us then consider Higgs fields of the form

$$\theta = \begin{pmatrix} 0 & \vartheta \\ \gamma & 0 \end{pmatrix}, \tag{3.28}$$

where γ is a holomorphic section of the line bundle corresponding to D, and ϑ is a quadratic differential.

3.7.2 Representation in terms of flat connections

It can be shown (see [34] for details) that the flat connection $\nabla = d + \mathcal{A}$ associated to such Higgs pairs may then be represented in the form

$$\mathcal{A} = \begin{pmatrix} -\frac{1}{2}\partial\varphi & R\,\frac{1}{\zeta}\,\vartheta\, e^{-\varphi} \\ R\,\frac{1}{\zeta}\,\gamma\, e^{\varphi} & +\frac{1}{2}\partial\varphi \end{pmatrix} dz + \begin{pmatrix} +\frac{1}{2}\bar{\partial}\varphi & R\,\zeta\,\bar{\gamma}\, e^{\varphi} \\ R\,\zeta\,\bar{\vartheta}\, e^{-\varphi} & -\frac{1}{2}\bar{\partial}\varphi \end{pmatrix} d\bar{z}. \tag{3.29}$$

It is manifest that this is a SU(1,1)-connection when $\zeta = 1$. The flatness is equivalent to

$$\partial\bar{\partial}\varphi = R^2(\gamma\bar{\gamma}\, e^{2\varphi} - \vartheta\bar{\vartheta}\, e^{-2\varphi}). \tag{3.30}$$

This is a variant of the Sinh-Gordon equation. It reduces to a variant of the Liouville-equation for $\vartheta = 0$. In this case, equation (3.30) implies that

$$ds^2 = \gamma\bar{\gamma}\, e^{2\varphi}\, dzd\bar{z} \tag{3.31}$$

is a metric of constant negative curvature on C. This metric has conical singularities with excess angle 2π at the zeros of γ.

3.7.3 Complex structures on the real slices

Let $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^d$ be the connected component in $\mathrm{Hom}(\pi_1(C),$ $\mathrm{PSL}(2,\mathbb{R}))$ that is described in this way. It will be important for us to note that there is a convenient description of $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^d$ as a complex analytic manifold associated to this description. It is proven in [34, Section 10], see also [29, Section 6.2], that $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^d$ has the structure of a holomorphic vector bundle over the symmetric power $\mathrm{Sym}^d(C)$, with fiber over $D \in \mathrm{Sym}^d(C)$ being the vector space

$$\{\vartheta \in H^0(C, K_c^2) | \mathrm{div}(\vartheta) \geq D\} \simeq \mathbb{C}^{3g-3+n-d}. \tag{3.32}$$

The relation to the representation in terms of Fuchsian differential equations described in Section 3.4 is easy to see: The divisor D is the collection $(w_1, \ldots, w_d)$ of apparent singularities. Equations (3.16) imply that there are d relations among the $3g - 3 + n$ holomorphic quadratic differentials.

3.7.4 Teichmüller component

Of particular interest and importance is the case where $d = 0$. The representations $\rho \in \mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$ are then *Fuchsian*, which means that quotient of the upper half-plane by the representation ρ produces a Riemann surface C with natural constant curvature metric induced from the hyperbolic metric of upper half-plane [29]. The component $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$ is therefore called the Teichmüller component.

The relation to the discussion above can easily be seen as follows. Setting $d = 0$ implies that γ can be set to unity in above equations. Each point in $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$ can be represented by a flat connection of the form (3.29). It is shown in [34, Theorem (11.2)] that any solution of the flatness condition (3.30) defines a metric of constant negative curvature via

$$ds^2 = e^{2\varphi}(dz + e^{-2\varphi}\bar{\vartheta}\, d\bar{z})(d\bar{z} + e^{-2\varphi}\vartheta\, dz). \tag{3.33}$$

We see that the quadratic differentials ϑ parameterize deformations of the constant negative curvature metric associated to the complex structure of C. The natural complex structure on the Teichmüller space $\mathcal{T}(C)$ of such deformations coincides with the complex structure on the Teichmüller component $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$ introduced in Section 3.7.3, as follows from the fact that $e^{-2\varphi}\bar{\vartheta}$ is the so-called harmonic Beltrami-differential associated to the quadratic differential ϑ from Teichmüller theory.

One should note, however, that in order to get the corresponding Fuchsian representative $\partial_y^2 + t(y)$, we need to set $\vartheta = 0$, as was observed above. It is not hard to show that $t(y)$ is then equal to the so-called energy-momentum tensor associated to the metric of constant negative curvature,

$$t(y) = -(\partial_z\varphi)^2 + \partial_z^2\varphi. \tag{3.34}$$

This means that the space of opers $\mathrm{Op}_{\mathfrak{sl}_2}(C)$ is another slice in $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ which intersects the real slice $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$ transversally. This fits naturally to our earlier observation that the space of opers is naturally isomorphic to the holomorphic cotangent space of $\mathcal{T}(C)$: While $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$ is naturally isomorphic to the Teichmüller space, the space $\mathrm{Op}_{\mathfrak{sl}_2}(C)$ represents the cotangent space of $\mathcal{T}(C)$. Both spaces are naturally isomorphic to each other, but this isomorphism is not holomorphic, as it involves the constant curvature metric $e^{2\varphi}dzd\bar{z}$.

3.8 Kähler potential on the real slices

The symplectic structure Ω_J on $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$, restricted to the real slices $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^d$ gives us the natural symplectic structure $\Omega_J^{\mathbb{R}}$ we will consider. We have seen, on the other hand, that the real slices $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^d$ have a natural complex structure related to the the complex structure from the Teichmüller theory.

For $d = 0$ it is known that the symplectic structure $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^d$ is *Kähler* w.r.t. this symplectic structure, with Kähler potential given by the Liouville action. The Liouville action functional $S^{\mathrm{cl}}[\varphi]$ is defined as

$$S_L[\varphi] = \frac{1}{2\pi}\int_{C_{g,n}} d^2z\left(\frac{1}{2}(\partial_a\varphi)^2 + 8\pi\mu b^2 e^{2\varphi}\right) + [\text{boundary terms}], \quad (3.35)$$

with a suitable choice of boundary terms which was determined in [68–70]. The Liouville action defines a natural symplectic form on $\mathcal{T}(C)$ as

$$\Omega_{\mathcal{T}} = 2i\,\partial\bar{\partial}S^{\mathrm{cl}}, \quad (3.36)$$

where ∂, $\bar{\partial}$ are the holomorphic and anti-holomorphic components of the de Rham differential on $\mathcal{T}_{g,n}$ respectively. It was shown in [68–72] that $\Omega_{\mathcal{T}}$ coincides with the Weil–Petersson symplectic form from the Teichmüller theory, which in turn is known to coincide [28, 34] with the symplectic structure $\Omega_J^{\mathbb{R}}$ on $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$. This implies that the Poisson structure on the real slices is still of the form (3.18), but the variables H are no longer independent, but rather given as functions of the variables q. A convenient reference projective connection P_S is e.g., given by the Schottky uniformization, and

$$P - P_S = \frac{1}{2}\partial S_L[\varphi], \quad (3.37)$$

for a suitable choice of boundary terms in the definition of the Liouville action functional [68–72].

For $g = 0$, $C_{0,n} = \mathbb{P}^1 \setminus \{z_1, \ldots, z_n\}$ one can represent P in the form

$$P = \partial_y^2 + t(y), \quad t(y) = \sum_{r=1}^{n}\left(\frac{\delta_r}{(y - z_r)^2} + \frac{H_r}{y - z_r}\right). \quad (3.38)$$

The parameters H_r are restricted by the relations

$$\sum_{r=1}^{n} H_r = 0, \quad \sum_{r=1}^{n}(z_r H_r + \delta_r) = 0, \quad \sum_{r=1}^{n}(z_r^2 H_r + 2\delta_r z_r) = 0. \quad (3.39)$$

The coordinates q_r conjugate to the H_r may be chosen as cross-ratios of the coordinates $z_1, \ldots, z_n$. Alternatively, one may set $z_n = \infty$, $z_{n-1} = 1$ and $z_{n-2} = 0$, and identify the remaining coordinates $q_1 \equiv z_1, \ldots, q_{n-3} \equiv z_{n-3}$ as the conjugates of $H_1, \ldots, H_{n-3}$, respectively.

For $d > 0$ one needs to take into account the fact that the quadratic differentials that are holomorphic on C are constrained by the relations (3.16).

For general g let us pick a set $\{\vartheta_1, \dots, \vartheta_{3g-3+n-d}\}$ of linearly independent holomorphic quadratic differentials. In order to generate all Teichmüller deformations one has to add d meromorphic quadratic differentials $\{\vartheta_1^{\rm mer}, \dots, \vartheta_d^{\rm mer}\}$, where $\vartheta_k^{\rm mer}$ has a pole at the point w_k. Expanding

$$P - P_0 = \sum_{r=1}^{3g-3+n-d} H_r\, \vartheta_r + \sum_{k=1}^{d} \kappa_k\, \vartheta_k^{\rm mer}, \tag{3.40}$$

The quadratic differentials define $(1,0)$-forms on $\mathcal{T}(C)$. There are corresponding local coordinates $q_1, \dots, q_{3g-3+n-d}$ and $w_1, \dots, w_d$ such that these $(1,0)$-forms are representable as dq_r and dw_k, respectively. The only non-vanishing Poisson brackets are then

$$\{\kappa_k, w_l\} = \delta_{r,s}, \quad \{H_r, q_s\} = \delta_{r,s}. \tag{3.41}$$

The coordinates w_k will parameterize the positions of the apparent singularities.

3.9 Limit to the Hitchin system

Let us now consider the limit $\zeta \to 0$, $R \to \infty$ such that $R\zeta$ stays constant. This implies in particular that the integrable holomorphic structure $\nabla''_\zeta = \bar\partial_A + \zeta R\theta$ is kept fixed in the limit. The ϵ-connection $\partial_\epsilon = \epsilon\partial - I$ becomes the Higgs field θ. In terms of opers, one may take the limit by rescaling $t(y) = \epsilon^{-2}\vartheta_\epsilon(y)$. The transformation of $\vartheta_\epsilon(y)$ is then

$$\vartheta_\epsilon(y) \mapsto (y'(w))^2\, \vartheta_\epsilon(y(w)) - \frac{\epsilon^2}{2}\{y, w\}. \tag{3.42}$$

For $\epsilon \to 0$ we get the transformation law of quadratic differentials. We may in this sense regard the space $\mathrm{Op}_{\mathfrak{sl}_2}(C)$ as a deformation $\mathcal{B}_\epsilon$ of the base $\mathcal{B}$ of the Hitchin fibration.

The complex structure $J^{(\zeta)}$ turns into the complex structure I characteristic for the Hitchin integrable system, and the symplectic structure ϖ_ζ becomes the symplectic structure Ω_I of $T^*\mathrm{Bun}_G$ in the sense that

$$\Omega_I = \operatorname*{Res}_{\zeta=0}(\varpi_\zeta).$$

One may furthermore study the isomonodromic deformation equations in this limit. This is slightly delicate, but the upshot is that isomonodromic deformation equations indeed reduce to the equations of motion of the Hitchin system in this limit [45].

We may naturally distinguish two types of observables, the Hamiltonians H_r on the one hand, and the traces $L_r = 2\cosh\frac{l_r}{2} = \mathrm{tr}(\rho_P(\gamma_r))$ of monodromies on the other hand. It seems natural to refer to them as local and non-local observables, respectively. The former are clearly related to the Hitchin Hamiltonians H_r in the limit under consideration. In order to study the asymptotics for $\epsilon \to 0$ of the latter, let us note that the leading Wentzel-Kramers-Brillouin (WKB) approximation to the solutions of the equation $(\epsilon^2\partial_y^2 + \vartheta(y))\chi(y) = 0$ can be constructed in terms of the differential dS introduced in (2.13) as

$$\chi_\pm(y) = \exp\left(\frac{i}{\epsilon}\int^y dz\; v_\pm\right), \tag{3.43}$$

where $v_\pm$ are two choices of a branch for the solution of the equation $v^2 = \vartheta(z)$. It follows easily from (3.43) that the parameters l_r are related to the action variables a_r introduced in (2.14) in the limit $\epsilon \to 0$,

$$l_r = \frac{4\pi}{\epsilon} a_r, \quad r = 1, \ldots, 3g-3+n, \tag{3.44}$$

as follows from (3.43). We may therefore regard the non-local observables L_r parameterizing the monodromies of the flat connections as deformations of the action variables a_r associated to the special geometry of the Hitchin fibration.

These remarks are supposed to clarify the meaning of the arrow marked $(\mathrm{A})_{\epsilon_2}$ in (1.1). In this regard let us note in particular that the relation between the isomonodromic deformations and the Hitchin system involves a hyperkähler rotation in the parameter ζ.

4 Quantization of the Hitchin system

4.1 Quantization scheme

The quantization of an algebraically integrable system like the Hitchin system can roughly be approached in the following way.

(a) Deform the space of (algebraic) functions on the phase space to a non-commutative algebra $\mathcal{A}$, whose elements are supposed to become the observables of the quantum theory. Of particular interest are the Hamiltonians whose proper definition will typically involve ordering issues. Integrability means that $\mathcal{A}$ should contain a commutative subalgebra $\mathcal{I}$ of "sufficient size" generated by the quantized Hamiltonians.

(b) Choose a Lagrangian subspace $\mathcal{L}$ of the phase space, here $\mathcal{M}_{\rm H}$, and represent the quantized algebra of observables as algebraic differential or difference operators on $\mathcal{L}$.

(c) Choose a $*$-structure on the algebra of observables and find a scalar product on the space of functions that realizes the $*$-structure via hermitian conjugation.

For any given value of E there is typically a finite-dimensional space of solutions to the eigenvalue equations $\mathsf{H}\Psi = E\Psi$ of the Hamiltonian $\mathsf{H} \in \mathcal{A}$ that have suitable analytic properties. Normalizability of the solutions w.r.t. the scalar product introduced in step (c) then selects in many cases a discrete subset of the possible values of E and thereby yields the quantization conditions.

In the case of the Hitchin system it is in most cases difficult to implement step (c) explicitly since the complex structure on the phase space typically depends on the complex structure of the underlying surface C, and is hard to describe explicitly, making the definition of a suitable scalar product difficult. The only known examples of Hitchin-type systems where it is know how to implement step (c) explicitly are the Calogero systems.

In the following, we shall describe basic elements of steps (a) and (b), but instead of implementing (c) we shall discuss another approach. Integrability means that the phase space in question has the structure of a torus fibration with base $\mathcal{B}$. We will (inspired by [53]) propose to replace step (c) by

(c′) Define a suitable "deformation" $\mathcal{B}_\epsilon$ of the base $\mathcal{B}$, and a function $\mathcal{W} : \mathcal{B}_\epsilon \to \mathbb{C}$ called Yang's potential whose critical points define the eigenvalues.

Such a procedure appears to be well-motivated in the case of algebraically integrables systems for the following reason. In some prototypical examples like the quantum Toda chain it is possible to prove that the quantization conditions obtained in step (c) can indeed be recast in the form (c′) for a suitable choice of the Yang's potential $\mathcal{W}$ [43]. As $\mathcal{W}$ depends analytically on all parameters, one may use the characterization of the spectrum in terms of the Yang's potential even in cases when step (c) is hard to implement.

In the case of the Hitchin system, the space $\mathcal{B}_\epsilon$ will be identified with the moduli space of opers. Our proposal will be to identify the Yang's potential with the semiclassical Liouville conformal blocks, which leads to a precise definition in terms of the theory of ordinary differential equations.

4.2 Semiclassical quantization of the separated variables

One possible approach to the quantization of the Hitchin system can be based on using the separated variables $(y,t) \equiv [(y_1,t_1),\dots,(y_h,t_h)]$ with symplectic form (2.8) as a starting point. In view of (2.8) it seems natural to regard the variables t_r as momenta, the y_r as coordinates. The quantization of equation (2.12) defining the spectral curve of the Hitchin system would then naturally lead to the differential equation

$$(\epsilon^2\partial_{y_r}^2 + \vartheta(y_r))\chi(y) = 0. \tag{4.1}$$

These equation will in the following be referred to as the Baxter equations.

The leading WKB approximation to the solutions of the Baxter equation (4.1) can be constructed in terms of the differential dS introduced in (2.13),

$$\chi_\pm(y) = \exp\left(\frac{i}{\epsilon}\int^y dz\; v_\pm\right), \tag{4.2}$$

where $t_\pm$ are two choices of a branch for the solution of the equation $v^2 = \vartheta(z)$.

There are a few natural possibilities one could discuss for the definition of quantization conditions.

4.2.1 Real quantization

It may happen that the integrable system of physical interest is actually a real slice of the algebraically integrable system under mathematical study. This is the case e.g., in the Calogero model, which is a special case of the Hitchin system, see e.g., the discussion in [53]. In this case one needs to impose a reality condition on the coordinate functions a_r (or a_r^{D}) of the base of the torus fibration. Combined with the Bohr–Sommerfeld quantization conditions one arrives at the conditions

$$a_r = 2\pi\epsilon\, n_r, \quad n_r \in \mathbb{Z}, \tag{4.3}$$

where a_s are the periods of the Seiberg–Witten differential $dS = dy\, v$ w.r.t. the cycles generating a canonical basis for $H_1(\Sigma,\mathbb{Z})$,

$$a_r = \int_{\alpha_r} dS, \quad a_s^D = \int_{\beta_s} dS. \tag{4.4}$$

The concrete choice of a basis $(\alpha_1,\dots,\alpha_{3g-3+n};\beta_1,\dots,\beta_{3g-3+n})$ may be tricky. For later convenience we will henceforth assume that the α_r coincide with a maximal set of simple closed curves defining a pants decomposition of $C_{g,n}$. Nekrasov and Shatashvili [53] discussed the quantization conditions (4.3) in a related context.

4.2.2 Complex quantization

In the present case, there is an interesting alternative one can discuss. The phase space in question has a complex structure, allowing one to require that the $*$-structure on the algebra of observables acts as complex conjugation. One may simply choose the Lagrangian subspace $\mathcal{L}$ to be a complex subspace, and assume that the algebra $\mathcal{A}$ of observables is realized both by holomorphic and anti-holomorphic differential operators. For $g = 0$ one thereby gets the $\mathrm{SL}(2,\mathbb{C})$-Gaudin model.

One of the basic requirements that an eigenfunctions of the Hitchin Hamiltonians should satisfy is single-valuedness. In order to find a single-valued solution of the eigenvalue equations we need to form linear combinations of the form

$$\phi(y,\bar{y}) = (\chi_+(y), \chi_-(y)) \cdot K \cdot \begin{pmatrix} \bar{\chi}_+(\bar{y}) \\ \bar{\chi}_-(\bar{y}) \end{pmatrix}. \tag{4.5}$$

Single-valuedness of ϕ leads to the Bohr–Sommerfeld quantization conditions $K = \mathrm{diag}(1,-1)$ and

$$\mathrm{Re}(a_r) = \pi\epsilon\, n_r, \quad \mathrm{Re}(a_s^D) = \pi\epsilon\, n_s^D, \tag{4.6}$$

where a_s and a_s^D are the periods of the Seiberg–Witten differential $dS = dy\, v$ as above. The derivation of (4.6) is discussed in detail for the closely related $\mathrm{SL}(2,\mathbb{C})$-XXX-model in [10].

Remark 4.1. It is interesting to note[2] that the conditions (4.6) coincide with the so-called attractor equations.

4.3 Quantization of the Hitchin Hamiltonians

A dense open subset of $\mathcal{M}_{\mathrm{H}}$ is isomorphic to $T^*\mathrm{Bun}_G$, the moduli space of stable G-bundles (here $G = SL(2)$) on C. This forms the basis to an alternative approach to the quantization of $\mathcal{M}_{\mathrm{H}}$, in which the Lagrangian subspace $\mathcal{L}$ taken to be (possibly a real slice of) Bun_G. States in the quantum theory can then be described in terms of functions (or sections of some line bundle) on Bun_G. Linear coordinates on the fibers of $T^*\mathrm{Bun}_G$ play the role of

[2] This fact has independently been remarked by S. Shatashvili, who had discussed it in various lectures long before this paper has appeared.

momenta and would consequently be realized as differential operators. The complex structure on Bun_G (which is coming from the complex structure on $C_{g,n}$) allows us to distinguish holomorphic and anti-holomorphic coordinates and the corresponding differential operators.

Hitchin's Hamiltonians are constructed from $\mathrm{tr}(\theta^2)$. As θ is holomorphic in complex structure I, they should become holomorphic differential operators on Bun_G after quantization. Beilinson and Drinfeld [3] constructed such differential operators from the representation theory of affine Kac-Moody algebras at the critical level, as will be reviewed in Section 7 below. Here we will discuss the example of $g = 0$ where Hitchin's Hamiltonians can be quantized in an elementary way.

4.3.1 Example: the $SL(2,\mathbb{C})$-Gaudin model

In the case of $g = 0$, $\mathcal{M}$ parameterizes the choices of parabolic structures at the marked points z_n. On an open dense subspace one may use the collection of complex numbers $(x_1, \ldots, x_n)$ modulo Moebius-transformations as coordinates for $\mathcal{M}$. The complex number x_r parameterizes a point in the flag manifold G/B "attached" to marked point z_r.

We will consider the tensor product of n principal series representations $\mathcal{P}_j$ of $SL(2,\mathbb{C})$. It corresponds to the tensor product of representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ generated by differential operators $\mathcal{J}_r^a$ acting on functions $\Psi(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)$ as

$$\mathcal{J}_r^- = \partial_{x_r}, \quad \mathcal{J}_r^0 = x_r\partial_{x_r} - j_r, \quad \mathcal{J}_r^+ = -x_r^2\partial_{x_r} + 2j_r x_r, \tag{4.7}$$

and the complex conjugate operators $\bar{\mathcal{J}}_r^a$. The Casimir of the representation $\mathcal{P}_{j_r}$ is parameterized via j_r as $j_r(j_r+1)$. The Gaudin Hamiltonians are defined as

$$\mathsf{H}_r \equiv \sum_{s\neq r} \frac{\mathcal{J}_{rs}}{z_r - z_s}, \quad \bar{\mathsf{H}}_r \equiv \sum_{s\neq r} \frac{\bar{\mathcal{J}}_{rs}}{\bar{z}_r - \bar{z}_s}, \tag{4.8}$$

where the differential operator $\mathcal{J}_{rs}$ is defined as

$$\mathcal{J}_{rs} := \eta_{aa'}\mathcal{J}_r^a\mathcal{J}_s^{a'} := \mathcal{J}_r^0\mathcal{J}_s^0 + \frac{1}{2}(\mathcal{J}_r^+\mathcal{J}_s^- + \mathcal{J}_r^-\mathcal{J}_s^+), \tag{4.9}$$

while $\bar{\mathcal{J}}_{rs}$ is the complex conjugate of $\mathcal{J}_{rs}$. The Gaudin Hamiltonians are mutually commuting,

$$[\mathsf{H}_r, \mathsf{H}_s] = 0, \quad [\mathsf{H}_r, \bar{\mathsf{H}}_s] = 0, \quad [\bar{\mathsf{H}}_r, \bar{\mathsf{H}}_s] = 0. \tag{4.10}$$

It is therefore natural to look for joint eigenfunctions of the Gaudin Hamiltonians in the space of wave-functions $\Psi(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)$ which satisfy the conditions

$$\sum_{r=1}^{n} \mathcal{J}_r^a \, \Psi = 0, \quad \sum_{r=1}^{n} \bar{\mathcal{J}}_r^a \, \Psi = 0, \tag{4.11}$$

for $a = -, 0, +$. It will be convenient for us to observe that the solutions of this problem are in one-to-one correspondence to the solutions of the slightly simplified problem which is found in the limit when $z_n \to \infty$, $x_n \to \infty$. The simplified problem can be defined directly by dropping the terms with $s = n$ from the expression (4.8) for the Hamiltonians H_r, $r = 1, \ldots, n-1$. The eigenvalue equations for the resulting Hamiltonians are supplemented by the equations $\sum_{r=1}^{n-1} \mathcal{J}_r^a \Psi = \delta_{a,0} \jmath_n$ and $\sum_{r=1}^{n-1} \bar{\mathcal{J}}_r^a \Psi = \delta_{a,0} \bar{\jmath}_n$ for $a = -, 0$. The equivalence of the two problems is seen by expressing the solutions to (4.11) in terms of functions ψ that depend only on the cross-ratios formed out of the variables $z_1, \ldots, z_n$ and $x_1, \ldots, x_n$. The same functions can be used to express the solutions of the simplified problem.

4.3.2 Eigenvalue problems?

It is not trivial to define a reasonable eigenvalue problem in the case of the Gaudin model. In order to illustrate the point, let us consider the $SL(2, \mathbb{R})$-Gaudin model, in which case the variables x_r are assumed to be real. Let us look at the simplest case $n = 4$ in some detail. In this case one may reduce the dependence on $x_1, \ldots, x_4$ to the cross-ratio x. There is only a single operator H to consider, which reduces to a second-order differential operator $\mathcal{D}_x^{(2)}$ in x of the form

$$\mathcal{D}_x^{(2)} = \frac{\mathcal{D}_{21}^{(2)}}{z} + \frac{\mathcal{D}_{32}^{(2)}}{1-z}, \tag{4.12}$$

with $\mathcal{D}_{21}^{(2)}$ and $\mathcal{D}_{32}^{(2)}$ being second-order differential operators that do not depend on z. One natural quantization problem to consider would be to assume $z \in \mathbb{R}$, and to look for a measure $d\nu(x)$ making $\mathcal{D}_x^{(2)}$ self-adjoint in $L^2(\mathbb{R}, d\nu(x))$. The problem is that the definition of self-adjoint extensions of $\mathcal{D}_x^{(2)}$ may require careful choice of boundary conditions at the singular points of $\mathcal{D}_x^{(2)}$. In this regard, let us note that $\mathcal{D}_x^{(2)}$ has regular singular points at $x = 0, z, 1, \infty$, respectively, as follows from

$$\mathcal{D}_x^{(2)} = x(x-1)(x-z)\frac{\partial^2}{\partial x^2} + \cdots, \tag{4.13}$$

up to terms with less derivatives with respect to x. Of particular interest is the singularity at $x = z$. We will return to this point later.

4.4 Quantum SOV

It is known that the quantization of the the separated variables and the quantization of the Hitchin Hamiltonians are equivalent even on the quantum level [64], as we shall now briefly recall. We will use the simplified formulation obtained by sending $z_n \to \infty$, $x_n \to \infty$ in the following, as was introduced at the end of Section 4.3.1.

The first step is to diagonalize J^- by means of the Fourier transformation

$$\tilde{\Psi}(\mu_1, \dots, \mu_{n-1}) = \frac{1}{\pi^{n-1}} \int d^2x_1 \dots \int d^2x_{n-1} \times \prod_{r=1}^{n-1} |\mu_r|^{2j_r+2} e^{\mu_r x_r - \bar{\mu}_r \bar{x}_r} \Psi(x_1, \dots, x_{n-1}). \tag{4.14}$$

The generators J_r^a are mapped to the differential operators D_r^a,

$$\mathrm{D}_r^- = \mu_r, \quad \mathrm{D}_r^0 = \mu_r \partial_{\mu_r}, \quad \mathrm{D}_r^+ = \mu_r \partial_{\mu_r}^2 - \frac{j_r(j_r+1)}{\mu_r}, \tag{4.15}$$

so that the Gaudin Hamiltonians get represented by

$$\mathsf{H}_r \equiv \sum_{s \neq r} \frac{\mathrm{D}_{rs}}{z_r - z_s}, \quad \mathrm{D}_{rs} := \eta_{aa'} \mathrm{D}_r^a \mathrm{D}_s^{a'}, \tag{4.16}$$

and their complex conjugates. Let us then define variables to $y_1, \dots, y_{n-3}, u$ related to the variables $\mu_1, \dots, \mu_{n-1}$ via

$$\sum_{i=1}^{n-1} \frac{\mu_i}{t - z_i} = u \frac{\prod_{j=1}^{n-3}(t - y_j)}{\prod_{i=1}^{n-1}(t - z_i)}. \tag{4.17}$$

Note that the constraints (4.11) imply $\sum_{r=1}^{n-1} \mu_r = 0$.

It was shown by Sklyanin [64] that the system of eigenvalue equations $\mathsf{H}_r \Psi = E_r \Psi$ is transformed by the change of variables $\mu_1, \dots, \mu_{n-1} \to y_1, \dots, y_{n-3}, u$ into the set of equations

$$(\partial_{y_k}^2 + t(y_k))\chi(y_k) = 0, \quad t(y) \equiv -\sum_{r=1}^{n-1} \left(\frac{j_r(j_r+1)}{(y_k - z_r)^2} - \frac{E_r}{y_k - z_r} \right). \tag{4.18}$$

The dependence with respect to the variables y_k has completely separated. Solutions to the Gaudin-eigenvalue equations $\mathsf{H}_r \Psi = E_r \Psi$ can therefore be constructed from solutions $\chi_k(y_k)$ of (4.18) by means of the ansatz

$$\Psi = \prod_{k=1}^{n-3} \chi_k(y_k; q). \tag{4.19}$$

Note, in particular that Baxter equations (4.18) reproduce (4.1) if $\delta_r = \mathcal{O}(\epsilon^{-2})$ so that $t(y) = \epsilon^{-2}\vartheta(y)$.

4.5 Quantization from single-valuedness

In the case of the complex quantization as discussed above, one may find strong constraints on the eigenvalues already from the condition of single-valuedness.

Sklyanin's observation allows us to write Ψ as a linear combination of solutions to the Fuchsian differential equations (4.18), which have the factorized form

$$\Psi(y_1, \bar{y}_1, \ldots, y_{n-3}, \bar{y}_{n-3}) = \prod_{a=1}^{n-3} \chi(y_a, \bar{y}_a). \tag{4.20}$$

We want to impose the condition of single-valuedness. Let us focus on the dependence of Ψ w.r.t. some $y \in \{y_1, \ldots, y_{n-3}\}$. $\chi(y, \bar{y})$ can be represented as a linear combination of the linearly independent solutions to the equation $(\partial_y^2 + t(y))\chi_i = 0$ and its complex-conjugate counterpart in the form,

$$\chi(y, \bar{y}) \equiv \chi(y, \bar{y}|z_1 \ldots z_{n-1}) = (\bar{\chi}_1(\bar{y}), \bar{\chi}_2(\bar{y})) \cdot K \cdot \begin{pmatrix} \chi_1(y) \\ \chi_2(y) \end{pmatrix}, \tag{4.21}$$

where K is a 2×2 matrix which is constrained by the condition of single-valuedness,

$$M_r^\dagger \cdot K \cdot M_r = K \quad \text{for all } r = 1, \ldots, n. \tag{4.22}$$

This is a highly overdetermined system of equations for the matrix K, which can not be satisfied for arbitrary monodromy matrices M_r.

We claim that it is necessary and sufficient that the representation of the fundamental group $\pi_1(\Sigma)$ which is generated by the matrices M_r is conjugate to a discrete subgroup of $SU(1,1) \subset SL(2,\mathbb{C})$. We may then use $K = \text{diag}(1, -1)$ to solve (4.22).

Given a single-valued solution $\chi(y, \bar{y})$ of $(\partial_y^2 + t(y))\chi = 0$ and the complex conjugate equation we may construct the metric $e^{2\varphi} dy d\bar{y}$ where $e^{-\varphi} = \chi$.

This metric has negative constant curvature since $\varphi = -\log\chi$ satisfies the Liouville equation $\partial\bar{\partial}\varphi = e^{2\varphi}$. The uniformization theorem ensures existence of such a metric, which implies existence of a solution to the problem to find a single-valued solution to $(\partial_y^2 + t(y))\chi_i = 0$ and its complex-conjugate counterpart.

We conclude that there exists a distinguished "state" $|q\,\rangle$ in the $\mathrm{SL}(2,\mathbb{C})$ Gaudin model corresponding to the metric of negative constant curvature on $C_{g,n}$.

5 The Liouville theory

Liouville theory is a field theory with conformal symmetry generated by the energy-momentum tensor with central charge c that will be parameterized in terms of a parameter b as

$$c = 1 + 6Q^2, \quad Q := b + b^{-1}. \tag{5.1}$$

It is characterized by the correlation functions of n primary fields $e^{2\alpha_r\phi(z_r,\bar{z}_r)}$ denoted as

$$\big\langle\big\langle e^{2\alpha_n\phi(z_n,\bar{z}_n)}\cdots e^{2\alpha_1\phi(z_1,\bar{z}_1)}\big\rangle\big\rangle_{C_q}. \tag{5.2}$$

C_q is a family of Riemann surfaces parameterized by a collection $q = (q_1,\dots,q_{3g-3+n})$ of complex-analytic local coordinates for the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces. The conformal dimension Δ_r of the primary field $e^{2\alpha_r\phi(z_r,\bar{z}_r)}$ is given as $\Delta_r \equiv \Delta_{\alpha_r} := \alpha_r(Q-\alpha_r)$. The correlation functions (5.2) can be represented in a holomorphically factorized form

$$\big\langle\big\langle e^{2\alpha_n\phi(z_n,\bar{z}_n)}\cdots e^{2\alpha_1\phi(z_1,\bar{z}_1)}\big\rangle\big\rangle_{C_q} = \int d\mu(p)|\mathcal{F}^{\sigma}_{\alpha,C_q}(p)|^2. \tag{5.3}$$

The conformal blocks $\mathcal{F}^{\sigma}_{\alpha,C_q}(q)$ are objects that are defined from the representation theory of the Virasoro algebra, as will be recalled in the following two subsections.

5.1 Virasoro conformal blocks

5.1.1 Definition of the conformal blocks

Let Vir_c be the Virasoro algebra with generators L_n, $n \in \mathbb{Z}$, and relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{5.4}$$

For the given set of highest weight representations $\mathcal{V}_r$, $r = 1, \dots, n$ of Vir_c with highest weights Δ_r, and Riemann surface C with n marked points at positions $z_1, \dots, z_n$ one defines the conformal blocks as linear functionals $\mathcal{F}_C : \mathcal{V}_{[n]} \equiv \otimes_{r=1}^n \mathcal{V}_r \to \mathbb{C}$ that satisfy the invariance property

$$\mathcal{F}_C(T[\chi] \cdot v) = 0 \quad \forall v \in \mathcal{R}_{[n]}, \quad \forall \chi \in \mathfrak{V}_{\rm out}, \tag{5.5}$$

where $\mathfrak{V}_{\rm out}$ is the Lie algebra of meromorphic differential operators on C with poles only at $z_1, \dots, z_n$. The action of $T[\chi]$ on $\otimes_{r=1}^n \mathcal{R}_r \to \mathbb{C}$ is defined by taking the Laurent expansions of χ at the points $z_1, \dots z_n$, w.r.t. local coordinates t_r which vanish at z_r,

$$\chi(t_r) = \sum_{k \in \mathbb{Z}} \chi_k^{(r)} \, t_r^{k+1} \, \partial_{t_r} \in \mathbb{C}((t_r))\partial_{t_r}, \tag{5.6}$$

to which we may associate the operator

$$T[\chi] = \sum_{r=1}^n \mathrm{id} \otimes \dots \otimes \underset{(\mathrm{r-th})}{L[\chi^{(r)}]} \otimes \dots \otimes \mathrm{id}, \quad L[\chi^{(r)}] := \sum_{k \in \mathbb{Z}} L_k \chi_k^{(r)} \in \mathrm{Vir}_c. \tag{5.7}$$

It can be shown that the central extension vanishes on the image of the Lie algebra $\mathfrak{V}_{\rm out}$ in $\bigoplus_{r=1}^n \mathrm{Vir}_c$, making the definition consistent. The defining invariance condition (5.5) has generically many solutions. We will denote the vector space of conformal blocks associated to the Riemann surface C with representations $\mathcal{V}_r$ associated to the marked points z_r, $r = 1, \dots, n$ by $\mathsf{CB}(\mathcal{V}_{[n]}, C)$.

Physicists may be more familiar with conformal blocks as expectation values of chiral vertex operators associated to the representations $\mathcal{V}_r$. State-operator correspondence associates a chiral vertex operator $\Phi(v_r|z_r)$ to each vector v_r in $\mathcal{V}_r$. The chiral vertex operators associated to highest weight vectors e_r in $\mathcal{V}_r$ are called primary fields, all other chiral vertex operators $\Phi(v_r|z_r)$ descendants. The functionals $\mathcal{F}_C$ represent the expectation values of a product of chiral vertex operators as

$$\mathcal{F}_C(v_n \otimes \dots \otimes v_1) = \left\langle \prod_{r=1}^n \Phi(v_r|z_r) \right\rangle_{\mathbb{G}}. \tag{5.8}$$

The subscript $\mathbb{G}$ indicates the parameters for the different ways to compose the chiral vertex operators, as will be made more explicit below. The defining invariance property (5.5) is a consequence of the Virasoro Ward identities that $\langle \prod_{r=1}^n \Phi(v_r|z_r) \rangle_{\mathbb{G}}$ is required to satisfy. We shall often use the notation on the right-hand side of (5.8), as it may be more appealing to the intuition of physicists.

5.1.2 Propagation of vacua

The vacuum representation $\mathcal{V}_0$ which corresponds to $\Delta_r = 0$ plays a distinguished role. If $\Phi_0(v_0|z_0)$ is the vertex operator associated to the vacuum representation, we have

$$\Phi_0(e_0|z_0) = \mathrm{id}, \quad \Phi_0(L_{-2}e_0|z_0) = T(z_0), \tag{5.9}$$

where $T(z)$ is the energy-momentum tensor. It can be shown that the spaces of conformal blocks with and without insertions of the vacuum representation are canonically isomorphic. The isomorphism between $\mathsf{CB}(\mathcal{V}_0 \otimes \mathcal{V}_{[n]}, C_{g,n+1})$ and $\mathsf{CB}(\mathcal{V}_{[n]}, C_{g,n})$ is simply given by evaluation at the vacuum vector $e_0 \in \mathcal{V}_0$

$$\mathcal{F}'_{C_{g,n+1}}(e_0 \otimes v_n \otimes \cdots \otimes v_1) \equiv \mathcal{F}_{C_{g,n}}(v_n \otimes \cdots \otimes v_1), \tag{5.10}$$

as is also obvious from (5.9). This fact is often referred to as the "propagation of vacua".

5.1.3 Deformations of the complex structure of X

A key point that needs to be understood about spaces of conformal blocks is the dependence on the complex structure of C. There is a canonical way to represent infinitesimal variations of the complex structure on the spaces of conformal blocks. By combining the definition of conformal blocks with the so-called "Virasoro uniformization" of the moduli space $\mathcal{M}_{g,n}$ of complex structures on $C = C_{g,n}$ one may construct a representation of infinitesimal motions on $\mathcal{M}_{g,n}$ on the space of conformal blocks.

The "Virasoro uniformization" of the moduli space $\mathcal{M}_{g,n}$ may be formulated as the statement that the tangent space $T\mathcal{M}_{g,n}$ to $\mathcal{M}_{g,n}$ at C can be identified with the double quotient

$$T\mathcal{M}_{g,n} = \Gamma(C \setminus \{x_1, \ldots, x_n\}, \Theta_C) \Big\backslash \bigoplus_{k=1}^{n} \mathbb{C}((t_k))\partial_k \Big/ \bigoplus_{k=1}^{n} \mathbb{C}[[t_k]]\partial_k, \tag{5.11}$$

where $\Gamma(C \setminus \{x_1, \ldots, x_n\}, \Theta_C)$ is the set of vector fields that are holomorphic on $C \setminus \{x_1, \ldots, x_n\}$, while $\mathbb{C}((t_k))$ and $\mathbb{C}[[t_k]]$ are formal Laurent and Taylor series respectively.

Let us then consider $\mathcal{F}_C(T[\eta] \cdot v)$ with $T[\eta]$ being defined in (5.7) in the case that η is an arbitrary element of $\bigoplus_{k=1}^n \mathbb{C}((t_k))\partial_k$ and $L_r v_k = 0$ for all $r > 0$ and $k = 1, \ldots, n$. The defining invariance property (5.5) together with $L_r v_k = 0$ allow us to define

$$\delta_\vartheta \mathcal{F}_C(v) = \mathcal{F}_C(T[\eta_\vartheta] \cdot v), \tag{5.12}$$

where δ_ϑ is the derivative corresponding to a tangent vector $\vartheta \in T\mathcal{M}_{g,n}$ and η_ϑ is any element of $\bigoplus_{k=1}^n \mathbb{C}((t_k))\partial_k$, which represents ϑ via (5.11). Generalizing these observations one is led to the conclusion that derivatives w.r.t. to the moduli parameters of $\mathcal{M}_{g,n}$ are (projectively) represented on the space of conformal blocks, the central extension coming from the central extension of the Virasoro algebra (5.4).

In the case of $g = 0$, and v_r being equal to the highest weight vector e_r of $\mathcal{V}_r$ for $r = 1, \ldots, n$, formula (5.12) is closely related to the familiar formula

$$\langle T(x)\Phi_n(z_n)\ldots\Phi_1(z_1)\rangle = \sum_{i=1}^n \left(\frac{\Delta_{\alpha_i}}{(x-z_i)^2} + \frac{1}{x-z_i}\frac{\partial}{\partial z_i} \right) \langle \Phi_n(z_n)\ldots\Phi_1(z_1)\rangle, \tag{5.13}$$

where we have abbreviated the primary fields $\Phi(e_r|z_r)$ as $\Phi_r(z_r)$.

5.1.4 Conformal blocks versus $\mathcal{D}$-modules

It may be worth noting the two possible ways to read (5.12). Having defined the action of the Virasoro algebra on $\mathcal{V}_{[n]}$, (5.12) tells us how the ring of holomorphic differential operators on $\mathcal{M}_{g,n}$ acts on the spaces of the conformal blocks. This makes the spaces of conformal blocks a (twisted) $\mathcal{D}$-module over $\mathcal{M}_{g,n}$.

On the other hand, given *any* holomorphic function $\mathcal{F}$ defined in an open subset $\mathcal{U} \subset \mathcal{M}_{g,n}$ one may use (5.12) recursively in order to construct the values of $\mathcal{F}(v)$ on arbitrary vectors $v \in \mathcal{V}_{[n]}$. The Virasoro uniformization (5.11) of $T\mathcal{M}_{g,n}$ describes the local structure of $\mathcal{M}_{g,n}$ in terms of the Lie algebra $\mathbb{C}((t))\partial_t$ of infinitesimal diffeomophisms of the circle, and (5.12) can be read as a description of the space of local holomorphic sections of a projective line-bundle over $\mathcal{M}_{g,n}$ in terms of the representation theory of the central extension of $\mathbb{C}((t_k))\partial_k$.

5.2 Gluing construction of conformal blocks

5.2.1 Gluing two boundary components

Let C be a (possibly disconnected) Riemann surface with marked points and choices of coordinates around the marked points. We can construct a new Riemann surface C' by picking two marked points z_0 and z_0' with non-intersecting annuli A and A' embedded in coordinate neighborhoods around the two points, choosing a bi-holomorphic mapping $I : A \to A'$, and by identifying the points that are mapped to each other under I, see e.g., [78] for more details.

Let us in particular consider a Riemann surface C_{21} that was obtained by gluing two surfaces C_2 and C_1 with $n_2 + 1$ and $n_1 + 1$ boundary components, respectively. Given an integer n, let sets I_1 and I_2 be such that $I_1 \cup I_2 = \{1, \ldots, n\}$. Let us consider conformal blocks $\mathcal{F}_{C_i} \in \mathsf{CB}(\mathcal{V}_i^{[n_i]}, C_i)$ where $\mathcal{V}_2^{[n_2]} = (\otimes_{r \in I_2} \mathcal{V}_r) \otimes \mathcal{V}_0$ and $\mathcal{V}_1^{[n_1]} = \mathcal{V}_0 \otimes (\otimes_{r \in I_1} \mathcal{V}_r)$ with the same representation $\mathcal{V}_0$ assigned to $z_{0,1}$ and $z_{0,2}$, respectively. Let $\langle\, .\,, .\, \rangle_{\mathcal{V}_0}$ be the invariant bilinear form on $\mathcal{V}_0$. For given $v_2 \in \otimes_{r \in I_2} \mathcal{V}_r$ let W_{v_2} be the linear form on $\mathcal{V}_0$ defined by

$$W_{v_2}(w) := \mathcal{F}_{C_2}(v_2 \otimes w), \quad \forall w \in \mathcal{V}_0, \tag{5.14}$$

and let $\mathsf{C}_1(q)$ be the family of linear operators $\mathcal{V}_1^{[n_1]} \to \mathcal{V}_0$ defined as

$$\mathsf{C}_1(q) \cdot v_1 := \sum_{e \in B(\mathcal{V}_0)} q^{L_0} e \, \mathcal{F}_{C_1}(\check{e} \otimes v_1), \tag{5.15}$$

where we have used the notation $B(\mathcal{V}_0)$ for a basis of the representation $\mathcal{V}_0$ and $\check{e}$ for the dual of an element e of $B(\mathcal{V}_0)$ defined by $\langle\, \check{e}, e' \,\rangle_{\mathcal{V}_0} = \delta_{e,e'}$. We may then consider the expression

$$\mathcal{F}_{C_{21}}(v_2 \otimes v_1) := W_{v_2}(\mathsf{C}_1(q) \cdot v_1). \tag{5.16}$$

We have thereby defined a new conformal block associated to the glued surface C_{21}, see [78] for more discussion. The insertion of the operator q^{L_0} plays the role of a regularization. It is not a priori clear that the linear form W_{v_2} is defined on infinite linear combinations such as $\mathsf{C}_1(q) \cdot v_1$. Assuming $|q| < 1$, the factor q^{L_0} will produce an suppression of the contributions with large L_0-eigenvalue, which renders the infinite series produced by the definitions (5.16) and (5.15) convergent.

5.2.2 Gluing from pairs of pants

One can produce any Riemann surface C by gluing pairs of pants. The different ways to obtain C in this way are labeled by cut systems $\mathcal{C}$, a collection of mutually non-intersecting simple closed curves on C. Using the gluing construction recursively leads to the definition of a family of conformal blocks denoted

$$\mathcal{F}^{\sigma}_{\beta,C_q}(p) \equiv \big\langle e^{2\alpha_n\phi(z_r)} \cdots e^{2\alpha_1(z_1)}\big\rangle_{C_q,\mathbb{G}} \tag{5.17}$$

depending on the following set of data:

- σ is a marking: A pants decomposition defined by a cut system $\mathcal{C}_\sigma$ together with three-valent graphs on the pairs of pants glued together to form a connected graph Γ_σ on C.
- q is an assignments $q : \gamma \mapsto q_\gamma \in \mathbb{U}$, defined for all curves $\gamma \in \mathcal{C}_\sigma$. q_γ are the gluing parameters q_γ entering the gluing construction from three-punctured spheres. They parameterize the complex structure of the family C_q of Riemann surfaces obtained in the gluing construction.
- p is an assignment $p : \gamma \mapsto p_\gamma \in \mathbb{R}$, defined for all curves $\gamma \in \mathcal{C}_\sigma$. The parameters p_γ determine the Virasoro representations $\mathcal{V}_{\Delta_\gamma}$ to be used in the gluing construction of the conformal blocks from pairs of pants via
$$\Delta_\gamma = \frac{Q^2}{4} + \frac{p_\gamma^2}{\hbar^2}. \tag{5.18}$$
- $\beta = (\beta_1, \ldots, \beta_n)$ is taken to parameterize the external representations $\mathcal{V}_1, \ldots, \mathcal{V}_n$ via
$$\alpha_r = \frac{\beta_r}{\hbar}. \tag{5.19}$$

The pair of data (σ, p) is condensed into the "gluing data" $\mathbb{G}$ in (5.17). While cut systems can be used to label boundary components in $\partial\mathcal{M}_{g,n}$, one may parameterize boundary components $\partial_\sigma\mathcal{T}_{g,n}$ of the Teichmüller space $\mathcal{T}_{g,n}$ with the help of markings σ. Using the markings allows one to properly take care of the multi-valuedness of the conformal blocks on $\mathcal{M}_{g,n}$ [78].

The conformal blocks $\mathcal{F}^{\sigma}_{\beta,C_q}(p)$ are entire analytic with respect to the variables β_r, meromorphic in the variables p_γ, $\gamma \in \mathcal{C}_\sigma$ with poles at the zeros of the Kac determinant, and the dependence on the gluing parameters q can be analytically continued over $\mathcal{T}_{g,n}$ [74, 78]. When the dependence on β is not important we will abbreviate $\mathcal{F}^{\sigma}_{q}(p) := \mathcal{F}^{\sigma}_{\beta,C_q}(p)$.

5.2.3 Change of pants decomposition

It turns out that the conformal blocks $\mathcal{F}^{\sigma_1}_{q_1}(p)$ constructed by the gluing construction in a neighborhood of the asymptotic region of $\mathcal{T}(C)$ that is determined by σ_1 have an analytic continuation $(\mathsf{A}^{\sigma_2}_{\sigma_1}\mathcal{F})^{\sigma_1}_{q_2}(p)$ to the asymptotic region of $\mathcal{T}(C)$ determined by a second marking σ_2. A fact [74,75,78][3]

[3] A full proof of the statements made here does not appear in the literature yet. It can, however, be assembled from building blocks that are published. By using the groupoid of

of foundational importance for the subject is that the analytically continued conformal blocks $(\mathsf{A}_{\sigma_1}^{\sigma_2}\mathcal{F})_{q_2}^{\sigma_1}(p)$ can be represented as a linear combination of the conformal blocks $\mathcal{F}_{q_2}^{\sigma_2}(p)$, which takes the form

$$(\mathsf{A}_{\sigma_1}^{\sigma_2}\mathcal{F})_{q_2}^{\sigma_1}(p) = \int d\mu(p')\, V_{\sigma_2\sigma_1}(p|p')\, \mathcal{F}_{q_2}^{\sigma_2}(p'). \tag{5.20}$$

The changes from one pants decomposition to another generate the modular groupoid ([50], see also [78] for non-rational cases). Having a representation of the modular groupoid via (5.20) makes the space of conformal blocks a representation of the mapping class group via

$$(\mathsf{A}_{\sigma}^{m.\sigma}\mathcal{F})_{q}^{\sigma}(p) = \int d\mu(p')\, V_{m.\sigma,\sigma}(p,p')\, \mathcal{F}_{q}^{\sigma}(p'), \tag{5.21}$$

where $m.\sigma$ is the image of the marking σ under $m \in \mathrm{MCG}(C)$.

To each marking σ one may associate a Hilbert space $\mathcal{H}_\sigma \simeq L^2((\mathbb{R}^+)^{3g-3+n}, d\mu)$ of complex valued functions $\psi_\sigma(p)$ on the space of assignments $p : \gamma \mapsto p_\gamma \in \mathbb{R}$, $\gamma \in \mathcal{C}_\sigma$ that are square-integrable w.r.t. μ. The scalar product is defined by means of the same measure μ that appears in the holomorphic factorization of the full correlation functions (5.3),

$$\|\psi\|^2 = \int d\mu(p)|\psi_\sigma(p)|^2. \tag{5.22}$$

The integral operators defined in (5.20) and (5.21) are unitary w.r.t. this scalar product, which is equivalent to crossing symmetry and modular invariance of the physical correlation functions constructed from the conformal blocks as in (5.3) [74, 78].

5.3 Degenerate fields as probes

5.3.1 Insertion of degenerate fields

An interesting way to probe the conformal blocks [2, 11] is to consider insertions of degenerate fields like

$$\langle \mathcal{O}_{n,l}\rangle_{\hat{\mathbb{G}}} \equiv \big\langle e^{2\alpha_n\phi(z_n)} \cdots e^{2\alpha_1\phi(z_1)} e^{-\frac{1}{b}\phi(y_l)} \cdots e^{-\frac{1}{b}\phi(y_1)}\big\rangle_{\hat{\mathbb{G}}}. \tag{5.23}$$

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changes of the markings it is sufficient to verify the claim for the cases $g = 0, n = 4$ and $g = 1, n = 1$, respectively. For $g = 0, n = 4$ this was done in [74], see also [76]. The case of $g = 1, n = 1$ was recently reduced to the case $g = 0, n = 4$ in [33].

The conformal blocks satisfy the null vector decoupling equations

$$\mathcal{D}_{y_k}^{\mathrm{BPZ}} \cdot \langle \mathcal{O}_{n,l} \rangle = 0, \quad \forall k = 1, \dots, l, \tag{5.24}$$

with differential operators $\mathcal{D}_{y_k}^{\mathrm{BPZ}}$ being for $g = 0$ given as

$$\begin{aligned}\mathcal{D}_{y_k}^{\mathrm{BPZ}} = b^2 \frac{\partial^2}{\partial y_k^2} + \sum_{r=1}^{n} \left(\frac{\Delta_r}{(y_k - z_r)^2} + \frac{1}{y_k - z_r} \frac{\partial}{\partial z_r} \right) \\ - \sum_{\substack{k'=1 \\ k' \neq k}}^{l} \left(\frac{3b^{-2} + 2}{4(y_k - y_{k'})^2} - \frac{1}{y_k - y_{k'}} \frac{\partial}{\partial y_{k'}} \right).\end{aligned}$$

Let us abbreviate the notation for the space of conformal blocks on $C_{g,n}$ to $\mathsf{CB}(C_{g,n})$ and let $\mathsf{CB}'(C_{g,n+l})$ be the space of conformal blocks on $C_{g,n+l}$ with l vertex operators $e^{-\frac{1}{b}\phi}$ assigned to the extra punctures $y_1, \dots, y_l$, respectively. It follows from (5.24) that the three point conformal blocks $\langle e^{-\frac{1}{b}\phi(z_3)} e^{2\alpha_2\phi(z_2)} e^{2\alpha_1\phi(z_1)} \rangle$ can only be non-zero if $\Delta_{\alpha_2} = \Delta_{\alpha_1 \mp 1/2b}$, which is symbolically expressed in the fusion rules

$$[e^{-\frac{1}{b}\phi}]\,[e^{2\alpha\phi}] \sim [e^{(2\alpha - 1/b)\phi}] + [e^{(2\alpha + 1/b)\phi}]. \tag{5.25}$$

This implies that $\mathsf{CB}'(C_{g,n+l})$ is isomorphic to $\mathsf{CB}(C_{g,n}) \otimes (\mathbb{C}^2)^{\otimes l}$ as a vector space.

5.3.2 Quantum loop operators

The key observation to be made is that for $l = 2$ there is a canonical embedding

$$\imath_{g,n} : \mathsf{CB}(C_{g,n}) \hookrightarrow \mathsf{CB}'(C_{g,n+2}), \tag{5.26}$$

coming from the fact that the fusion of the two degenerate fields $V_{-1/2b}$ contains the vacuum representation, and that insertions of the vacuum representation do no alter the space of conformal blocks (propagation of vacua). It follows from the existence of the embedding (5.26) that the mapping class group action on $\mathsf{CB}'(C_{g,n+2})$ can be projected onto $\mathsf{CB}(C_{g,n})$. The mapping class group $\mathrm{MCG}(C_{g,n+2})$ contains in particular the monodromies generated by moving the insertion point of one of the vertex operators $e^{-\frac{1}{b}\phi}$ along a closed curve γ on $C_{g,n}$. The projection of the action of these elements on $\mathsf{CB}'(C_{g,n+2})$ down to $\mathsf{CB}(C_{g,n})$ defines operators on $\mathsf{CB}(C_{g,n})$. Let us denote the operator associated to a generator γ of the fundamental group $\pi_1(C_{g,n})$ by L_γ. We will call L_γ a quantum loop operator.

The conformal blocks $\mathcal{F}_q^\sigma(p)$ defined above generate a basis for $\mathsf{CB}(C_{g,n})$. This basis is such the operators L_γ associated to the curves $\gamma \in \mathcal{C}_\sigma$ in the cut system corresponding to σ are represented diagonally,

$$\mathsf{L}_\gamma \cdot \mathcal{F}_q^\sigma(p) = 2\cosh(2\pi p_\gamma/\epsilon_1)\, \mathcal{F}_q^\sigma(p). \tag{5.27}$$

This means that the operators L_γ can be used to "measure" the intermediate representation that has been used in the construction of conformal blocks by summing over complete sets of vectors from given representations. The parameterization in terms of the data σ and p is therefore equivalent to a parameterization in terms of the eigenvalues of the quantum loop operators L_γ, $\gamma \in \mathcal{C}_\sigma$.

5.4 Parameterizing conformal blocks with degenerate fields

In order to get a parameterization for the space of solutions to (5.24), we shall consider representations for the Riemann surface $C_{g,n+l}$ which are obtained as follows. Let us call a marked point special if it will be the insertion point of a degenerate field, non-special otherwise. We may then consider representations for $C_{g,n+l}$ obtained by gluing surfaces T_ν, $\nu = 1, \ldots, 2g-2+n$, of genus zero with l_ν special marked points and exactly three non-special ones. For each surface T_ν we may then pick a pants decomposition which is such that each pair of pants contains at most one special marked point. We may therefore view the markings $\hat{\sigma}$ on $C_{g,n+l}$ that have pants decomposition of this type as certain refinements of a marking σ on the surface $C_{g,n}$ obtained from $C_{g,n+l}$ by "forgetting" the insertion points of the degenerate fields. We will in the following restrict attention to markings of this type.

Conformal blocks can then be defined by the gluing construction. This defines solutions to (5.24) denoted as

$$\mathcal{F}_{q,y}^{\hat{\sigma}}(p,\delta) := \left\langle e^{2\alpha_n \phi(z_n)} \cdots e^{2\alpha_1 \phi(z_1)} e^{-\frac{1}{b}\phi(y_l)} \cdots e^{-\frac{1}{b}\phi(y_1)} \right\rangle_{C_{q,y},\hat{\mathbb{G}}}. \tag{5.28}$$

These conformal blocks are parameterized by the data p and q associated to the underlying marking σ on $C_{g,n}$ in the same way as explained in Section 5.2.2, together with the following additional data

- δ is a map which assigns a sign δ_k to each of the special marked points y_k, which determines the change of representation label according to the fusion rules (5.25). Noting that p determines the choice of representations associated to the non-special marked points of T_ν it is easy to see that this allows one to determine all representations involved in the gluing construction unambigously.

- y is the collection of gluing parameters involved in the gluing construction of T_ν from three-punctured spheres.

In the notation on the left-hand side of (5.28), we have displayed the gluing data $\hat{\mathbb{G}} = (\hat{\sigma}, p, \delta)$ more explicitly.

The conformal blocks (5.28) form a *complete* set of solutions to equations (5.24) in the sense that the solutions associated to a given marking $\hat{\sigma}_1$ can be analytically continued to the boundary component $\partial_{\hat{\sigma}_2}\mathcal{T}_{g,n+l}$ of the Teichmüller space $\mathcal{T}_{g,n+l}$ which is associated to any other marking $\hat{\sigma}_2$, and that the analytically continued solutions associated to $\hat{\sigma}_1$ can be represented as a linear combination of the solutions representable as power series in gluing parameters in a neighborhood of $\partial_{\hat{\sigma}_2}\mathcal{T}_{g,n+l}$.

5.5 Quantum Hitchin system from the semiclassical limit of Liouville theory

5.5.1 Eigenfunctions of Hitchin's Hamiltonians from classical conformal blocks

Let us now consider the limit $\epsilon_2 \to 0$ of the conformal blocks (5.23), keeping ϵ_1 finite in the case $g = 0$. This means that $\hbar \to 0$ while $b \to \infty$. The sum over k' in the expression for $\mathcal{D}_y^{\rm BPZ}$ becomes subleading in this limit. To leading order we can factorize the solutions $\langle\, \mathcal{O}_{n,l}\,\rangle_{\hat{\mathbb{G}}}$ to (5.24) in the form

$$\big\langle \mathcal{O}_{n,l} \big\rangle_{\hat{\mathbb{G}}} = \exp(-b^2\mathcal{W}(q)) \prod_{k=1}^{l} \chi_k(y_k; q), \tag{5.29}$$

where $\chi_k(y) \equiv \chi_k(y; q)$ are solutions to equation

$$(\partial_y^2 + t(y))\chi_k(y) = 0, \quad t(y) = \sum_{r=1}^{n}\left(\frac{\delta_r}{(y-z_r)^2} + \frac{H_r}{y-z_r}\right), \tag{5.30}$$

with $\delta_r = \lim_{b\to\infty} b^{-2}\Delta_r$, and

$$H_r = -\frac{\partial}{\partial z_r}\mathcal{W}(q). \tag{5.31}$$

In (5.29) and (5.31), we are using the notation q for the collection of variables $(z_1, \ldots, z_n)$, which determine the complex structure of the underlying Riemann surface $C_{0,n} = \mathbb{P}^1 \setminus \{z_1, \ldots, z_n\}$.

On the right-hand side of (5.29), we may for $l = n - 3$ and $z_n \to \infty$ recognize an eigenfunction of the Gaudin Hamiltonians as explained in Section 4.4. It does not satisfy the quantization conditions as discussed in Section 4.5, in general.

Considering the limit $b \to \infty$ of the basis elements $\mathcal{F}_q^\sigma(p)$ we are led to the conclusion that the following limit exists:

$$\mathcal{W}_q^\sigma(l) \equiv -\lim_{b\to\infty} b^{-2} \log \mathcal{F}_q^\sigma(p) \tag{5.32}$$

where the parameters $l = (l_1, \ldots, l_{n-3})$ and $p = (p_1, \ldots, p_{n-3})$ are related via

$$l_r = \frac{p_r}{4\pi\epsilon_2}. \tag{5.33}$$

and that the monodromy group of the oper $(\partial_y^2 + t(y))\chi(y) = 0$,

$$t(y) = \sum_{r=1}^{n} \left(\frac{\delta_r}{(y - z_r)^2} + \frac{H_r(l,q)}{y - z_r} \right), \quad H_r(l,q) = -\frac{\partial}{\partial z_r} \mathcal{W}_q^\sigma(l) \tag{5.34}$$

satisfies (3.19). Let us note in particular that the parameterization of the conformal blocks in terms of eigenvalues of quantum loop operators introduced in Section 5.3 turns into the parameterization of the opers in terms of the traces of their monodromies introduced in (3.19).

5.5.2 Semiclassical limit of the full correlation functions

Let us now consider the classical limit $b \to \infty$ of full correlation functions (5.2). We may assume that the measure $d\mu(p)$, which appears in the holomorphically factorized representation (5.3) is just the usual Lebesque measure, $d\mu(p) = \prod_{r=1}^{n-3} dp_r$. This is related to the more conventional representation in which $d\mu(p)$ is constructed from the product of three-point functions by a change of normalization for the conformal blocks, see e.g., [1] for explicit formulae. The leading behavior of the integrand in (5.3) is $e^{-2b^2 \mathrm{Re}(\mathcal{W}_q^\sigma(l))}$, as follows from (5.32). The integral in the holomorphically factorized representation (5.3) of the full correlation functions will therefore be dominated by a saddle point $p_{\mathrm{s}} = (p_{1,\mathrm{s}} \ldots p_{n-3,\mathrm{s}})$,

$$\langle\, \mathcal{O} \,\rangle \sim e^{-b^2 S_{\mathrm{L}}(q)}, \quad S_{\mathrm{L}}(q) = 2\mathrm{Re}(\mathcal{W}_q^\sigma(l_s)), \tag{5.35}$$

with l and p related via (5.33), and the value $l_s = l_s(q, \bar{q})$ at the saddle point is determined by

$$\frac{\partial}{\partial l_r} \mathrm{Re}(\mathcal{W}_q^\sigma(l)) \bigg|_{l=l_{\mathrm{s}}} = 0. \tag{5.36}$$

More explicit analysis of the case $n = 4$ can be found in [80].

5.5.3 Single-valued Gaudin eigenfunctions from Liouville correlation functions

Let us now consider the semiclassical limit of full correlation functions containing $n-3$ insertions of degenerate fields $e^{-b\phi(y_k,\bar{y}_k)}$. By the same arguments as used before we find that

$$\left\langle\left\langle \prod_{r=1}^{n} e^{2\alpha_n\phi(z_r,\bar{z}_r)} \prod_{k=1}^{n-3} e^{-b\phi(y_k,\bar{y}_k)} \right\rangle\right\rangle \sim e^{-b^2 S_{\mathrm{L}}(q)} \prod_{k=1}^{n-3} \chi_r(y_k,\bar{y}_k), \tag{5.37}$$

where $S_{\mathrm{L}}(q)$ was introduced in (5.35). On the right-hand side of (5.37), we recognize [59] the solutions (4.20) to the eigenvalue equations for the Gaudin model in the SOV representation. They are automatically single-valued both with respect to the variables y_k and q as the correlation function on the left-hand side of (5.37) has this property. We see that the distinguished state $|q\rangle$ of the Gaudin model introduced in Section 4.5 is reproduced in the semiclassical limit of a Liouville correlation function.

5.5.4 Yang's potential from classical conformal blocks?

Recall that the space of all differential operators of the form $\partial_y^2 + t(y)$ parameterizes via the quantum SOV the commutative algebra of differential operators on Bun_G generated by $\mathsf{H}_r - E_r$. This space can be viewed as a "deformation" $\mathcal{B}_\epsilon$ of the base $\mathcal{B}$ of the Hitchin fibration. Within $\mathcal{B}_\epsilon$ we want to identify isolated points representing the quantized eigenvalues with the help of a function $\mathcal{W}$ on $\mathcal{B}_\epsilon$ called Yang's potential. We are now going to point out that our discussion of the relation between the semiclassical limit of the Liouville correlation functions and the complex quantization of the Hitchin system above suggests that the classical Liouville conformal blocks are natural candidates for the Yang's potentials associated to the complex quantization of the Hitchin system as discussed in Section 4.

For the case $g=0$, $C_{0,n} = \mathbb{P}^1 \setminus \{z_1,\dots,z_n\}$ under consideration, let $l = (l_1,\dots,l_{n-3})$ be the coordinates for the space of opers introduced in (3.19) above. Our discussion of the semiclassical limit of the complex quantization of the Hitchin system in 4.2 implies that to leading order in ϵ we may identify the Yang's function $\mathcal{W}(l)$ with the prepotential $\mathcal{F}(a)$, where l and a are related via (3.44). The standard relation $a_s^D = \partial_{a_s}\mathcal{F}(a)$ then allows us to reformulate the Bohr–Sommerfeld quantization conditions (4.6) to leading order in ϵ in terms of $\mathcal{W}(l)$. This suggests that the exact quantization conditions could likewise be formulated in terms of a potential $\mathcal{W}(l)$, and that they should include the conditions

$$\frac{\partial}{\partial l_r}\mathrm{Re}(\mathcal{W}(l))\bigg|_{l=l(\mathbf{k})} = \pi\epsilon k_r, \quad \text{for } r=1,\dots,n-3, \tag{5.38}$$

where $\mathbf{k}=(k_1,\dots,k_{n-3})$ is a given vector of integers.

In our above discussion, we had observed that the quantization condition in the case of the distinguished state $|q\,\rangle$ can be formulated as the saddle-point condition (5.36). This invites us to identify

$$\mathcal{W}(l) \equiv \mathcal{W}_q^\sigma(p), \tag{5.39}$$

with l and p related by (5.33). The saddle-point condition (5.36) would then correspond to the special case $\mathbf{k}=(0,\dots,0)$ of (5.38). It remains to be seen if other single-valued eigenstates of the Gaudin–Hamiltonians can be characterized in terms of the conditions (5.38).

5.5.5 Characterization of Yang's potential in terms of opers

At the end of Section 3.5, we had defined $H_r(l,q)$ as the accessory parameters which give the oper a monodromy characterized by the parameters l. It follows from (5.39) and (5.34) that $\mathcal{W}(l)\equiv\mathcal{W}(l,q)$ satisfies the equations

$$H_r(l,q) = -\frac{\partial}{\partial z_r}\mathcal{W}(l,q). \tag{5.40}$$

Equations (5.40) define $\mathcal{W}(l,q)$ up to addition of q-independent functions of l.[4]

The formulation of the quantization conditions in terms of the Yang's potential via (5.38) will only work for a suitable choice of the l-dependence in $\mathcal{W}(l,q)$. Such a choice is implied in the identification (5.39) with the classical conformal blocks. The freedom to add q-independent functions of the variables l is via (5.39) related to the freedom to multiply the conformal blocks $\mathcal{F}_q^\sigma(p)$ by functions of the parameters p. The latter freedom is fixed if one requires, as has been done above, that the *single-valued* correlation functions (5.2) are constructed from the conformal blocks by an expression of the form (5.3) with measure $d\mu(p)$ being the standard Lebesque measure. This amounts to absorbing the three-point functions into the conformal blocks. We see that the correct choice of the q-independent functions of the variables l in the definition of $\mathcal{W}(l,q)$ is ultimately determined by the single-valuedness

[4]This corrects an inaccurate statement in a previous version of this paper that has been pointed out by S. Shatashvili.

of the Liouville correlation functions (5.2) which determines the measure $d\mu(p)$, as discussed e.g., in [78]. Explicit formulae can easily be found with the help of [80]. This single-valuedness is directly related to the single-valuedness of the eigenfunctions of the Gaudin–Hamiltonians via (5.37).

The two different formulations of the quantization conditions — from single-valuedness of the wave-functions on the one hand, and in terms of $\mathcal{W}(l, q)$ on the other hand — are unified in the condition of single-valuedness of the Liouville correlation functions appearing on the left-hand side of (5.37) above. These relations fit into a Langlands-duality scheme similar to our diagram (1.9) above, in which the single-valued Gaudin eigenvectors would appear in the upper right box, and the points on $\mathcal{B}_\epsilon$ determined from $\mathcal{W}$ should be placed into the upper left box.

5.5.6 Quantization conditions in real quantization?

In Section 4.2, we had also considered the quantization of a real slice in the phase space in the semiclassical limit. It is suggestive to observe that both in the real and complex quantization schemes discussed in Section 4.2 it is the *same* function (the prepotential), which appears in the formulation of the leading semiclassical quantization conditions. This suggests that the quantization conditions in real quantization can be formulated as the equations

$$\frac{\partial}{\partial l_r} \mathcal{W}(l, q)\bigg|_{l=l(\mathbf{k})} = 2\pi\epsilon k_r, \quad \text{for } r = 1, \ldots, n-3, \tag{5.41}$$

where $\mathbf{k} = (k_1, \ldots, k_{n-3})$. The critical point(s) of $\mathcal{W}(l(\mathbf{k}), q)$ give the eigenvalues E_r of the Hitchin Hamiltonians via

$$E_r = H_r(l(\mathbf{k}), q). \tag{5.42}$$

As partially discussed in Section 4 we will need further investigations to properly define the eigenvalue problem in the real quantization and to check if it can be reformulated in the form (5.41).

5.5.7 Further remarks

The identification of Yang's potential with the semiclassical limit of conformal blocks can also be arrived at by combining the discussion of [53] with the observations of [1]. It is proposed in [53] that the Yang's potential is obtained from Nekrasov's partition function $\mathcal{Z}(a, \epsilon_1, \epsilon_2; q)$ in the limit $\epsilon_2 \to 0$. One of the main observations made in [1] is the coincidence of the Nekrasov partition functions for the theories of interest with Liouville conformal blocks.

This holds in particular in the case of the $N = 2^*$-theory discussed in [53] for which the Nekrasov partition function coincides according to [1,15] with the Liouville conformal blocks on the one-punctured torus.

The observations discussed above appear to be deeply related to the recent work of Nekrasov and Shatashvili [54].

5.6 Degenerate fields as heavy sources

We shall now consider more general Liouville conformal blocks of the form

$$\langle \mathcal{O}_{n,m,l} \rangle \equiv \left\langle \prod_{s=1}^{n} e^{2\alpha_s \phi(z_s)} \prod_{r=1}^{m} e^{-b\phi(w_r)} \prod_{k=1}^{l} e^{-\frac{1}{b}\phi(y_k)} \right\rangle_{\hat{\mathbb{G}}} . \tag{5.43}$$

The conformal blocks (5.28) satisfy the null vector decoupling equations

$$\mathcal{D}^{\rm BPZ}_{y_q} \cdot \langle \mathcal{O}_{n,m,l} \rangle = 0, \quad \tilde{\mathcal{D}}^{\rm BPZ}_{w_r} \cdot \langle \mathcal{O}_{n,m,l} \rangle = 0, \tag{5.44}$$

where for $g = 0$

$$\begin{aligned}
\mathcal{D}^{\rm BPZ}_{y_k} &= b^2 \frac{\partial^2}{\partial y^2} + \sum_{s=1}^{n} \left(\frac{\Delta_s}{(y_k - z_s)^2} + \frac{1}{y_k - z_s} \frac{\partial}{\partial z_s} \right) \\
&\quad - \sum_{r=1}^{m} \left(\frac{3b^2 + 2}{4(y_k - w_r)^2} - \frac{1}{y_k - w_r} \frac{\partial}{\partial w_r} \right) \\
&\quad - \sum_{\substack{k'=1 \\ k' \neq k}}^{l} \left(\frac{3b^{-2} + 2}{4(y_k - y_{k'})^2} - \frac{1}{y_k - y_{k'}} \frac{\partial}{\partial y_{k'}} \right),
\end{aligned} \tag{5.45}$$

$$\begin{aligned}
\tilde{\mathcal{D}}^{\rm BPZ}_{w_r} &= \frac{1}{b^2} \frac{\partial^2}{\partial w_r^2} + \sum_{s=1}^{n} \left(\frac{\Delta_r}{(w_r - z_s)^2} + \frac{1}{w_r - z_s} \frac{\partial}{\partial z_s} \right) \\
&\quad - \sum_{k=1}^{l} \left(\frac{3b^{-2} + 2}{4(w_r - y_k)^2} - \frac{1}{w_r - y_k} \frac{\partial}{\partial y_k} \right) \\
&\quad - \sum_{\substack{r'=1 \\ r' \neq r}}^{m} \left(\frac{3b^2 + 2}{4(w_r - w_{r'})^2} - \frac{1}{w_r - w_{r'}} \frac{\partial}{\partial w_r} \right).
\end{aligned} \tag{5.46}$$

Equations (5.44) imply the fusion rules

$$[V_{-b/2}] \cdot [V_\alpha] = [V_{\alpha - b/2}] + [V_{\alpha - b/2}]. \tag{5.47}$$

Bases for the space of conformal blocks of the type (5.43) can be parameterized in a similar way as described in Section 5.4.

As above in Section 5.3 we may now consider the insertions of the degenerate fields $e^{-\frac{1}{b}\phi(y_q)}$ as probes. The key observation to be made is that the monodromy of $e^{-\frac{1}{b}\phi(y)}$ around any of the degenerate fields $e^{-b\phi(w_k)}$ is minus the identity matrix, therefore projectively trivial. This can easily be verified with the help of the well-known expressions for the fusion and braiding matrices of the degenerate field $e^{-\frac{1}{b}\phi(y)}$ as recollected e.g., in [11, Appendix B]. The procedure explained in Section 5.3 can therefore be used to construct an operator L_γ acting on the space of conformal blocks (5.43) for each generator γ of the fundamental group $\pi_1(C_{g,n})$. This operator is insensitive to the insertions of $e^{-b\phi(w_k)}$, and "measures" via a formula analogous to (5.27) the intermediate dimensions p used in the gluing construction of the conformal blocks only.

5.7 Isomonodromic deformations from the semiclassical limit of Liouville theory

Let us now consider the limit $\epsilon_2 \to 0$ keeping ϵ_1 fixed, which corresponds to $\hbar \to 0$ and $b \to \infty$. Analyzing the differential equations satisfied by $\langle\, \mathcal{O}_{n,m,l}\, \rangle$ in this limit we find that

(i) the following limits exist

$$\mathcal{W}^{\hat{\sigma}}_{q,w}(p,\delta) \equiv \lim_{b\to\infty} b^{-2}\log\langle\, \mathcal{O}_{n,m}\, \rangle_{\hat{\mathbb{G}}}, \tag{5.48}$$

$$\Psi(y) \equiv \lim_{b\to\infty} [\langle\, \mathcal{O}_{n,m}\, \rangle_{\hat{\sigma}}]^{-1}\langle\, \mathcal{O}_{n,m,l}\, \rangle_{\hat{\mathbb{G}}}, \tag{5.49}$$

(ii) $\Psi(y)$ factorizes as

$$\Psi(y) = \prod_{k=1}^{l} \chi_k(y_k), \tag{5.50}$$

where $\chi_k(y_k)$ satisfy an equation of the form $(\partial_y^2 + t(y))\chi_k(y) = 0$ with

$$t(y) = \sum_{s=1}^{n}\left(\frac{\delta_s}{(y-z_s)^2} + \frac{H_s}{y-z_s}\right) - \sum_{r=1}^{m}\left(\frac{3}{4(y-w_r)^2} - \frac{\kappa_r}{y-w_r}\right), \tag{5.51}$$

(iii) the residues H_s and κ_k are constrained by the relations (3.24).

(iv) the residues $H_s = H_s(p, \delta|q, w)$ and $\kappa_r = \kappa_r(p, \delta|q, w)$ of $t(y)$ introduced in (5.51) are related to $\mathcal{W}^{\hat{\sigma}}_{\beta}(p, \delta|q, w)$ as

$$E_s = -\frac{\partial}{\partial z_s}\mathcal{W}^{\hat{\sigma}}_{q,w}(p, \delta), \quad \kappa_r = -\frac{\partial}{\partial w_r}\mathcal{W}^{\hat{\sigma}}_{q,w}(p, \delta). \tag{5.52}$$

In the case $m = n - 3$ we may note that equations (3.24) coincide with equations (3.16) and that (5.52) are the the relations defining the isomonodromic tau-function.

6 Liouville theory as a quantum theory of the space of local systems

6.1 Overview

The results of the previous sections have demonstrated that Liouville theory has many relations to the moduli spaces of local systems — it deforms key geometrical structures of these moduli spaces. We now want to show that the main features of Liouville theory can be understood in terms of the *quantization* of real slices in $\mathcal{M}_{\rm H}$.

It is very important that the structure of $\mathcal{M}_{\rm H} \simeq \mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ as a complex algebraic variety has a natural deformation that is realized within the quantization of its real slices. The ring $\mathcal{O}$ of regular functions on $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ is generated from the traces of holonomies $\mathrm{tr}(\rho(\gamma))$. It is natural that the algebra $\mathcal{O}_b$ of quantized observables should be generated from the quantum operators H_γ associated to the classical observables $\mathrm{tr}(\rho(\gamma))$. A natural integrable structure is obtained by choosing a maximal set of non-intersecting closed curves γ_r, $r = 1, \dots, 3g - 3 + n$. The corresponding observables $\mathsf{L}_r \equiv \mathsf{L}_{\gamma_r}$ commute, $[\mathsf{L}_r, \mathsf{L}_s] = 0$ for all $r, s = 1, \dots, 3g - 3 + n$, so that the subalgebra $\mathcal{I} \subset \mathcal{O}_b$ generated by the L_r represents the integrable structure of the quantum theory of $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$.

However, non-compactness of the moduli spaces $\mathrm{Loc}_{\mathrm{PSL}(2,\mathbb{C})}(C)$ implies that the elements of the algebra $\mathcal{O}_b$ can not be realized by bounded operators on a Hilbert space $\mathcal{H}$. It is therefore important to consider the maximal common domain of definition for the elements of $\mathcal{O}_b$ within $\mathcal{H} = \mathcal{H}(C_{g,n})$. This defines a natural analog $\mathcal{S}_{\mathcal{O}_b}$ of the Schwartz-space of smooth, rapidly decreasing functions on the real line. The common eigenstates of the Hamiltonians L_r are elements of the Hermitian dual $\mathcal{S}^{\dagger}_{\mathcal{O}_b}$ of $\mathcal{S}_{\mathcal{O}_b}$. Let us denote by $\langle\, p|$ the element of $\mathcal{S}^{\dagger}_{\mathcal{O}_b}$ which satisfies

$$\langle\, p|\mathsf{L}_r = 2\cosh(2\pi b p_r/\epsilon_1)\langle\, p|, \quad \forall r = 1, \dots, 3g - 3 + n. \tag{6.1}$$

The spectrum of the operators L_r is exhausted by considering $p_r \in \mathbb{R}^+$. It will be important for us to note that the eigenstates $\langle p |$ can be meromorphically continued to arbitrary *complex* values of p, in the sense that $\psi(p) = \langle p | \psi \rangle$ can be meromorphically continued w.r.t. p for all $\psi \in \mathcal{S}_{\mathcal{O}_b}$. The wave-functions $\psi(p)$ give a concrete representation for the elements of $\mathcal{S}_{\mathcal{O}_b}$.

The action of $\mathcal{O}_b$ on the space $\mathcal{S}_{\mathcal{O}_b}$ can be represented as the action of a ring of finite difference operators on the wave-functions $\psi(p)$. This furnishes a concrete realization of the quantization of the ring of regular functions on $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ as a non-commutative ring of difference operators acting on $\mathcal{S}_{\mathcal{O}_b}$.

The conformal blocks of Liouville theory are found to be wave-functions of certain states $|C\rangle$ associated to the Riemann surface C. The conformal blocks $\mathcal{F}_q^{\sigma}(p)$, for example, are nothing but the wave-functions $\langle p | C_q \rangle$ of states $|C_q\rangle$ associated to a family of surfaces C_q with complex structure parameterized by $q = (q_1, \ldots, q_{3g-3+n})$ in the representation introduced above. Using this dictionary it is possible to see that the Liouville loop operators L_γ introduced in Section 5.3 are mapped precisely to the difference operators which represent the Hamiltonians L_γ on the wave-functions $\psi(p)$. Parameterizing conformal blocks in terms of the eigenvalues of the Liouville loop operators corresponds to labeling the eigenstates $\langle p |$ by their eigenvalues, (6.1).

The Liouville correlation functions (5.2) represent the norm squared of $|C_q\rangle$,

$$\langle C_q | C_q \rangle = \big\langle\big\langle\, e^{2\alpha_n \phi(z_n, \bar{z}_n)} \cdots e^{2\alpha_1 \phi(z_1, \bar{z}_1)} \,\big\rangle\big\rangle_{C_q}, \tag{6.2}$$

and the holomorphic factorization (5.3) is the representation of the scalar product on $\mathcal{H}$ in the representation where the operators L_r, $r = 1, \ldots, 3g - 3 + n$ are diagonal.

6.2 Fock–Goncharov coordinates

Let τ be a triangulation of the surface C such that all vertices coincide with marked points on C. An edge e of τ separates two triangles defining a quadrilateral Q_e with corners being the marked points $P_1, \ldots, P_4$. For a given local system $(\mathcal{E}, \nabla')$, let us choose four sections s_i, $i = 1, 2, 3, 4$ that are holomorphic in Q_e, obey the condition

$$\nabla' s_i = \left(\frac{\partial}{\partial y} + M(y) \right) s_i = 0, \tag{6.3}$$

and are eigenvectors of the monodromy around P_i. Out of the sections s_i form [18, 24, 30]

$$\mathcal{X}_e^\tau := \frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}, \tag{6.4}$$

where all sections are evaluated at a common point $P \in Q_e$. It is not hard to see that $\mathcal{X}_e^\tau$ does not depend on the choice of P.

The Poisson structure is particularly simple in terms of these coordinates,

$$\{\mathcal{X}_e^\tau, \mathcal{X}_{e'}^\tau\} = \langle\, e, e' \,\rangle\, \mathcal{X}_{e'}^\tau\, \mathcal{X}_e^\tau, \tag{6.5}$$

where $\langle\, e, e' \,\rangle$ is the number of faces e and e' have in common, counted with a sign.

A real slice $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))$ in $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ can be defined by the conditions $\mathcal{X}_e^* = \mathcal{X}_e$. Recall that the real slice $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))$ decomposes into different connected components, and that one of these components is canonically isomorphic to the Teichmüller space of deformations of C. This component is characterized by the property that the functions $\mathcal{X}_e$ are all positive.

6.3 Holonomy variables

Assume given a path ϖ_γ on the fat graph homotopic to a simple closed curve γ on $C_{g,n}$. Let the edges be labelled e_i, $i = 1, \dots, r$ according to the order in which they appear on ϖ_γ, and define σ_i to be 1 if the path turns left at the vertex that connects edges e_i and e_{i+1}, and to be equal to -1 otherwise. Consider the following matrix,

$$\mathrm{X}_\gamma = \mathrm{V}^{\sigma_r}\mathrm{E}(z_{e_r}) \cdots \mathrm{V}^{\sigma_1}\mathrm{E}(z_{e_1}), \tag{6.6}$$

where $z_e = \log X_e$, and the matrices $\mathrm{E}(z)$ and V are defined respectively by

$$\mathrm{E}(z) = \begin{pmatrix} 0 & +e^{+\frac{z}{2}} \\ -e^{-\frac{z}{2}} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.7}$$

Taking the trace of X_γ one gets the hyperbolic length of the closed geodesic isotopic to γ via [21]

$$L_\gamma \equiv 2\cosh\left(\frac{1}{2} l_\gamma\right) = |\mathrm{tr}(\mathrm{X}_\gamma)|. \tag{6.8}$$

We may observe that the classical expression for $L_\gamma \equiv 2\cosh\frac{1}{2}l_\gamma$ as given by formula 6.8 is a linear combination of monomials in the variables $u_e^{\pm 1} \equiv e^{\pm\frac{z_e}{2}}$ of a very particular form,

$$L_\gamma = \sum_{\nu\in\mathcal{F}} C_{\tau,\gamma}(\nu)\prod_e u_e^{\nu_e} \tag{6.9}$$

where the summation is taken over a finite set $\mathcal{F}$ of vectors $\nu \in \mathbb{Z}^{3g-3+2n}$ with components ν_e. The coefficients $C_{\tau,\gamma}(\nu)$ are positive integers.

It is proven in [19, Theorem 12.3] that the products of traces of monodromies of finite laminations form a basis for the vector spaces of regular functions on $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$. These objects are therefore natural generators for the algebra $\mathcal{O}$ of observables of interest.

6.4 Quantization of the Teichmüller component

The simplicity of the Poisson brackets (6.5) makes part of the quantization quite simple. To each edge e of a triangulation of a Riemann surface $C_{g,n}$ associate a quantum operator z_e corresponding to the classical phase space function $z_e = \log\mathcal{X}_e$. Canonical quantization of the Poisson brackets (6.5) yields an algebra $\mathcal{A}_\tau$ with generators z_e and relations

$$[\mathsf{z}_e, \mathsf{z}_{e'}] = 2\pi i b^2\langle\, e, e'\,\rangle. \tag{6.10}$$

The algebra $\mathcal{A}_\tau$ has a center with generators c_a, $a = 1,\dots,n$ defined by $\mathsf{c}_a = \sum_{e\in E_a}\mathsf{z}_e$, where E_a is the set of edges in the triangulation that emanates from the a^{th} boundary component. The representations of $\mathcal{A}_\tau$ that we are going to consider will therefore be such that the generators c_a are represented as the operators of multiplication by real positive numbers $l_a/2$. Geometrically one may interpret l_a as the geodesic length of the a^{th} boundary component [21]. The vector $l = (l_1,\dots,l_n)$ of lengths of the boundary components will figure as a label of the representation of the algebra $\mathcal{A}_\tau$.

Recall furthermore that the variables $\mathcal{X}_e$ are positive for the Teichmüller component. The scalar product of the quantum theory should realize the phase space functions $z_e = \log\mathcal{X}_e$ as self-adjoint operators z_e, $\mathsf{z}_e^\dagger = \mathsf{z}_e$. By choosing a maximal set of commuting generators for the algebra A_τ one may naturally define a Schrödinger type representation of the algebra $\mathcal{A}_\tau$ in terms of multiplication and differentiation operators. It is realized on the Hilbert space $\mathcal{H}_\tau \simeq L^2(\mathbb{R}^{3g-3+n})$.

Less trivial is the fact that one can define on $\mathcal{H}_\tau$ a projective unitary representation of the mapping class group $\mathrm{MCG}(C_{g,n})$. It is generated by unitary operators $\mathsf{W}_\tau(m) : \mathcal{H}_\tau \to \mathcal{H}_\tau,\ m \in \mathrm{MCG}(C_{g,n})$ constructed in [7, 41, 77].

The resulting quantum theory does not depend on the underlying triangulation in an essential way. This follows from the existence of a family of unitary operators $\mathsf{U}_{\tau_2,\tau_1}$ that satisfy

$$\mathsf{U}_{\tau_2\tau_1}^{-1} \cdot \mathsf{W}_{\tau_1}(m) \cdot \mathsf{U}_{\tau_2\tau_1}^{-1} = \mathsf{W}_{\tau_2}(m). \tag{6.11}$$

The operators $\mathsf{U}_{\tau_2,\tau_1}$ describe the change of representation when passing from the quantum theory associated to triangulation τ_1 to the one associated to τ_2 [7, 20, 41, 77]. They allow us to identify $\mathcal{H}_{\tau_2} \simeq \mathcal{H}_{\tau_1} =: \mathcal{H}(C_{g,n})$.

6.5 Quantizing regular functions on $\mathbf{Hom}(\pi_1(C), \mathbf{PSL}(2, \mathbb{C}))$

6.5.1 Quantizing traces of holonomies

In order to define a set of generators for the quantized algebra $\mathcal{O}_b$ of observables one needs to define the length operators $\mathsf{L}_{\tau,\gamma}$ associated to simple closed curves γ. The operators $\mathsf{L}_{\tau,\gamma}$ should be representable as Laurent polynomials in the variables $\mathsf{u}_e = e^{\pm\frac{1}{2}\mathsf{z}_e}$ with positive coefficients that reproduce the expressions (6.9) in the classical limit. It is important to ensure that the definition of the operators $\mathsf{L}_{\tau,\gamma}$ is independent of the triangulation in the sense that

$$\mathsf{U}_{\tau_2\tau_1}^{-1} \cdot \mathsf{L}_{\tau_1,\gamma} \cdot \mathsf{U}_{\tau_2\tau_1}^{-1} = \mathsf{L}_{\tau_2,\gamma},$$

where $\mathsf{U}_{\tau_2\tau_1}$ is the unitary operator relating the representation associated to triangulation τ_1 to the one associated to τ_2. This ensures that the collection of length operators $\mathsf{L}_{\tau,\gamma}$ associated to the different triangulations τ ultimately defines an operator L_γ that is *independent* of the triangulation. A general construction of length operators which fulfills this requirement was given in [77]. This construction coincides with the earlier constructions in [7, 8] whenever both can be applied.

6.5.2 The length representation

It can be shown that the length operators associated to non-intersecting simple closed curves commute with each other. This together with the self-adjointness of the length operators allows one to introduce bases of eigenfunctions for the length operators.

One gets one such basis for each marking σ of $C_{g,n}$. A key result for the connection between quantum Liouville and quantum Teichmüller theory is that for each marking σ there exists a basis for $\mathcal{H}_{g,n} \equiv \mathcal{H}(C_{g,n})$ spanned by ${}_\sigma\langle l|$, $l = (l_1, \dots, l_{3g-3+n})$ which obeys the factorization rules of conformal field theory [77]. This means in particular that for any pair σ_2, σ_1 of markings one can always decompose the unitary transformation $\mathsf{V}_{\sigma_2\sigma_1}$ which relates the representation corresponding to marking σ_1 to the one corresponding to σ_2 as a product of operators which represent the elementary fusion, braiding and modular transformation moves introduced in [50]. The unitary transformation $\mathsf{V}_{\sigma_2\sigma_1}$ can be represented as an integral operator of the form

$$\psi_{\sigma_2}(l_2) = \int d\mu(l_1)\; V_{\sigma_2\sigma_1}(l_2, l_1)\, \psi_{\sigma_1}(l_1). \tag{6.12}$$

The explicit expressions for the kernel $V_{\sigma_2\sigma_1}(l_2, l_1)$ are known for the cases where σ_2 and σ_1 differ by one of the elementary moves.

With the help of (6.12) we may describe the unitary operators representing the action of the mapping class group as integral operators of the form

$$\psi_{\sigma}(l_2) = \int d\mu(l_1)\; V_{m.\sigma,\sigma}(l_2, l_1)\, \psi_{\sigma}(l_1), \tag{6.13}$$

where $m.\sigma$ is the image of the marking σ under $m \in \mathrm{MCG}(C)$, and we are taking advantage of the fact that the length representations for $\mathcal{H}_{g,n}$ associated to markings σ and $m.\sigma$ are canonically isomorphic.

6.6 Kähler quantization of the Teichmüller component

6.6.1 Quantization of local observables

In analogy to the coherent state representation of quantum mechanics it is natural to consider a quantization scheme in which states are represented by holomorphic multi-valued wave-functions[5]

$$\Psi(q) = \langle\, q|\Psi\,\rangle, \quad q = (q_1, \dots, q_{3g-3+n}), \tag{6.14}$$

in which the operators q_r corresponding to the observables q_r introduced in Section 3 are represented as multiplication operators, and the operators H_r associated to the conjugate “momenta” H_r should be represented by the differential operators $b^2\partial_{q_r}$ in such a representation,

$$\mathsf{q}_r\Psi(q) = q_r\Psi(q), \quad \mathsf{H}_r\Psi(q) = b^2\frac{\partial}{\partial q_r}\Psi(q). \tag{6.15}$$

[5] More precisely sections of a projective line bundle on $\mathcal{T}_{g,n}$.

The state $\langle q |$ introduced in (6.14) is thereby identified as an analog of a coherent state (eigenstate of the "creation operators" q_i) in quantum mechanics.

Formulae (6.15) turn the space of holomorphic wave-functions obtained in the Kähler quantization of the Teichmüller spaces into a module over the ring of holomorphic differential operators on $\mathcal{T}_{g,n}$. Let P be the projective connection $\partial_y^2 + t(y)$, and let the difference $P - P_S$ w.r.t. a reference projective connection P_S be expanded as

$$t(y) - t_S(y) = \sum_{r=1}^{3g-3+n} \vartheta_r(y)\, H_r.$$

We may then represent the corresponding quantum operator obtained in the Kähler quantization of the Teichmüller spaces as

$$\mathsf{T}(y) - b^{-2} t_S(y) = \sum_{r=1}^{3g-3+n} \vartheta_r(y) \frac{\partial}{\partial q_r}. \tag{6.16}$$

The operator $\mathsf{T}(y)$ may be called the "quantum energy-momentum tensor". For $g = 0$ we will find the following operator as the counterpart of the classical energy-momentum tensor $b^{-2} t_\varphi$,

$$\mathsf{T}(y) = \sum_{r=1}^{n-1} \left(\frac{\Delta_r}{(y - z_r)^2} + \frac{1}{y - z_r} \frac{\partial}{\partial z_r} \right), \tag{6.17}$$

where, as before, $z_{n-1} = 1$ and $z_{n-2} = 0$. We have introduced the quantum conformal dimensions Δ_r which are related to the δ_r by $\delta_r = b^2 \Delta_r + \mathcal{O}(b^2)$. This should be compared with the Virasoro Ward identities (5.13). Comparison of (6.17) and (5.13) indicates that the $\mathcal{D}$-module structure on $\mathcal{M}_{0,n}$ produced by the the Kähler quantization of $\mathcal{T}_{0,n}$ can be identified with the $\mathcal{D}$-module structure on the space of Virasoro conformal blocks.

6.6.2 Relation between length representation and Kähler quantization

The relation between length representation and the Kähler quantization is described by means of the wave functions

$$\Psi_l^\sigma(q) \equiv \langle\, q \mid l \,\rangle_\sigma. \tag{6.18}$$

The following characterization of these matrix elements was obtained in [76]:

$$\Psi_l^\sigma(q) = \mathcal{F}_q^\sigma(p), \tag{6.19}$$

where $\mathcal{F}_q^{\sigma}(p)$ is the Liouville conformal block associated to a marking σ with fixed intermediate dimensions given by the parameters p_γ, $\gamma \in \mathcal{C}_\sigma$. These parameters are related to the lengths c_a of the boundary components and to the lengths l_γ around the curves defining the pants decomposition respectively as

$$\beta_s = \frac{Q}{2} + i\frac{c_s}{4\pi\epsilon_1}, \quad p_\gamma = \frac{l_\gamma}{4\pi\epsilon_1}, \tag{6.20}$$

where $s = 1, \ldots, n$ and $\gamma \in \mathcal{C}_\sigma$.

Let me quickly recall the argument which leads to the identification (6.19). It is based on the observation that the wave-function $\Psi_l^{\sigma}(q) \equiv \langle\, q \,|\, l \,\rangle$ can be characterized as the unique solution of the following Riemann–Hilbert-type problem:

- The mapping class group element m acts on the wave-functions $\Psi(z)$ in the Kähler quantization in the natural way as a deck transformation. This means if U_m is the operator representing an element m of the mapping class group, we should have $(\mathsf{U}_m\Psi)(z) \equiv \Psi(m.z)$, with $\Psi(m.z)$ being the analytic continuation of $\Psi(z)$ along the path associated to m. We may, on the other hand, describe the action of U_m on $\Psi_l^{\sigma}(z)$ by means of (6.13). The consistency of these two descriptions implies that the monodromy action $\Psi_{l_2}^{\sigma}(m.z)$ can be represented as

$$\Psi_{l_2}^{\sigma}(m.z) = \int d\mu(l_1)\; V_{m.\sigma,\sigma}(l_2, l_1)\; \Psi_{l_1}^{\sigma}(z).$$

- The asymptotic behavior of $\Psi_l^{\sigma}(z)$ can be determined by quantizing the classical relation

$$q_\gamma H_\gamma \sim \left(\frac{l_\gamma}{4\pi}\right)^2 - \frac{1}{4},$$

  which is valid to leading order in the limit $l_\gamma \to 0$ if q_γ is the gluing parameter that vanishes when $l_\gamma \to 0$, and H_γ is the corresponding accessory parameter. We refer to [76] for more details and references.

This defines a Riemann–Hilbert-type problem that characterizes the left-hand side of (6.19) uniquely. It remains to show that the right-hand side of (6.19) is a solution to this Riemann–Hilbert problem. This was done in [74, 75].

6.7 Intermediate summary

It may be helpful to summarize the main arguments in a schematic form. On the one hand, we have seen that the Kähler quantization, which can be understood as the quantization of the holomorphic infinitesimal structure of $\mathcal{T}_{g,n}$, produces the action of the the ring of holomorphic differential operators on $\mathcal{T}_{g,n}$ realized on the wave-functions of the quantum Teichmüller theory, in other words

Quantization of local observables like H_r	$\rightarrow$	Ward identities / $\mathcal{D}$-module structure

The complex structure used here is the one from the Teichmüller theory. The canonical quantization of $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$, on the other hand, yields

Quantization of global observables like L_γ	$\rightarrow$	Representation of the algebra $\mathcal{O}_b$ of quantum loop operators

The realization of the algebra $\mathcal{O}_b$ deforms the structure of the ring $\mathcal{O}$ of algebraic functions on $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))$ in a natural way. The quantization of the global observables represents a quantization of $\mathrm{Hom}(\pi_1(C),$ $\mathrm{PSL}(2,\mathbb{C}))$ in complex structure J with symplectic form Ω_J.

Different representations for the resulting Hilbert space are obtained by diagonalizing different maximal subsets of commuting loop operators. Such subsets are in correspondence with pants decompositions. The resulting representation of the groupoid of changes of pants decompostion (more precisely markings) induces canonically a representation of the mapping class group via (6.13).

Classically, there is a natural isomorphism between $\mathrm{Hom}(\pi_1(C),$ $\mathrm{PSL}(2,\mathbb{R}))^0$ and the Teichmüller space $\mathcal{T}(C)$. Compatibility of canonical quantization of $\mathrm{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{R}))^0$ and Kähler quantization of $\mathcal{T}_{g,n}$ then defines a Riemann–Hilbert-type problem as discussed in Section 6.6.2. The Virasoro conformal blocks are the unique solution of this Riemann–Hilbert-type problem.

6.8 Quantization of the isomonodromic deformation problem

6.8.1 Quantization of the local observables

Let us return to the example of the Garnier system discussed in Section 3.6. The phase space in question can be identified with $\mathrm{Hom}(\pi_1(C_{0,n}), \mathrm{PSL}(2,\mathbb{C}))$ via the monodromy map for the differential operator $\partial_y^2 + t(y)$. It will be useful to start by considering the cases $0 < d < n-3$ first. We may then parameterize $t(y)$ in terms of $H_1, \dots, H_{n-3-d}$ and $\kappa_1, \dots, \kappa_d$, and the corresponding conjugate coordinates $z_1, \dots, z_{n-3+d}$, $w_1, \dots, w_d$. The remaining variables $H_{n-2-d}, \dots, H_{n-3}$ are determined by the constraints (3.16), and everything depends on the parameters $z_{n-2-d}, \dots, z_{n-3}$.

Contemplating a possible Kähler quantization of the Hitchin moduli space defined by the complex structure J and the symplectic structure ϖ_1, we are lead to propose a quantization scheme in which states are represented by holomorphic multi-valued wave-functions

$$\Psi(w,z) = \langle\, w, z \,|\, \Psi \,\rangle, \quad w = (w_1, \dots, w_d), \quad z = (z_1, \dots, z_{n-3-d}), \tag{6.21}$$

such that the operators w_r corresponding to the classical observables w_r are represented as multiplication operators, and the operators k_r associated to the momenta κ_r should be represented by the differential operators $b^2\partial_{w_r}$ in such a representation,

$$\mathsf{w}_r\Psi(w,z) = w_r\Psi(w,z), \quad \mathsf{k}_r\Psi(w,z) = b^2\frac{\partial}{\partial w_r}\Psi(w,z). \tag{6.22}$$

The quantum operators z_s and H_s representing z_s and H_s, respectively, should likewise be represented as

$$\mathsf{z}_s\Psi(w,z) = z_s\Psi(w,z), \quad \mathsf{H}_s\Psi(w,z) = b^2\frac{\partial}{\partial z_s}\Psi(w,z), \tag{6.23}$$

for $s = 1, \dots, n-3-d$. The constraints (3.16) are quantized as

$$\begin{aligned}&\sum_{s=1}^{n}\left(\frac{b^2\Delta_s}{(w_r - z_s)^2} + \frac{1}{w_r - z_s}\mathsf{H}_s\right)\\ &\quad + b^4\frac{\partial^2}{\partial w_r^2} + \sum_{\substack{r'=1\\ r'\neq r}}^{d}\left(b^2\frac{1}{w_r - w_{r'}}\frac{\partial}{\partial w_{r'}} - \frac{3+2b^2}{4(w_r - w_{r'})^2}\right) = 0,\end{aligned} \tag{6.24}$$

for $r = 1, \dots, d$. These equations reproduce equations (3.16) or equivalently (3.24) in the limit $b \to \infty$. The quantum correction proportional to

b^2 was introduced in the numerator of the last terms to ensure commutativity of the operators in (6.24). Equations (6.24) define the Hamiltonians $\mathsf{H}_{n-2-d}, \ldots, \mathsf{H}_{n-3}$ as functions of the remaining variables.

The wave-functions will depend on $z_{n-2-d}, \ldots, z_{n-3}$ as parameters. We propose that this dependence should be expressed by equations of the form

$$b^2 \frac{\partial}{\partial z_r} \Psi(w, z) = \mathsf{H}_r \Psi(w, z), \tag{6.25}$$

with $\mathsf{H}_{n-2-d}, \ldots, \mathsf{H}_{n-3}$ defined by (6.24). Indeed, let us note that we could equally well have chosen other subsets of $\{H_1, \ldots, H_n\}$ and $\{z_1, \ldots, z_n\}$ as independent sets of conjugate variables. The consistency with (6.23) requires (6.25).

The system of equations (6.24) is then equivalent to the equations

$$\Bigg[\sum_{s=1}^{n} \left(\frac{\Delta_r}{(w_r - z_s)^2} + \frac{1}{w_r - z_s} \frac{\partial}{\partial z_s} \right) + b^2 \frac{\partial^2}{\partial w_r^2} - \sum_{\substack{r'=1 \\ r' \neq r}}^{d} \left(\frac{3 + 2b^2}{4b^2 (w_r - w_{r'})^2} - \frac{1}{w_r - w_{r'}} \frac{\partial}{\partial w_{r'}} \right) \Bigg] \Psi(w, z) = 0, \tag{6.26}$$

which are equivalent to the null vector decoupling equations satisfied by the Liouville conformal blocks (5.43).

In the case $d = n - 3$ we may regard the second-order differential operators H_r as natural quantization of the Hamiltonian functions of the Garnier system. The differential equations (6.25) represent the change of the wave-function under the change of representation induced by a change of the underlying complex structure, analogous to the way the KZ equations were derived by Hitchin in [35]. We will see later that equations (6.25) are indeed essentially equivalent to the KZ equations in the SL(2) WZNW model.

6.8.2 Quantization of the global observables

In the maximal case $d = n - 3$, it seems natural to identify the space of states with the space spanned by a complete set of solutions to equations (6.23). We have previously seen in Section 5.4 how to identify a set of solutions to (6.23) that is complete in the sense that changes of the pants decomposition are realized by linear transformations from one set of solutions to another. The conformal blocks $\mathcal{F}^{\sigma}_{q,w}(p,\delta)$ generate a set of solutions which has simple asymptotic behavior in the boundary component of $\mathcal{T}_{0,n}$ corresponding to the marking σ. The analytic continuation of $\mathcal{F}^{\sigma_1}_{q,w}(p,\delta)$ into the boundary component of $\mathcal{T}_{0,n}$ corresponding to the marking σ_2 can be represented as a linear combination of the solutions $\mathcal{F}^{\sigma_2}_{q,w}(p,\delta)$.

There is a natural Hermitian form on this space of solutions that is invariant under the action of the mapping class (braid) group, given by the Liouville-correlation functions in a similar way as in (6.2). At the moment it is not clear to the author if this Hermitian form is positive definite for $d > 0$. For $d = 0$ it certainly is.

As in Section 5.3 one can define quantum loop operators acting on the space of states defined above. These are realized as difference operators. In the classical limit $b \to \infty$ we get a distinguished point in the real slice in $\mathrm{Hom}(\pi_1(C_{0,n}), \mathrm{PSL}(2,\mathbb{C}))$ defined by the extremum of the absolute value squared of $\mathcal{F}^{\sigma}_{q,w}(p,\delta)$. This point lies in the component of $\mathrm{Hom}(\pi_1(C_{0,n}), \mathrm{PSL}(2,\mathbb{R}))$ labeled by the integer d. The quantum theory described above can therefore be interpreted as a quantization of this component of $\mathrm{Hom}(\pi_1(C_{0,n}), \mathrm{PSL}(2,\mathbb{R}))$.

We arrive at a very natural interpretation of the parameterization of the wave-functions in terms of their asymptotic behavior at the boundaries of $\mathcal{M}_{0,n}$. The "interactions" between degrees of freedom in the isomonodromic deformation system go to zero near the boundaries of $\mathcal{M}_{0,n}$. One may therefore classify the elements of a basis for the space of states in terms of the asymptotics of the eigenvalues of the quantized Hamiltonians. The representation of the space of states in terms of asymptotic eigenvalues coincides with the representation for the space of conformal blocks in terms of the eigenvalues of the quantum loop operators. The operators representing the transition from one pants decomposition to another are thereby interpreted as analogs of scattering operators relating "In"- and "Out"-representations of the space of states.

7 Geometric Langlands correspondence and conformal field theory

In this section, we will try to explain some of the relevant features of the conformal field theory approach to the geometric Langlands correspondence initiated by Beilinson, Drinfeld, Feigin and Frenkel to physicists, following mostly the review [24].

7.1 Geometric Langlands correspondence and quantization of the Hitchin system

The correspondence between opers and the Hitchin eigenvalue equations is part of the geometric Langlands correspondence, for the case at hand schematically

$$\boxed{{}^{\mathrm{L}}\mathfrak{g} - \text{opers}} \quad \longrightarrow \quad \boxed{\mathcal{D} - \text{modules on } \mathrm{Bun}_G} \tag{7.1}$$

The $\mathcal{D}$-modules on Bun_G in question are in the case of $g = 0$ generated by the differential operators $\mathcal{D}_r = \mathsf{H}_r - E_r$. For the case $n = 0$, Beilinson and Drinfeld construct $3g - 3$ differential operators $\mathcal{H}_r$, $r = 1, \ldots, 3g - 3$ on the line bundle $K^{\frac{1}{2}}$ on Bun_G, which are mutually commuting and have the Hitchin Hamiltonians as their leading symbols.

7.2 Conformal blocks for the current algebra

7.2.1 Definition of the conformal blocks

Let $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_{2,k}$ be the central extension of the loop algebra of $\mathfrak{sl}_2$ with level k. The generators of $\hat{\mathfrak{g}}$ will be denoted J^a_n, $n \in \mathbb{Z}$, $a = -, 0, +$, the relations are

$$\begin{aligned} [\, J^0_n,\, J^0_m\,] &= \frac{k}{2} n\delta_{n+m,0}, \\ [\, J^0_n,\, J^\pm_m\,] &= \pm J^\pm_{n+m}, \end{aligned} \qquad [\, J^+_n,\, J^-_m\,] = 2J^0_{n+m} + kn\delta_{n+m,0}. \tag{7.2}$$

For representations $\mathcal{R}_r$, $r = 1, \ldots, n$ of $\hat{\mathfrak{g}}$ and Riemann surface C with n marked points at positions $z_1, \ldots, z_n$, one defines the conformal blocks as linear functionals $\mathcal{G} : \mathcal{R}_{[n]} \equiv \otimes_{r=1}^n \mathcal{R}_r \to \mathbb{C}$ that satisfy the invariance property

$$\mathcal{G}(\eta \cdot v) = 0 \quad \forall v \in \mathcal{R}_{[n]}, \quad \forall \eta \in \mathfrak{g}_{\mathrm{out}}, \tag{7.3}$$

where $\mathfrak{g}_{\mathrm{out}}$ is the Lie algebra of $\mathfrak{g}$-valued meromorphic functions on C with poles only at $z_1, \ldots, z_n$. The action of η on $\otimes_{r=1}^n \mathcal{V}_r \to \mathbb{C}$ is defined by taking the Laurent expansions of η at the points $z_1, \ldots z_n$, w.r.t. local coordinates t_r,

$$\eta(t) = \sum_{k \in \mathbb{Z}} \sum_{a=1}^{\dim(\mathfrak{g})} t_r^k J^a\, \eta^a_{r,k} \in \mathfrak{g} \otimes \mathbb{C}((t_r)), \tag{7.4}$$

to which we may associate the element

$$J[\eta_r] := \sum_{k\in\mathbb{Z}} \sum_{a=1}^{\dim(\mathfrak{g})} J_k^a \, \eta_{r,k}^a \in \hat{\mathfrak{g}}_k, \tag{7.5}$$

Denoting by $J_r[\eta_r]$ the operator which acts on $\mathcal{R}_{[n]}$ non-trivially only on the rth tensor factor of $\mathcal{R}_{[n]}$, where the action is given by $J[\eta_r]$, we finally get $\eta = \sum_{r=1}^{n} J_r[\eta_r]$. It can be shown that the central extension vanishes on the image of the Lie algebra $\mathfrak{g}_{\rm out}$ in $\bigoplus_{r=1}^{n} \hat{\mathfrak{g}}_k$, making the definition consistent.

7.2.2 Twisted conformal blocks

In order to obtain differential equations for the conformal blocks from the conformal Ward identities one possible solution is to modify the definition (7.3) by twisting $\mathfrak{g}_{\rm out}$ by an element $\mathcal{E}$ of Bun_G, which means to use (7.3) with $\mathfrak{g}_{\rm out}$ replaced by

$$\mathfrak{g}_{\rm out}^{\mathcal{E}} = \Gamma(C \setminus \{z_1, \dots, z_n\}, \, \mathfrak{g}_{\mathcal{E}}), \quad \mathfrak{g}_{\mathcal{E}} = \mathcal{E} \underset{G}{\times} \mathfrak{g}. \tag{7.6}$$

The space of linear functionals that satisfy the invariance conditions in (7.3) with $\eta \in \mathfrak{g}_{\rm out}^{\mathcal{E}}$ will be denoted $\mathsf{CB}(\mathcal{R}_{[n]}, C, \mathcal{E})$.

Concerning the dependence on the choice of $\mathcal{E}$ one can a priori only say that one has defined the conformal blocks as a sheaf over Bun_G. This means that locally over Bun_G we assign to each bundle $\mathcal{E}$ the vector space $\mathsf{CB}(\mathcal{R}_{[n]}, C, \mathcal{E})$, but the spaces assigned to "neighboring" bundles $\mathcal{E}$ and $\mathcal{E}'$ do not need to have the same dimension. The key observation to be made here is that the twisting of conformal blocks by elements of Bun_G offers a canonical way to define an action of the differential operators on Bun_G on the sheaf of conformal blocks. In mathematical language this is expressed as the statement that the space of conformal blocks becomes a $\mathcal{D}$-module. In physicists terms this can e.g., be expressed more concretely as follows. Let us consider conformal blocks with $n+1$ marked points $z_0, \dots, z_n$, where the vacuum representation is assigned to the marked point z_0. Then for each differential operator $\mathcal{D}_\eta$ on Bun_G there exists an element $J[\eta] \in \hat{\mathfrak{g}}_k$ such that

$$\mathcal{D}_\eta \cdot \left\langle \Phi_0(v_0|z_0) \prod_{r=1}^{n} \Phi_r(v_r|z_r) \right\rangle_{C_{g,n+1}}^{\mathcal{E}} = \left\langle \Phi_0(J_\eta v_0|z_0) \prod_{r=1}^{n} \Phi_r(v_r|z_r) \right\rangle_{C_{g,n+1}}^{\mathcal{E}}. \tag{7.7}$$

The point is that (7.7) is to be read as the *definition* of the action of the differential operator $\mathcal{D}_\eta$ on the conformal blocks. The construction of the differential operators $\mathcal{D}_\eta$ in (7.7) is non-trivial in general. In the mathematical literature there is a construction named "localization functor" which produces the corresponding sheafs of twisted[6] differential operators on Bun_G under rather general conditions.

In general it is not possible to exponentiate the infinitesimal action of of the affine Lie algebra $\hat{\mathfrak{g}}$ given by (7.7) to a projective representation of the corresponding loop group. This means that in general one cannot define a parallel transport that would allow one to regard the locally defined spaces of conformal blocks as a vector bundle over Bun_G. For the cases of interest, however, it will turn out that the Lie algebra action (7.7) can be exponentiated at least locally, away from a certain divisor of singularities in Bun_G.

7.2.3 More concrete representation of twisted conformal blocks

In the cases where the Lie algebra action on the vacuum representation $\mathcal{R}_0$ exponentiates to a projective representation of the corresponding loop group, one may represent the relation between twisted and untwisted conformal blocks more concretely e.g., for $n = 1$

$$\big\langle \Phi_0(v_0|z_0)\big\rangle_{C_{g,1}}^{\mathcal{E}} = \big\langle \Phi_0\big(e^{J[\eta]}v_0|z_0\big)\big\rangle_{C_{g,1}}, \tag{7.8}$$

where $J[\eta] = \sum_n \sum_a J_n^a \eta_n^a$. $e^{J[\eta]}$ is an operator, which represents an element of the (centrally extended) loop group on $\mathcal{V}_0$. $e^{J[\eta]}$ can be factorized as $e^{J[\eta]} = Ne^{J[\eta_<]}G_{\mathrm{in}}$, where $G_{\mathrm{in}}e_0 = e_0$ and $N \in \mathbb{C}$. Note that Bun_G can be represented as double quotient,

$$\mathrm{Bun}_G \simeq G_{\mathrm{out}} \setminus G(\!(t_0)\!) \;/\; G[[t_0]], \tag{7.9}$$

where G_{out} is the group of algebraic maps $C_{g,n}\setminus \{z_0\} \to G$, and t_0 is a local coordinate around z_0 vanishing there. The representation (7.9) follows from the fact that any G-bundle can be trivialized on the complement of a disc $\mathbb{D}_0$ cut out of the surface C. This means that the transition function can be represented by means of an element of the loop group assigned to the boundary of the disc $\mathbb{D}_0$. The double quotient representation (7.9) implies a similar representation for the tangent space $T_{\mathcal{E}}\mathrm{Bun}_G$ as $\mathfrak{g}_{\mathrm{out}} \setminus \mathfrak{g}(\!(z_0)\!)/\mathfrak{g}[[z_0]]$. We may therefore represent tangent vectors from $T_{\mathcal{E}}\mathrm{Bun}_G$ in terms of derivatives w.r.t. the parameters η_n^a introduced in (7.8), which explains how (7.7) comes about.

[6]That means roughly "taking care of the central extension".

If all representations $\mathcal{R}_r$, $r = 1, \ldots, n$ are integrable one may similarly introduce the twisting via

$$\left\langle \prod_{r=1}^{n} \Phi_r(v_r|z_r) \right\rangle_{C_{g,n}}^{\mathcal{E}} = \left\langle \prod_{r=1}^{n} \Phi_r(e^{J[\eta_r]} v_r|z_r) \right\rangle_{C_{g,n}}. \tag{7.10}$$

In this case one should replace (7.9) by

$$\mathrm{Bun}_G \simeq G_{\mathrm{out}} \setminus \prod_{r=1}^{n} G(\!(t_r)\!) \ / \ \prod_{r=1}^{n} G[[t_r]], \tag{7.11}$$

where t_r are local coordinates around the points z_r. The representation (7.11) comes from the existence of a trivialization of the bundle $\mathcal{E}$ on the complement of the union $\bigcup_{r=1}^{n} \mathbb{D}_r$ of small discs around the points z_r.

7.2.4 Conformal blocks versus functions on subsets of Bun_G

It will also be important for our aims that the twisting allows us to express the values of the conformal blocks $\mathcal{G}^{\mathcal{E}}$ on arbitrary vectors $v \in \mathcal{R}_{[n]}$ in terms of derivatives on Bun_G. This means that for each $v \in \mathcal{R}_{[n]}$ there exists a differential operator $\mathcal{D}_{\mathcal{E}}(v)$ on Bun_G such that

$$\mathcal{G}_{\mathcal{E}}(v) = \mathcal{D}_{\mathcal{E}}(v)\, \mathcal{G}_{\mathcal{E}}(e_{[n]}), \tag{7.12}$$

where $e_{[n]} = e_n \otimes \cdots \otimes e_1$ is the product of highest weight vectors.

Given a holomorphic bundle $\mathcal{E}$, a neighborhood $\mathcal{U}$ of $\mathcal{E}$ in Bun_G and a holomorphic function $\mathcal{G}$ on $\mathcal{U}$ we may turn (7.12) around and use it to *define* a conformal block. This means that large classes of conformal blocks actually come from (locally defined) functions on Bun_G. The point is that the double quotient representation (7.9) of Bun_G identifies this space as a locally symmetric space of the loop group, with infinitesimal structure given by the loop algebra $\mathfrak{g} \otimes \mathbb{C}(\!(t)\!)$. The relation (7.12) describes how a holomorphic function $\mathcal{G}$ can be described in terms of this infinitesimal symmetry.

This suggests that one can use conformal blocks as a basis for the space of holomorphic "functions", or rather sections of bundles, on Bun_G. One could thereby put conformal field theory in analogy to the harmonic analysis on locally symmetric spaces. The issue raised by this point of view is the possibility to extend these structures globally over Bun_G or some compactification thereof, possibly allowing "controllable" singular behavior at some divisors.

7.3 Realization of the geometric Langlands correspondence from conformal field theory

The representation theory of $\hat{\mathfrak{g}}_k$ at the critical level $k = -2$ has remarkable features. The universal enveloping algebra $\mathcal{U}_{\rm crit}(\widehat{\mathfrak{sl}}_2) \equiv \mathcal{U}(\widehat{\mathfrak{sl}}_2)/(k+2)$ has a large center generated by the modes t_n of the rescaled energy-momentum tensor

$$t(y) = -\frac{1}{k+2}T(y) = \sum_{n\in\mathbb{Z}} t_n y^{-n-2}. \tag{7.13}$$

This means that there exist representations π_t in which all the generalized Casimir elements t_n are realized as multiples of the identity. The generating function $t(y) = \sum_{n\in\mathbb{Z}} t_n y^{-n-2}$ can be used to parameterize such representations.

One may then attempt to construct the conformal blocks with insertions from this class of representations,

$$\left\langle \prod_{r=1}^{n} \Phi_{r,t_r}(v_r|z_r) \right\rangle, \tag{7.14}$$

where Φ_{r,t_r} is the vertex operator associated to a representation π_{t_r} with fixed choice of a generating function $t_r(y)$. The key point to observe about such conformal blocks is that they can be non-vanishing if, and only if, the generating functions $t_r(y)$ are the Laurent expansions near the marked points z_r of an oper $\partial_y^2 + t(y)$ which is *globally defined* on the surface C.

The correspondence between this oper $\partial_y^2 + t(y)$ and the *space* of conformal blocks associated to C and the choice of a collection of representations assigned to the marked points z_r,

$$\boxed{{}^{\rm L}\mathfrak{g} - \text{opers}} \quad \longrightarrow \quad \boxed{\text{conformal blocks of } \hat{\mathfrak{g}}_{\rm crit}} \tag{7.15}$$

is the origin of the geometric Langlands correspondence in the approach of Beilinson and Drinfeld. It remains to remember that spaces of conformal blocks canonically represent $\mathcal{D}$-modules to arrive at (7.1). The differential equations following from (7.7) include in particular the eigenvalue equations for the quantized Hitchin Hamiltonians. For $g = 0$ one finds that the eigenvalues E_r are given given by the residues of the oper $\partial_y^2 + t(y)$ at z_r.

7.3.1 Hecke action

There is a class of natural operations on the $\mathcal{D}$-modules on Bun_G called Hecke functors. We refer to [24] for more discussion of the Hecke functors and their realization on spaces of conformal blocks at the critical level. For the moment let us only remark that in the cases where the $\mathcal{D}$-modules are produced by the conformal blocks of $\hat{\mathfrak{g}}_k$ at the critical level $k = -2$ one may describe the Hecke functors as the modification of the conformal blocks by the insertion of certain representations with rather special properties. We will later (in Section 8.5) discuss natural analogs of the Hecke functors on the spaces of conformal blocks for $\hat{\mathfrak{g}}_k$ at the non-critical level.

Restricting to $\mathfrak{g} = \mathfrak{sl}_2$ for simplicity, the representations in question are labeled by half-integers j and denoted $\mathcal{W}_j$. As representations of the affine algebra $\hat{\mathfrak{g}}_{\rm crit}$ these representations are just the vacuum representation $\mathcal{R}_0$, but they come equipped with a $2j+1$-dimensional "multiplicity"-space V_j which is a module for the Lie algebra $\mathfrak{sl}_2$,

$$\mathcal{W}_j \simeq \mathcal{R}_0 \otimes V_j. \tag{7.16}$$

The Lie algebra $\mathfrak{sl}_2$ that V_j is a module of has no direct relation with the $\mathfrak{sl}_2$-subalgebra of the affine algebra $\widehat{\mathfrak{sl}}_{2,k}$ that we started from. It is identified as $\mathfrak{sl}_2$-representation by its categorical properties, in particular by its behavior under taking tensor products. In the case that one is considering a general affine algebra $\hat{\mathfrak{g}}_{\rm crit}$ one finds similarly

$$\mathcal{W}_\lambda \simeq \mathcal{R}_0 \otimes V_\lambda. \tag{7.17}$$

with V_λ being a module of the Langlands dual Lie algebra ${}^L\mathfrak{g}$ to $\mathfrak{g}$.

One may then consider conformal blocks with the representations $\mathcal{W}_j$ inserted,

$$\left\langle \Xi_j(v|y) \prod_{r=1}^{n} \Phi_r(v_r|z_r) \right\rangle_{\mathcal{E}} . \tag{7.18}$$

The special properties of the representations $\mathcal{W}_j$ imply that the spaces of conformal blocks with and without insertion of $\mathcal{W}_j$ are related as

$$\mathsf{CB}_{\mathcal{E}}(\mathcal{W}_j \otimes \mathcal{R}_{[n]}) \simeq V_j \otimes \mathsf{CB}_{\mathcal{E}}(\mathcal{R}_{[n]}). \tag{7.19}$$

The crucial Hecke eigenvalue property of the geometric Langlands correspondence can loosely speaking be described as the statement that under a variation of the insertion point y of $\mathcal{W}_j$ the local isomorphisms (7.19) glue together to generate a local system E. If $\mathsf{CB}(\mathcal{R}_{[n]}, C, \mathcal{E}, P)$ is the space of conformal blocks associated to a given oper $P = \partial_y^2 + t(y)$ one gets the local system corresponding to the monodromy representation of $\partial_y^2 + t(y)$. The local system E associated to the oper $\partial_y^2 + t(y)$ therefore plays a role analogous to an eigenvalue. This is roughly what is called the Hecke eigenvalue property in the context of the geometric Langlands correspondence.

8 Quantum geometric Langlands correspondence

8.1 The KZ equations

In the case of non-critical level $k \neq -2$ we can use the Sugawara construction to realize the generators L_n, $n \in \mathbb{Z}$ of the Virasoro algebra within the universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}}_k)$. Recall that the Virasoro algebra uniformizes infinitesimally the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces in a way that is similar to the way the current algebra uniformizes Bun_G, as expressed in (5.11) above. In the same way as described in Section 7.2 for the case of the current algebra one may use (5.11) in order to define a (twisted) action of the differential operators on $\mathcal{M}_{g,n}$ on the spaces of conformal blocks.

The fact that the Virasoro generators L_n can be expressed as bilinear expressions in the generators J_n^a implies relations between the differential operators representing the action of vector fields on $\mathcal{M}_{g,n}$ and Bun_G, respectively, which take the form of the KZB equations, schematically

$$(k+2)\frac{\partial}{\partial z_r}\Phi(x|z) = \mathsf{H}_r\,\Phi(x|z). \tag{8.1}$$

These equations allow us to "parallel transport" conformal blocks over $\mathcal{M}_{g,n}$. Any given solution to equations (8.1) in open subsets of $\mathrm{Bun}_G \times \mathcal{M}_{g,n}$ defines a conformal block according to our discussion in Section 7.2. We will see, however, that equations (8.1) have for fixed point in $\mathcal{M}_{g,n}$ regular singularities in Bun_G. This is related to the fact that the action of $\hat{\mathfrak{g}}_k$ on the spaces of conformal blocks defined in (7.7) does not exponentiate to a group action in general. However, away from the singularities of equation (8.1) it is certainly possible to integrate equations (8.1) in order to extend local solutions to solutions defined on some covering space of $\mathrm{Bun}_G \times \mathcal{M}_{g,n} \setminus \mathcal{S}$, where $\mathcal{S}$ is a certain divisor of singularities.

8.2 Conformal blocks for genus zero

8.2.1 Twisting parameters in genus zero

We will discuss conformal blocks for the SL(2)-WZNW model in $g = 0$ denoted as

$$\mathcal{G}(x|z) \equiv \big\langle \Phi^{j_n}(x_n|z_n)\cdots\Phi^{j_1}(x_1|z_1)\big\rangle. \tag{8.2}$$

The parameters x_r represent a non-minimal twisting of the conformal blocks as in (7.10). In the cases where the representation $\mathcal{R}_r$ has a highest weight vector e_r we may introduce the dependence on the variables x_r via $\Phi^{j_r}(x_r|z_r) \equiv \Phi^{j_r}(e^{x_r J_0^-} e_r|z_r)$. The parameters x_r represent the choice of parabolic structures near the marked points z_r. As vector bundles on a surface of genus zero are always trivial, we can take the coordinates x_r to parameterize an open dense subset of $\mathrm{Bun}_G(C_{0,n})$. The current algebra Ward identities now take the familiar form

$$\big\langle J^a(t)\, \Phi^{j_n}(x_n|z_n) \dots \Phi^{j_1}(x_1|z_1)\big\rangle = \sum_{r=1}^{n} \frac{\mathcal{J}_r^a}{t - z_r} \big\langle \Phi^{j_n}(x_n|z_n) \dots \Phi^{j_1}(x_1|z_1)\big\rangle, \tag{8.3}$$

where $\mathcal{J}_r^a$ are the differential operators defined in (4.7). The conformal blocks (8.2) satisfy the KZ equations (8.1) with differential operators H_r being explicitly given in (4.8).

8.2.2 More general classes of representations

So far we had assumed that the representations $\mathcal{R}_r$ of the current algebra are all of highest weight type. It is worth noting that the formalism easily allows one to cover representations of principal or complementary series type, too. Let, for example, $\mathcal{R}_r$ be a representation of $\widehat{\mathfrak{sl}}_{2,k}$ induced from a principal series representation of $\mathrm{SL}(2,\mathbb{R})$. We may assume that the zero mode sub-algebra $\mathfrak{sl}_2 \subset \widehat{\mathfrak{sl}}_{2,k}$ generated by the J_0^a is realized on functions $f(x_r) \in \mathcal{S}_r$ by the differential operators $\mathcal{J}_r^a$ defined in (4.7), with $\mathcal{S}_r$ being the Schwartz space of smooth functions on $\mathbb{R}$ with rapid decay. The dual space of distributions $\mathcal{S}_r^\dagger$ contains the delta-distributions δ_x with support at x. In this case, we should identify $\Phi^j(x|z)$ with $\Phi^j(\delta_x|z)$, with $\Phi^j(v|z)$ being the vertex operator associated to a vector $v \in \mathcal{R}_j^\dagger$, where $\mathcal{R}_j^\dagger$ is the Hermitian dual of $\mathcal{R}_j$.

Correlation functions as considered in (8.2) above are then to be understood as distributions on a Schwartz space of functions in n variables $x_1, \dots, x_n$. The type of representation one wants to consider will determine the precise space of solutions of the KZ equations that may be relevant for physical applications. It may, in general, contain distributional solutions supported on subspaces of $\mathrm{Bun}_G(C)$.

8.2.3 Singularities

In the case $g = 0$ it is possible to analyze the singularities of the differential equations (8.1) which prevent one to extend a local solution unambigously over $\mathrm{Bun}_G(C_{0,n})$ in detail. In the case $n = 4$, for example, one may recall the singularity at $x = z$ found in Section 4.3.1. This is the simplest example of a phenomenon that has also been discussed in the context of the geometric Langlands correspondence, where it figures under the name of the "global nilpotent cone", see [24, Section 9.5] for a discussion and further references. The global nilpotent cone is the locus in $\mathrm{Bun}_G(C)$ where all Hitchin Hamiltonians can vanish. Noting that the leading symbol of the differential operators H_r in the KZ equations coincides with the Hitchin–Hamiltonians [35], we are led to identify the singularity at $x = z$ exhibited in (4.13) with the global nilpotent cone in the example $g = 0$, $n = 4$.

8.2.4 The Whittaker model

By means of (formal) Fourier transformation $\mu^{j+1} \int dx_r \, e^{\mu_r x_r}$ one can pass to a representation in which the current $J^-(t)$ is represented diagonally,

$$\big\langle J^-(t)\, \tilde{\Phi}^{j_n}(\mu_n|z_n) \ldots \tilde{\Phi}^{j_1}(\mu_1|z_1) \big\rangle = \sum_{r=1}^{n} \frac{\mu_r}{t - z_r} \big\langle \tilde{\Phi}^{j_n}(\mu_n|z_n) \ldots \tilde{\Phi}^{j_1}(\mu_1|z_1) \big\rangle. \tag{8.4}$$

This representation will be called the Whittaker model.

The precise definition of the Fourier-transformation is delicate since the dependence of the conformal blocks on the variable x_r is multivalued in general. One would need to choose an appropriate branch. We plan to discuss this important issue in more detail elsewhere.

This sublety does not affect the relation between the *differential equations* characterizing the conformal blocks in the two representations. The conformal blocks must in particular satisfy the KZ equations (8.1) with differential operators H_r represented via (4.16) and (4.15). The subtleties coming from additional singularities in the dependence on the variables x_r will have counterparts in this representation as well. However, as will be explained below, there will now be a neat way to handle these singularities in this representation.

8.2.5 Gluing construction

We are interested in the class of solutions that are properly factorizable in the sense that they have power series representations in terms of the gluing parameters defined by a pants decomposition of the surface $C_{g,n}$. We will in the following construct sets of properly factorizable solutions that are complete in a suitable sense. It is possible to construct such solutions by means of a gluing construction which is analogous to the one discussed in

Section 5.2 for the Virasoro algebra. However, in order to get sufficiently large families of solutions, one also needs to consider representations of principal series type, as discussed in a related case in [73]. We plan to discuss this important point in more detail elsewhere.

8.3 Solutions to the KZ equations from solutions to null vector decoupling equations

In what follows we will describe a construction of a sufficiently large set of factorizable solutions to the KZ-equations (8.1) from the solutions to the BPZ-equations (5.24). In order to formulate it, we shall again take advantage of the fact that projective invariance allows us to reconstruct the conformal blocks introduced in (8.2) from their limits when $z_n \to \infty$, $x_n \to \infty$. The Fourier-transformation with respect to the remaining $n-1$ variables $x_1, \dots, x_{n-1}$ will be denoted as $\tilde{\mathcal{G}}(\mu|z)$, $\mu = (\mu_1, \dots, \mu_{n-1})$, $z = (z_1, \dots, z_{n-1})$ in the following. The main claim is that the ansatz

$$\tilde{\mathcal{G}}(\mu|z) = u\,\delta\left(\sum_{i=1}^{n-1}\mu_i\right)\Theta_n(y|z)\,\mathcal{F}(y|z), \tag{8.5}$$

yields a solution to the KZ-equations (8.1) from any given solution $\mathcal{F}(y|z)$ to the BPZ-equations (5.24). The function $\Theta_n(y|z)$ that appears here is defined as

$$\Theta_n(y|z) = \prod_{r<s\leq n-1} z_{rs}^{\frac{1}{2b^2}} \prod_{k<l\leq n-3} y_{kl}^{\frac{1}{2b^2}} \prod_{r=1}^{n-1}\prod_{k=1}^{n-3}(z_r - y_k)^{-\frac{1}{2b^2}}. \tag{8.6}$$

The claim will hold provided that the respective variables are related as follows:

(1) The variables $\mu_1, \dots, \mu_{n-1}$ are related to $y_1, \dots, y_{n-3}, u$ via

$$\sum_{r=1}^{n-1} \frac{\mu_r}{t - z_r} = u\frac{\prod_{k=1}^{n-3}(t - y_k)}{\prod_{r=1}^{n-1}(t - z_r)}. \tag{8.7}$$

In particular, since $\sum_{r=1}^{n-1}\mu_r = 0$, we have $u = \sum_{r=1}^{n-1}\mu_r z_r$.

(2) The Liouville parameter b is identified with the H_3^+ parameter $b^2 = -(k+2)^{-1}$.

(3) The Liouville momenta are given by

$$\alpha_r \equiv \alpha(j_r) := b(j_r + 1) + \frac{1}{2b}. \tag{8.8}$$

The fact that (8.5) solves the KZ equations (8.1) is a simple generalization of Sklyanin's observation described in Section 4.4 [59, 66].

Remark 8.1. Comparing with [59] one should note that the formulae in this paper yield the formulae above in the limit $z_n \to \infty$ and $x_n \to \infty$. It is interesting to note that the resulting formulae look very similar except that we have only $n-3$ variables y_k here rather than $n-2$ in [59]. In order to understand the relation between the two representations note that the solutions constructed in [59] automatically satisfy the constraints of invariance under the global SL(2). This follows indirectly from the proof of the main result in [59]. To see how this works one may start by considering the case $n = 3$. In this case, one may note that the condition $\sum_{r=1}^{n-1} \mathrm{D}_r^+ \tilde{\mathcal{G}}(\mu|z) = 0$ is a second-order differential equation on the variables μ_r, which is true as a consequence of the fact that the corresponding Liouville conformal block satisfies a BPZ null vector decoupling equation. The case of arbitrary n can be reduced to $n = 3$ by means of the factorization argument used in [59].

This being understood, we will in the following mostly use the formulation of reference [59]. The relevant formulae are obtained from the formulae above by the replacement $n \to n+1$.

8.3.1 Bases for the space of conformal blocks from the gluing construction

We may then *define* a family of $\widehat{\mathfrak{sl}}_{2,k}$-conformal blocks by means of the formula

$$\begin{aligned}&\left\langle \Phi^{j_n}(\mu_n|z_n) \dots \Phi^{j_1}(\mu_1|z_1) \right\rangle_{\hat{\mathbb{G}}} \\ &\quad = \delta\left(\sum_{i=1}^{n} \mu_i\right) u\, \Theta_n(y|z) \left\langle e^{2\alpha_n\phi(z_n)} \dots e^{2\alpha_1\phi(z_1)}\, e^{-\frac{1}{b}\phi(y_{n-2})} \dots e^{-\frac{1}{b}\phi(y_1)} \right\rangle_{\hat{\mathbb{G}}},\end{aligned} \tag{8.9}$$

where the conformal blocks on the right-hand side have been defined in Section 5.4.

We are looking for properly factorizable solutions i.e., solutions that have a simple behavior at the boundary component of Teichmüller space corresponding to a chosen marking σ. Consider e.g., a degeneration where $z_2 - z_1 = \mathcal{O}(\epsilon)$ with $\epsilon \to 0$. Considering formula (8.7) for values of t such that $t - z_1 = \mathcal{O}(\epsilon)$, we may note that the left-hand side is of order $\mathcal{O}(\epsilon^{-1})$, whereas the right-hand side would be of order $\mathcal{O}(\epsilon^{-2})$ unless there is an index j such that $y_j - z_1 = \mathcal{O}(\epsilon)$. Considering a degeneration of $\mathbb{P}^1 \setminus \{z_1, \dots, z_n\}$ into pairs of pants connected by thin tubes we would similarly find that each pair of pants will contain exactly one of the insertion points y_j of the degenerate fields $e^{-\frac{1}{b}\phi(y_j)}$ in (8.9). Specializing the parameterization of the solutions to the BPZ-equations introduced in Section 5.4 accordingly allows us to get a parameterization for a complete set of solutions to the KZ equations (8.1).

8.3.2 Critical level limit

We want to explain how formula (8.5) reproduces the geometric Langlands correspondence in the limit $b^2 \to \infty$ corresponding to $k \to -2$. We may, on the one hand, note that in the limit $k \to -2$ we may solve the KZ equations in the form

$$\tilde{\mathcal{G}}(\mu|z) \sim \exp(-b^2 S(z))\Psi(\mu|z)(1 + \mathcal{O}(b^{-2})) \tag{8.10}$$

provided that $\Psi(x|z)$ is a solution to the Gaudin eigenvalue equations $\mathsf{H}_r\Psi = E_r\Psi$ with E_r given in terms of $S(z)$ by $E_r = -\partial_{z_r} S(z)$. The system of these eigenvalue equations represents the $\mathcal{D}$-module on the right-hand side of (7.1).

Considering the right-hand side of (8.5), on the other hand, we may use the discussion of the semiclassical limit of Liouville conformal blocks in Section 5.5. It shows how the opers on the right-hand side of (7.1) are reproduced.

8.4 Modular duality

8.4.1 A dual WZNW model from Liouville theory

An interesting consequence pointed out in [27] of the duality of Liouville theory under $b \to b^{-1}$ is that one can build a second, dual WZNW model from Liouville theory by replacing (8.9) by

$$\begin{aligned} &\big\langle \tilde{\Phi}^{j_n}(\mu_n|z_n) \dots \tilde{\Phi}^{j_1}(\mu_1|z_1) \big\rangle^{\text{dual}}_{\mathbb{G}} \\ &\quad = \delta\left(\sum_{i=1}^{n} \mu_i\right) u\, \tilde{\Theta}_n(y|z) \big\langle e^{2\alpha_n\phi(z_n)} \dots e^{2\alpha_1\phi(z_1)}\, e^{-b\phi(y_{n-2})} \dots e^{-b\phi(y_1)} \big\rangle_{\mathbb{G}}, \end{aligned} \tag{8.11}$$

where $\tilde{\Theta}_n$ is obtained from the definition (8.6) by replacing $b \to 1/b$ on the left-hand side, and the parameters j_r are related to the α_r via

$$\alpha_r = b^{-1}(j_r + 1) + \frac{b}{2}. \tag{8.12}$$

The conformal blocks $\check{\mathcal{G}}(\mu, z) \equiv \langle\, \Phi^{j_n}(\mu_n|z_n) \dots \Phi^{j_1}(\mu_1|z_1)\rangle_{\mathbb{G}}^{\rm dual}$ on the left-hand side satisfy KZ equations of the form

$$(\check{k} + 2)\frac{\partial}{\partial q_r}\,\check{\mathcal{G}}(\mu, z) = \mathsf{H}_r\,\check{\mathcal{G}}(\mu, z), \tag{8.13}$$

which are the KZ equations for the SL(2)-WZNW model with level $\check{k}$ related to k via

$$\check{k} + 2 = \frac{1}{k+2} = -b^2. \tag{8.14}$$

The limit $b \to \infty$ corresponds to the classical limit of the dual SL(2)-WZNW model.

8.4.2 Local systems from the classical limit of WZNW conformal blocks

Let us consider the classical limit where $k \to \infty$ corresponding to $b \to 0$ in the WZNW model. Let us consider, in particular, conformal blocks like

$$\mathcal{G}(x, u|y, z) \;:=\; \big\langle \Phi^{+}_{(2,1)}(x|y)\, \Phi^{j_n}(u_n|z_n) \dots \Phi^{j_1}(u_1|z_1)\big\rangle_{\mathbb{G}}. \tag{8.15}$$

The null vector decoupling equation for the degenerate field $\Phi^{+}_{(2,1)}(x|y)$ is simply

$$\partial_x^2\, \Phi^{+}_{(2,1)}(x|y) = 0, \tag{8.16}$$

which means that $\Phi^{+}_{(2,1)}(x|y)$ transforms in the 2D representation of $\mathfrak{sl}_2$. Let $\mathcal{G}(x, u|y, z) = \mathcal{G}_+(u, y, z) + x\mathcal{G}_-(u, y, z)$, and let $\mathbf{G} = (\mathcal{G}_+, \mathcal{G}_-)^t$. The system of KZ equations satisfied by the conformal blocks (8.15) can the be written in the form

$$\begin{aligned}
-\frac{1}{b^2}\frac{\partial}{\partial y}\,\mathbf{G}(u, y, z) &= \sum_{r=1}^{n} \eta_{aa'}\frac{\sigma^a \mathcal{J}_r^{a'}}{y - z_r}\,\mathbf{G}(u, y, z),\\
-\frac{1}{b^2}\frac{\partial}{\partial z_r}\,\mathbf{G}(u, y, z) &= \sum_{\substack{s=1\\ s\neq r}}^{n} \eta_{aa'}\frac{\mathcal{J}_r^{a}\mathcal{J}_s^{a'}}{z_r - z_s}\,\mathbf{G}(u, y, z) + \eta_{aa'}\frac{\mathcal{J}_r^{a}\sigma^{a'}}{z_r - y}\,\mathbf{G}(u, y, z),
\end{aligned} \tag{8.17}$$

with σ^a being the matrices representing $\mathfrak{sl}_2$ in the 2D representation, and $\mathcal{J}_r^a$ being the differential operators introduced in (4.7). Let us assume that

$j_r = \mathcal{O}(\epsilon_1^{-1})$, which implies that $\mathcal{J}_r^a = \mathcal{O}(\epsilon_1^{-1})$, where $\epsilon_1 = \hbar b$. Note that this corresponds to $\alpha_r = \mathcal{O}(b^{-1})$ in terms of the Liouville parameters. We can assume that in the limit where $b \to 0$, $\hbar \to 0$ with $\epsilon_2 = \hbar/b$ fixed

$$I_r^a \,:=\, \lim_{\epsilon_2 \to 0} \epsilon_1 \frac{\mathcal{J}_r^a G_\pm(u,y,z)}{G_\pm(u,y,z)}, \tag{8.18}$$

is independent of y and the choice of component, and define

$$I(y) \,:=\, \sum_{r=1}^{n} \frac{\eta_{aa'}\sigma^a I_r^{a'}}{y - z_r}. \tag{8.19}$$

The first equation in (8.17) then implies that the vector $\mathbf{S}(y) \equiv \mathbf{S}(y|u,z)$,

$$\mathbf{S}(y) := \frac{\mathbf{G}(u,y,z)}{F(u,z)}, \tag{8.20}$$

where $F(u,z) := \langle \Phi^{j_n}(u_n|z_n) \dots \Phi^{j_1}(u_1|z_1) \rangle$ satisfies the equation

$$(\epsilon_2 \partial_y + I(y))\mathbf{S}(y) = 0. \tag{8.21}$$

$I(y)$, by definition, depends on z. However, the monodromy of the degenerate field $\Phi^+_{(2,1)}(x|y)$ inserted in (8.15) is completely defined in terms of gluing parameters $\mathbb{G}$. It follows that the monodromy of the ϵ_2-connection $\epsilon_2\partial_y + I(y)$ stays unchanged under variations of z. That is the dual way the isomonodromic deformation problem is recovered from the classical limit of Liouville theory which is related to the observations [32, 58] identifying the KZ equations as a formal quantization of the isomonodromic deformation problem.

8.5 Insertions of degenerate fields as quantum Hecke functors

Consideration of the relation between Liouville theory and the WZNW-model in cases where the representations $\mathcal{R}_j$ of $\widehat{\mathfrak{sl}}_{2,k}$ contain null vectors will reveal important further aspects of the relation with the geometric Langlands correspondence. Recall that the Verma modules $\mathcal{V}_{j,k}$ of the affine algebra $\widehat{\mathfrak{sl}}_{2,k}$ become degenerate whenever the representation of the zero mode subalgebra $\mathfrak{sl}_2$ has Casimir eigenvalue $j(j+1)$ with $j = j^{\epsilon}_{(k,l)}$, where

$$j^{+}_{(k,l)} = \frac{k-1}{2} + \frac{l-1}{2b^2}, \quad j^{-}_{(k,l)} = -\frac{k+1}{2} - \frac{l}{2b^2} \tag{8.22}$$

with $m, n = 1, 2, \ldots$. In the following, we are going to explain how the representations with $j = j^{-}_{(1,1)} = \frac{k}{2}$ and $j = j^{+}_{(1,2)} = 1/2b^2$ are related to the so-called Hecke functors.

8.5.1 Bundle modifications in conformal field theory

In Section 7.2 we have described how to assign spaces $\mathsf{CB}(\mathcal{R}_{[n]}, C_{g,n}, \mathcal{E})$ of conformal blocks to a Riemann surface $C_{g,n}$, a collection of representations $\mathcal{R}_1, \ldots, \mathcal{R}_n$ assigned to the marked points $z_1, \ldots, z_n$ of $C_{g,n}$ and a holomorphic G-bundle $\mathcal{E}$ on $C_{g,n}$. We now want to discuss how modifications of the bundle lead to modifications of $\mathsf{CB}(\mathcal{R}_{[n]}, C_{g,n}, \mathcal{E})$. Modifications of the bundle $\mathcal{E}$ can be described e.g., by cutting out a small disc $\mathbb{D}_0$ around a point $z_0 \in C_{g,n}$ and taking an element g_0 of the loop group LG associated to the boundary of $\mathbb{D}_0$ as the new transition function between $\mathbb{D}_0$ and the rest of $C_{g,n}$.

Our discussion in Section 7.2 suggests a simple realization of such bundle modifications in conformal field theory: Use the propagation of vacua to represent a conformal block $\mathcal{G} \in \mathsf{CB}(\mathcal{R}_{[n]}, C_{g,n}, \mathcal{E})$ by means of $\hat{\mathcal{G}} \in \mathsf{CB}(\mathcal{R}_{[n+1]}, C_{g,n+1}, \mathcal{E}')$ with an insertion of the vacuum e_0 at the point z_0, and then replace e_0 by a "twisted vacuum vector" $\tilde{e}_0$, which is a vacuum vector w.r.t. the generators $\tilde{J}^a_n$ obtained from the J^a_n by acting with the automorphism of $\widehat{\mathfrak{sl}}_{2,k}$ induced by the element g_0 of the loop group, which represents the transition function between $\mathbb{D}_0$ and the rest of $C_{g,n}$. We are thereby lead to define the modified conformal blocks $\mathcal{G}'$ as

$$\mathcal{G}'(v_{[n]}) = \hat{\mathcal{G}}(v_{[n]} \otimes \tilde{e}_0). \tag{8.23}$$

If, for example, the automorphism is represented as $\tilde{J}^a_n = \hat{g}_0 \, J^a_n \, \hat{g}_0^{-1}$ with $\hat{g}_0$ being an element of the central extension of the loop group corresponding to the Lie algebra $\widehat{\mathfrak{sl}}_{2,k}$, and if the vacuum representation exponentiates to a representation of this Lie group, we recover the description of the twisting of conformal blocks given in Section 7.2.

8.5.2 Hecke modifications

In order to get more interesting bundle modifications we need to consider a slightly more general setup. Instead of considering transition functions taking values in SL(2) let us consider transition function with values in GL(2). Let us in particular consider transition functions of the form

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & u_0 \\ 0 & 1 \end{pmatrix}, \tag{8.24}$$

where t is a local coordinate inside of $\mathbb{D}_0$ vanishing at z_0. Bundle modifications of this form are called Hecke modifications. The determinant of the modified bundle vanishes at z_0.

On an ϵ_2-connection, conjugation by the element $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$ induces the improper gauge transformation

$$\epsilon_2\,\partial_t + \tilde{I} := t^{-\frac{1}{2}} h \cdot (\epsilon_2 \partial_t + I) \cdot h^{-1} t^{\frac{1}{2}} = \epsilon_2\,\partial_t + \begin{pmatrix} I^0 + \frac{\epsilon_2}{2t} & \frac{1}{t} I^+ \\ tI^- & -I^0 - \frac{\epsilon_2}{2t} \end{pmatrix}. \tag{8.25}$$

The factors $t^{\pm\frac{1}{2}}$ were inserted to restore the SL(2)-form of the ϵ_2-connection. In terms of the modes defined by $\tilde{I}^a(t) = \sum_n t^{-n-1} \tilde{I}^a_n$ and $I^a(t) = \sum_n t^{-n-1} I^a_n$ this is equivalent to

$$I^\pm_n \to \tilde{I}^\pm_n := I^\pm_{n\mp 1}, \quad I^0_n \to \tilde{I}^0_n := I^0_n + \frac{\epsilon_2}{2}\delta_{n,0}. \tag{8.26}$$

There is an essentially unique counterpart of this transformation called spectral flow for the centrally extended Lie algebra $\widehat{\mathfrak{sl}}_{2,k}$,

$$J^\pm_n \to \tilde{J}^\pm_n \equiv J^\pm_{n\mp 1}, \quad J^0_n \to \tilde{J}^0_n \equiv J^0_n - \frac{k}{2}\delta_{n,0}. \tag{8.27}$$

The spectral flow (8.27) reduces to (8.26) in the classical limit $k \to \infty$ as considered in Section 8.4.2. The Hecke-modified conformal blocks can then be represented in terms of ordinary conformal blocks which have at the point z_0 a vector $\tilde{e}_0$ with the modified vacuum property

$$\tilde{J}^\pm_n\, \tilde{e}_0 = J^\pm_{n\mp 1}\, \tilde{e}_0 = 0, \quad \tilde{J}^0_n\, \tilde{e}_0 = J^0_n\, \tilde{e}_0 = 0, \quad n \geq 0. \tag{8.28}$$

There is no vector $\tilde{e}_0$ with such properties in the vacuum representation, but there is a distinguished representation $\mathcal{R}_{k/2}$ of $\widehat{\mathfrak{sl}}_{2,k}$ which has a highest weight vector $\tilde{e}_0 := e_{k/2}$ that satisfies (8.28). The representation is obtained as the quotient of the Verma module $\mathcal{V}_{k/2,k}$ by the submodule generated by the null-vector $J^+_{-1}\, e_{k/2}$ at level 1.

Inserting the representation $\mathcal{R}_{k/2}$ at z_0 is in the vertex operator notation represented by inserting the vertex operator $\Phi^{\frac{k}{2}}(u_0|z_0)$. It depends on the extra variable u_0 which parameterizes a choice of a parabolic subgroup at z_0.

In order to eliminate this dependence and in order to strengthen analogies with the definitions of the Hecke operators in other circumstances (see e.g., [23] for a discussion) let us consider

$$\Xi(w) := \Xi(0|w), \quad \Xi(u|w) := (\mathcal{I}_{\frac{k}{2}}\Phi^{\frac{k}{2}})(u|w), \tag{8.29}$$

where $\mathcal{I}_j$ is the $\mathfrak{sl}_2$-intertwining operator

$$(\mathcal{I}_j\Phi^j)(u|w) = -\frac{2j+1}{\pi}\int du'\ |u-u'|^{-2j-2}\Phi^j(u'|w). \tag{8.30}$$

It is worth noting that the vertex operator $\Xi(u|w)$ transforms under the $\mathfrak{sl}_2$-subalgebra as a representation with $j' = -1 - \frac{k}{2} = 1/2b^2 = j^+_{(1,2)}$, which vanishes at the critical level. In this case, the definition of $\Xi(w)$ simplifies to

$$\Xi(w) \equiv \int dx\ \Phi^{\frac{k}{2}}(x|w) \equiv \tilde{\Phi}^{\frac{k}{2}}(0|w). \tag{8.31}$$

The representation corresponding to the operator $\Xi(w)$ will become a multiple of the vacuum representation at the critical level, as is necessary to make contact with the discussion in Section 7.3.

8.5.3 GL(2)-twisted conformal blocks

An SL(2)-bundle $\mathcal{E}$ can be represented in many ways by means of a GL(2)-bundle $\hat{\mathcal{E}}$ with fixed determinant $\mathcal{O}(D)$, where D is an effective divisor of degree d. Let $w_1,\dots,w_d$ be the points of the divisor D, and let $\mathbb{D}_1,\dots,\mathbb{D}_d$ be small discs around the points $w_1,\dots,w_d$, respectively. If $\mathcal{E}$ is represented by the transition functions g_k at $\partial\mathbb{D}_k$, $k=1,\dots,d$, we may represent $\hat{\mathcal{E}}$ by the transition functions $\hat{g}_k$ defined by

$$\hat{g}_k = g_k\begin{pmatrix}1 & 0\\ 0 & t_k\end{pmatrix}\begin{pmatrix}1 & u_k\\ 0 & 1\end{pmatrix}. \tag{8.32}$$

We are lead to consider a natural family of generalizations of the space of conformal blocks $\mathsf{CB}(\mathcal{R}_{[n]},C_{g,n},\mathcal{E})$, which will be denoted $\mathsf{CB}^{[d]}(\mathcal{R}_{[n]},C_{g,n},\hat{\mathcal{E}})$. It is defined as the space of linear functionals $\mathcal{G}$ on $\mathcal{R}_{[n]}$, which can be represented in terms of conformal blocks $\mathcal{G}'\in\mathsf{CB}(\mathcal{R}_{[n+d]},C_{g,n+d},\hat{\mathcal{E}})$ of the form

$$\mathcal{G}(v_{[n]}) = \mathcal{G}'\big(v_{[n]}\otimes\tilde{e}_0^{\otimes d}\big). \tag{8.33}$$

It seems reasonable to regard the elements of $\mathsf{CB}^{[d]}(\mathcal{R}_{[n]},C_{g,n},\hat{\mathcal{E}})$ as natural generalizations of the twisted conformal blocks if the twisting by elements $\mathcal{E}$

of $\mathrm{Bun}_{\mathrm{SL}(2)}(C)$ is generalized to twisting by elements $\hat{\mathcal{E}}$ of $\mathrm{Bun}_{\mathrm{SL}(2)}^{\mathcal{O}(D)}(C)$, which are SL(2)-bundles represented by GL(2)-bundles $\hat{\mathcal{E}}$ with $\det(\hat{\mathcal{E}}) \simeq \mathcal{O}(D)$.

Of particular interest will also be the conformal blocks that are obtained by inserting the vertex operators $\Xi(w)$, like

$$\big\langle\big\langle \Phi^{j_n}(x_n|z_n)\cdots\Phi^{j_1}(x_1|z_1)\big\rangle\big\rangle := \big\langle \Xi(w_1)\cdots\Xi(w_d)\Phi^{j_n}(x_n|z_n)\cdots\Phi^{j_1}(x_1|z_1)\big\rangle \tag{8.34}$$

These linear functionals can of course not be canonically identified with the conformal blocks $\langle\Phi^{j_n}(x_n|z_n)\cdots\Phi^{j_1}(x_1|z_1)\rangle$ for non-critical level, but the fact that $\Xi(w)$ becomes proportional to the vacuum for $k=-2$ will imply that they become proportional to the conformal blocks $\langle\Phi^{j_n}(x_n|z_n)\cdots \Phi^{j_1}(x_1|z_1)\rangle$ *at* the critical level, as will be shown below.

8.5.4 Representation of Hecke modifications in terms of Liouville conformal blocks

Let us consider conformal blocks for the SL(2)-WZNW model with d insertions of $\Phi^{\frac{k}{2}}(u|w)$

$$\Phi(u,x|w,z) \equiv \big\langle \Phi^{\frac{k}{2}}(u_1|w_1)\cdots\Phi^{\frac{k}{2}}(u_d|w_d)\,\Phi^{j_n}(x_n|z_n)\cdots\Phi^{j_1}(x_1|z_1)\big\rangle. \tag{8.35}$$

After Fourier transformation to the μ-representation we get

$$\tilde{\Phi}(\nu,\mu|w,z) \equiv \big\langle \tilde{\Phi}^{\frac{k}{2}}(\nu_1|w_1)\cdots\tilde{\Phi}^{\frac{k}{2}}(\nu_d|w_d)\,\tilde{\Phi}^{j_n}(\mu_n|z_n)\cdots\tilde{\Phi}^{j_1}(\mu_1|z_1)\big\rangle. \tag{8.36}$$

Note that in the case $j=k/2$ the formula (8.8) gives $\alpha(k/2)=Q$. The Virasoro representation with $\alpha=Q$ has conformal weight zero, it therefore corresponds to the vacuum representation. The transformed conformal block (8.36) may therefore be represented in terms of Liouville conformal blocks as

$$\begin{aligned}&\left\langle \prod_{r=1}^{d}\tilde{\Phi}^{\frac{k}{2}}(\nu_r|w_r)\prod_{s=1}^{n}\tilde{\Phi}^{j_s}(\mu_s|z_s)\right\rangle_{\mathbb{G}}\\ &\quad=\delta\left(\sum_{s=1}^{n}\mu_s+\sum_{r=1}^{d}\nu_r\right)\,u\,\Theta_{n+d}(y|z)\,\left\langle\prod_{s=1}^{n}e^{2\alpha_s\phi(z_s)}\prod_{k=1}^{n+d-2}e^{-\frac{1}{b}\phi(y_k)}\right\rangle_{\mathbb{G}},\end{aligned} \tag{8.37}$$

where $u = \sum_{i=1}^{n} \mu_s z_s + \sum_{r=1}^{d} \nu_r w_r$ and

$$\sum_{s=1}^{n} \frac{\mu_s}{t - z_s} + \sum_{r=1}^{d} \frac{\nu_r}{t - w_r} = u \frac{\prod_{j=1}^{n+d-2}(t - y_j)}{\prod_{s=1}^{n}(t - z_s) \prod_{r=1}^{d}(t - w_r)}. \tag{8.38}$$

We see that an additional insertion of $\Phi^{\frac{k}{2}}(\nu|w)$ produces an extra degenerate field $e^{-\frac{1}{b}\phi}(y)$, but without producing any other insertion as would be the case for $\Phi^j(u|w)$ with $j \neq k/2$. It follows in particular from the fusion rules (5.25) that the spaces of conformal blocks with and without an insertion of $\Phi^{\frac{k}{2}}(\nu|w)$ are related as

$$\mathsf{CB}(\mathcal{R}_{k/2} \otimes \mathcal{R}_{[n]}) \simeq \mathbb{C}^2 \otimes \mathsf{CB}(\mathcal{R}_{[n]}). \tag{8.39}$$

The isomorphism (8.39) is not canonical. A useful way to describe it uses the markings introduced in Section 5.4. We will get something more canonical in the case of the Hecke functors at the critical level.

In the case where d is even one may on the one hand use the fact that the vacuum representation appears in the fusion rules $[e^{-\frac{1}{b}\phi}]\,[e^{-\frac{1}{b}\phi}] \sim [1] + [e^{-\frac{2}{b}\phi}]$. Subspaces of the space of conformal blocks of the form (8.35) are therefore naturally isomorphic to the original space of conformal blocks with $d = 0$. We may, on the other hand, regard the conformal blocks with d insertions of fields $\Phi^{\frac{k}{2}}(u_k|w_k)$ as conformal blocks associated to a bundle $\hat{\mathcal{E}}$ obtained from an original bundle $\mathcal{E}$ by means of d Hecke modifications. These facts can be used to represent at least a part of the dependence of the conformal blocks on the twisting bundle in terms of the variables $(u_1, \ldots, u_d)$ and $(w_1, \ldots, w_d)$ introduced in (8.35).

8.5.5 Representation of Hecke vertex operators in terms of Liouville conformal blocks

In order to describe conformal blocks with Hecke vertex operators $\Xi(w)$ it suffices to set $\nu_r = 0$ for $r = 1, \ldots, d$ in (8.37), as follows from (8.31). Note that setting $\nu_r = 0$ in (8.38) means that the expression on the right-hand side does not have a pole at $t = z_r$, which is only possible if one of the variables y_a coincides with z_r so that the apparent pole on the right-hand side is canceled. Noting that Θ_{n+d} simplifies to Θ_n in this case we arrive at the formula

$$\left\langle \prod_{k=1}^{d} \Xi(w_k) \prod_{s=1}^{n} \tilde{\Phi}^{j_s}(\mu_s|z_s) \right\rangle_{\mathbb{G}}$$
$$= \delta\left(\sum_{s=1}^{n} \mu_s\right) u\, \Theta_n(y|z) \left\langle \prod_{s=1}^{n} e^{2\alpha_s\phi(z_s)} \prod_{r=1}^{n-2} e^{-\frac{1}{b}\phi(y_r)} \prod_{k=1}^{d} e^{-\frac{1}{b}\phi(w_k)} \right\rangle_{\mathbb{G}}, \tag{8.40}$$

where $u = \sum_{i=1}^{n} \mu_s z_s$ and

$$\sum_{s=1}^{n} \frac{\mu_s}{t - z_s} = u \frac{\prod_{r=1}^{n-2}(t - y_r)}{\prod_{s=1}^{n}(t - z_s)}. \tag{8.41}$$

This means that inserting $\Xi(y)$ into an SL(2)-WZNW conformal blocks simply maps to the insertion of an extra degenerate field $e^{-\frac{1}{b}\phi(y)}$ on the Liouville side.

We had previously noted that the $\mathfrak{sl}_2$ representation under which the vertex operator $\Xi(w)$ transforms is proportional to the vacuum representation. This can not be the full story since insertion of $\Xi(y)$ modifies the space of conformal blocks as described by (8.39). However, from the discussion of the semiclassical limit of Liouville conformal blocks in Section 5.5 it follows that the insertions of $\Xi(w_k)$ will factor out in this limit, which leads to the formula

$$\left\langle \prod_{k=1}^{d} \Xi(w_k) \prod_{s=1}^{n} \Phi^{j_s}(x_s|z_s) \right\rangle_{\mathbb{G}} = \prod_{k=1}^{d} \chi_k(w_k) \left\langle \prod_{s=1}^{n} \Phi^{j_s}(x_s|z_s), \right\rangle_{\mathbb{G}} \tag{8.42}$$

where $\chi_k(w_k)$ are solutions to the differential equation $(\partial_w^2 + t(w))\chi_k = 0$. Which of the two linearly independent solution of the second-order differential equation one gets, depends on the choice of intermediate representation in the gluing construction of the relevant Liouville conformal blocks. This phenomenon is closely related to the Hecke eigenvalue property in the geometric Langlands correspondence as discussed in Section 7.3.

8.5.6 Quantum local systems

Monodromies of an extra insertion $\Xi(y)$ define operators on the space of conformal blocks as follows. Elements of the fundamental group $\pi_1(C_{g,n+d-2})$

are canonically identified with edge paths on the graph $\Gamma_{\hat{\sigma}}$. Moving $e^{-\frac{1}{b}\phi(y)}$ along a cycle γ representing a generator of the fundamental group corresponds to moving on a path on the marking graph $\Gamma_{\hat{\sigma}}$ described as a sequence of edges such that consecutive edges are connected at vertices. There is a standard way described in [2, 11] to associate to this edge path a composition of the elementary fusion and braiding moves [50]. Having returned to the point we started from, one may use the isomorphism (8.39) to define a two-by-two matrix M_γ of operators acting on the space conformal blocks with $n+d-2$ fields inserted. It is easy to see that the change of the choices involved in the definition of M_γ will change M_γ by conjugation with a possibly operator-valued matrix. Considering the operators M_γ associated to the generators γ of the fundamental group up to conjugation therefore defines a representation of the fundamental group by operator-valued matrices M_γ whose matrix elements are operators acting on the space of conformal blocks. Considering cycles γ, which are homotopic to the curves defining the pants decomposition corresponding to the marking $\hat{\sigma}$ one finds operator-valued matrices that act diagonally. Taking the trace of M_γ defines operators on $\mathsf{CB}(\mathcal{R}_{[n]}, C_{g,n}, \mathcal{E})$ that up to a phase factor are identical to the operators on $\mathsf{CB}(\mathcal{R}_{[n]}, C_{g,n}, \mathcal{E})$ defined by the construction described in Section 5.3. We have a correspondence

$$\boxed{\begin{array}{l}\text{Eigenvalues of}\\ \mathsf{M}_\gamma,\quad \gamma\in\mathcal{C}_\sigma\end{array}} \quad\longrightarrow\quad \boxed{\begin{array}{l}\text{Elements of a}\\ \text{basis for } \mathsf{CB}(\mathcal{R}_{[n]}, C, \mathcal{E})\end{array}} \tag{8.43}$$

We will call the operator-valued matrices M_γ quantum monodromies, and the representation of the fundamental group generated by the monodromies of the extra insertion $\Xi(y)$ a quantum local system. Parameterizing the space of conformal blocks by means of quantum local systems may be seen as a natural quantum analog of the geometric Langlands correspondence.

8.5.7 Critical level limit

Note that the operator-valued matrices M_γ will turn into the matrices $\rho(\gamma)$ representing the monodromy of the corresponding oper. We see that the quantum local systems turn into the classical local systems representing the opers. The representation $\mathcal{R}_{k/2}$ gets identified with the representation $\mathcal{W}_{\frac{1}{2}}$ representing the elementary Hecke functor on spaces of conformal blocks at the critical level according to the discussion in Section 7.3.1. Note furthermore that the eigenvalues of M_γ are parameterized by the variables p_r, which in the limit $b\to\infty$ get identified via (5.33) with the coordinates l_r for the space $\mathrm{Op}_{\mathfrak{sl}_2}(C_{g,n})$ of opers. We conclude that the correspondence (8.43) reduces to the geometric Langlands correspondence (7.1) in this limit. This is part of our motivation for calling (8.43) the quantum geometric Langlands correspondence.

8.5.8 Quantum Drinfeld–Sokolov reduction

Let us finally point out that the insertion of the fields $\Phi^{\frac{k}{2}}(u|w)$ representing the Hecke modifications not only allows us to raise the number of degenerate fields $e^{-\frac{b}{2}\phi(y)}$ in the Liouville-representation, it also allows us to lower this number. In order to see how this works, let us consider conformal blocks like

$$\begin{aligned}&\big\langle \tilde{\Phi}^{j}(\mu|w)\tilde{\Phi}^{j_n}(\mu_n|z_n)\dots\tilde{\Phi}^{j_1}(\mu_1|z_1)\big\rangle_{\mathbb{G}}\\ &\quad=\delta\left(\mu+\sum_{i=1}^{n}\mu_i\right)\,u_{n+1}\,\Theta_{n+1}(y|z)\\ &\qquad\times\big\langle e^{2\alpha\phi(w)}\,e^{2\alpha_n\phi(z_n)}\dots e^{2\alpha_1\phi(z_1)}\,e^{-\frac{1}{b}\phi(y_{n-2})}\dots e^{-\frac{1}{b}\phi(y_0)}\big\rangle_{\mathbb{G}},\end{aligned}\tag{8.44}$$

where $u_{n+1}=\sum_{r=1}^{n}\mu_r z_r+\mu w$ and

$$\frac{\mu}{t-w}+\sum_{r=1}^{n}\frac{\mu_r}{t-z_r}=u\frac{\prod_{k=0}^{n-2}(t-y_k)}{(t-w)\prod_{r=1}^{n}(t-z_r)}.\tag{8.45}$$

In the limit $\mu\to 0$ we find from (8.45) that one of the y_r, in the following taken to be y_0 must approach w to cancel the pole at $t=w$ of the right-hand side. It follows that the limit $\mu\to 0$ can be analyzed using the Liouville OPE:

$$\begin{aligned}e^{-\frac{1}{b}\phi(y_0)}e^{2\alpha\phi(w)}&\sim(y_0-w)^{b^{-1}\alpha}e^{(2\alpha-b^{-1})\phi(w)}(1+\mathcal{O}(y_0-w))\\ &\quad+C(\alpha)(y_0-w)^{b^{-1}(Q-\alpha)}e^{(2\alpha+b^{-1})\phi(w)}(1+\mathcal{O}(y_0-w)).\end{aligned}\tag{8.46}$$

In this way, it is straightforward to check that (8.46) implies that

$$\tilde{\Phi}^{j}(\mu|w)\underset{\mu\to 0}{\sim}\mu^{j+1}\Phi^{j}_{+}(w)(1+\mathcal{O}(\mu))+\mu^{-j}\Phi^{j}_{-}(w)(1+\mathcal{O}(\mu)).\tag{8.47}$$

The vertex operator $\Phi^{j}_{-}(w)$ is proportional to $\lim_{x\to\infty}x^{-2j}\Phi^{j}(x|w)$, as is simplest seen by noting that both are annihilated by J_0^- bearing in mind the representations (4.7) and (4.15). It has the *lowest weight* property $J^-(y)\Phi^j_-(w) = \text{regular}$.

In the case $j = k/2$ one has $\alpha = 0$. This implies that the term proportional to μ^{-j} in (8.47) would be absent unless one of the y_r, $r = 1, \dots, n$ happens to be at w. This is equivalent to the constraint

$$J^-(w) \equiv \sum_{r=1}^{n} \frac{\mu_r}{w - z_r} = 0. \tag{8.48}$$

We conclude that conformal blocks like $\big\langle \Phi^{\frac{k}{2}}_-(w)\, \tilde{\Phi}^{j_n}(\mu_n|z_n) \dots \tilde{\Phi}^{j_1}(\mu_1|z_1) \big\rangle$ can be defined as distributions with support given by (8.48). In the resulting representation by Liouville conformal blocks we will now find instead of $e^{2\alpha\phi(w)}$ one of the degenerate fields $e^{-\frac{1}{b}\phi(y_r)}$ with $y_r = w$ in (8.46). In this case, the second term in (8.46) will be proportional to the identity field. This leads to a representation of the form

$$\begin{aligned}&\left\langle \Phi^{\frac{k}{2}}_-(w) \prod_{r=1}^{n} \tilde{\Phi}^{j_r}(\mu_r|z_r) \right\rangle_{\mathbb{G}} \\ &\quad = \delta\left(\sum_{i=1}^{n} \mu_i\right) \delta\left(\sum_{r=1}^{n} \frac{\mu_r}{w - z_r}\right) u_n\, \Theta_n(y|z) \left\langle \prod_{r=1}^{n} e^{2\alpha_r\phi(z_r)} \prod_{k=1}^{n-3} e^{-\frac{1}{b}\phi(y_k)} \right\rangle_{\mathbb{G}},\end{aligned}$$

where $u_n = \sum_{r=1}^{n} \mu_r z_r$ and

$$\sum_{r=1}^{n} \frac{\mu_r}{t - z_r} = u(t - w) \frac{\prod_{k=1}^{n-3}(t - y_k)}{\prod_{r=1}^{n}(t - z_r)}. \tag{8.49}$$

The result is related to earlier work [16, 48] on the spectral flow in the SL(2)-WZNW model, and in particular to the description proposed in [60] for correlation functions in the SL(2)-WZNW model with winding number violation. The most important lesson for our purposes is the fact that the insertion of $\tilde{\Phi}^{\frac{k}{2}}(0|w)$ represents imposing the constraint (8.48) which is equivalent to $J^-(w) = 0$. Imposing this constraint effectively removes one of the degenerate fields $e^{-\frac{1}{b}\phi(y_k)}$ from the representation in terms of Liouville conformal blocks. The conformal blocks with maximal number of insertions of $\tilde{\Phi}^{\frac{k}{2}}(0|w)$ are proportional to the Liouville conformal blocks *without* degenerate fields.

8.6 Generalization of the geometric Langlands correspondence — from opers to more general local systems

It was proposed by Beilinson and Drinfeld (see [24, Section 9.6] for a discussion) to view the correspondence above as special case of a correspondence

$$\boxed{{}^{\mathrm{L}}G - \text{local systems}} \quad \longrightarrow \quad \boxed{\mathcal{D} - \text{modules on } \mathrm{Bun}_G} \tag{8.50}$$

In order to realize an example for this generalized version of the geometric Langlands correspondence, let us consider instead of (8.9) the following family of conformal blocks:

$$\left\langle \prod_{s=1}^{n} \Phi^{j_s}(x_s|z_s) \prod_{r=1}^{m} \Phi^{+}_{(2,2)}(u_r|w_r) \right\rangle_{\mathbb{G}} , \tag{8.51}$$

where $\Phi^{+}_{(2,2)}$ is the field corresponding to the degenerate representation corresponding to $j = j^{+}_{(2,2)} = \frac{1}{2}(1 + b^{-2})$.

8.6.1 Critical level limit of KZ equations

In the critical level limit $b \to \infty$, we may note that

$$j^{+}_{(2,2)} = \frac{1}{2}(1 + b^{-2}) \to \frac{1}{2}. \tag{8.52}$$

This implies that the null vector decoupling equation for the degenerate field $\Phi^{+}_{(2,2)}(x|w)$ simplifies in the critical level limit to

$$\partial_x^2 \, \Phi^{+}_{(2,2)}(x|w) = 0. \tag{8.53}$$

Representing the 2D space of solutions of (8.53) as $\mathbb{C}^2$ allows us to represent the conformal blocks (8.51) in terms of a vector-valued function $\mathbf{G}(z, w|x) \in (\mathbb{C}^2)^{\otimes m}$ as explained in Section 8.4. In the critical level limit, the KZ equations produce the pair of eigenvalue equations

$$\mathsf{H}_s \mathbf{G} = E_s \mathbf{G}, \quad \mathsf{k}_r \mathbf{G} = \kappa_r \mathbf{G}, \tag{8.54}$$

where

$$\begin{aligned} \mathsf{H}_s &= \sum_{s' \neq s} \frac{\mathcal{J}_s^a \mathcal{J}_{s'}^{a'}}{z_s - z_{s'}} \eta_{aa'} + \sum_{r} \frac{\mathcal{J}_s^a \sigma_r^{a'}}{z_s - w_r} \eta_{aa'}, \\ \mathsf{k}_r &= \sum_{s} \frac{\sigma_s^a \mathcal{J}_r^{a'}}{w_r - z_s} \eta_{aa'} + \sum_{r' \neq r} \frac{\sigma_r^a \sigma_{r'}^{a'}}{w_r - w_{r'}} \eta_{aa'}, \end{aligned} \tag{8.55}$$

with σ_r^a being the 2×2-matrices, which represent the action of $\mathfrak{sl}_2$ on the rth tensor factor in $(\mathbb{C}^2)^{\otimes m}$. The system of differential equations (8.55) will represent the $\mathcal{D}$-module to appear on the right-hand side of (8.50).

8.6.2 Classical limit of corresponding Liouville conformal blocks

Let us, on the other hand, analyze the conformal blocks (8.51) in the μ-representation obtained by Fourier-transformation over the variables x_s and u_r,

$$\begin{aligned}\left\langle \prod_{s=1}^{n}\tilde{\Phi}^{j_s}(\mu_s|z_s)\prod_{r=1}^{m}\tilde{\Phi}^{+}_{(2,2)}(\nu_r|w_r)\right\rangle_{\hat{\mathbb{G}}} &= \delta\left(\sum_{s=1}^{n}\mu_s+\sum_{r=1}^{m}\nu_r\right)\,u\,\Theta_{n+m}(y|z)\\ &\times\left\langle \prod_{s=1}^{n}e^{2\alpha_s\phi(z_s)}\prod_{r=1}^{m}e^{-b\phi(w_r)}\prod_{q=1}^{n+m-2}e^{-\frac{1}{b}\phi(y_q)}\right\rangle_{\hat{\mathbb{G}}},\end{aligned}\tag{8.56}$$

where $u=\sum_{i=1}^{n}\mu_s z_s+\sum_{r=1}^{m}\nu_r w_r$ and

$$\sum_{s=1}^{n}\frac{\mu_s}{t-z_s}+\sum_{r=1}^{m}\frac{\nu_r}{t-w_r}=u\frac{\prod_{j=1}^{n+m-2}(t-y_j)}{\prod_{s=1}^{n}(t-z_s)\prod_{r=1}^{m}(t-w_r)}.\tag{8.57}$$

The null vector decoupling equation (8.53) becomes $\mu^2\tilde{\Phi}^{+}_{(2,2)}(\mu|w)=0$ after the Fourier-transformation to the μ-representation. The conformal blocks (8.56) must therefore be distributions supported at $\nu_r=0$. Formula (8.57) implies that m of the variables y_q, here taken as $y_{n-1},\dots,y_{n+m-2}$, must equal one of $w_1,\dots,w_m$, respectively. The expectation values of the remaining fields $e^{-\frac{1}{b}\phi(y_q)}$ factor out in this limit, producing a factor $\prod_{q=1}^{n-2}\chi_q(y_q)$, with functions $\chi_q(y)$ that satisfy $(\partial_y^2+t(y))\chi_q(y)=0$ with $t(y)$ of the form

$$t(y)=\sum_{s=1}^{n}\left(\frac{\delta_s}{(y-z_s)^2}+\frac{E_s(p,z)}{y-z_s}\right)-\sum_{r=1}^{m}\left(\frac{3}{4(y-w_r)^2}-\frac{\kappa_r(p,z)}{y-z_r}\right).\tag{8.58}$$

The local system associated to this differential equation will appear on the left-hand side of (8.50).

8.6.3 The correspondence

We arrive at another interesting example for the geometric Langlands correspondence as the correspondence between the local systems corresponding to the differential equation $(\partial_y^2+t(y))\chi_q(y)=0$ with $t(y)$ of the form

(8.58) and the system of differential equations on $\mathrm{Bun}_G(C_{0,n})$. Note that for $m = n-3$ the number of parameters in $t(y)$ coincides with the dimension of $\mathrm{Loc}_{\mathrm{PSL}(2,\mathbb{C})}(C_{0,n})$.

This example exemplifies the abstract construction sketched in [24, Section 9.6]. It was noted there that the generalization beyond the case of opers requires introduction of additional parameters, which here are represented by the variables w_r. It was conjectured in this reference that the resulting system of differential equations is in a suitable sense independent of the choices of w_r. In this regard, we may observe that the dependence on w_r is controlled by the relation with Liouville semiclassical blocks in the following way: the function

$$\mathcal{W}_{\hat{\mathbb{G}}}(z,w) = \lim_{b\to\infty} b^{-2}\log\left\langle \prod_{s=1}^{n} e^{2\alpha_s\phi(z_s)} \prod_{r=1}^{m} e^{-b\phi(w_r)} \right\rangle_{\hat{\mathbb{G}}} \tag{8.59}$$

is a potential for $E_s = E_s(p,\delta|z,w)$ and $\kappa_r = \kappa_r(p,\delta|z,w)$ in the sense that

$$E_s = -\frac{\partial}{\partial z_s}\mathcal{W}_{\hat{\mathbb{G}}}(z,w), \quad \kappa_r = -\frac{\partial}{\partial w_r}\mathcal{W}_{\hat{\mathbb{G}}}(z,w). \tag{8.60}$$

The knowledge of $\mathcal{W}_{\hat{\mathbb{G}}}$ in principle allows us to compute how the parameters κ_r in the differential equations (8.54) have to be varied if one modifies the positions w_r of the additional singularities, keeping the local system fixed.

9 Concluding remarks

9.1 Relation with gauge theory

We believe that the results of this paper can help understanding the relation between gauge theory and Liouville theory suggested in [55] more precisely. They may thereby contribute to uncovering the deeper reasons for the correspondence between instanton partition functions and Liouville conformal blocks proposed in [1].

In this regard, let us note that the gauge theory set-up considered in [55] produces a Hilbert space $\mathcal{H}_{\epsilon_1\epsilon_2}$ of open strings which has a representation in terms of holomorphic sections of a line bundle on the space of opers. Locally these sections should be representable as holomorphic functions of the accessory parameters. There is no natural structure of non-commutative algebra on this space coming from quantization of a symplectic form on the space of opers. There are, however, two commuting actions on $\mathcal{H}_{\epsilon_1\epsilon_2}$ of quantized algebras of functions on $\mathrm{Loc}_{\mathrm{SL}(2)}(C)$. The deformation parameters can naturally be identified with ϵ_1 and ϵ_2, respectively. This is what strongly suggests that the Hilbert space of open strings produced by gauge theory can be identified with the space of Liouville conformal blocks [55].

The discussion in Section 3.8 suggests that the space of holomorphic sections of the line bundle produced by the gauge-theory setup of [55] should be seen as a sort of "momentum-representation" which is dual to the Kähler-quantization of Teichmüller space discussed here, in the sense that one works in a representation in which the conjugate momenta (the accessory parameters) of the Teichmüller moduli are diagonalized. Although such a quantization scheme remains to be developed in detail, we hope that these observations may help to clarify the relation between the Hilbert space of open strings coming from gauge theory and the space of conformal blocks in Liouville theory.

In any case, in order to understand the conjecture of [1] along such lines one should ultimately work in a third representation, which is the representation in which the a maximal set of commuting global observables (length operators) is diagonal. As pointed out in [11], one would thereby naturally explain the form that the gauge theory loop operator expectation values take according to [56], as discussed and generalized in [2, 11].

It is furthermore intriguing to note [6] that the conformal Ward identities have a counterpart in the context of the gauge-theoretical instanton counting: Variations of gauge coupling constants are described by means of insertions of $\mathrm{tr}(\phi^2)$. This observation should be compared to the fact that the Hamiltonians H_r obtained from the Higgs field θ via (2.3) and (2.4) end up being the generators of infinitesimal variations of the moduli of C in our approach.

9.2 Generalization to higher rank

Of obvious interest is the generalization of this picture when $\mathfrak{sl}_2$ is replaced by a Lie algebras $\mathfrak{g}$ of higher rank. We may anticipate the following picture.

The natural higher rank analogs of the Liouville theory are the conformal Toda theories denoted $\mathsf{Toda}_k(\mathfrak{g})$. The conformal symmetry of Liouville theory is extended to symmetry under the W-algebra $W_k(\mathfrak{g})$. Let us also consider the Toda theory $\mathsf{Toda}_{\check{k}}({}^{\mathrm{L}}\mathfrak{g})$ where ${}^{\mathrm{L}}\mathfrak{g}$ is the Langlands dual Lie algebra ${}^{\mathrm{L}}\mathfrak{g}$ with a Cartan matrix that is transpose of the Cartan matrix of $\mathfrak{g}$, while $\check{k}$ is related to k via

$$(k + h^\vee)r^\vee = (\check{k} + \check{h}^\vee)^{-1}. \tag{9.1}$$

$r^\vee$ is the lacing number of $\mathfrak{g}$, the maximal number of edges connecting two nodes of the Dynkin diagram, and $h^\vee$ is the dual Coxeter number. It was proven in [18] that the W-algebras $W_{\check{k}}({}^{\rm L}\mathfrak{g})$ and $W_k(\mathfrak{g})$ are isomorphic,

$$W_k(\mathfrak{g}) \simeq W_{\check{k}}({}^{\rm L}\mathfrak{g}). \tag{9.2}$$

It follows that the Toda theories $\mathsf{Toda}_k(\mathfrak{g})$ and $\mathsf{Toda}_{\check{k}}({}^{\rm L}\mathfrak{g})$ are dual to each other in the sense that the conformal blocks in the two theories coincide. This naturally suggests the conjecture [78] that there exist modular functors associated to $\mathsf{Toda}_k(\mathfrak{g})$ and $\mathsf{Toda}_{\check{k}}({}^{\rm L}\mathfrak{g})$, respectively, which are dual to each other if the levels are related by (9.1).

Let us now assume that there is a way to construct the conformal blocks in $\mathsf{WZNW}_k(\mathfrak{g})$ from those of $\mathsf{Toda}_k(\mathfrak{g})$, generalizing what was described above for the case $\mathfrak{g} = \mathfrak{sl}_2$. As in the $\mathfrak{g} = \mathfrak{sl}_2$-case discussed in this paper, we could then construct the conformal blocks of two different WZNW models from those of $W_k(\mathfrak{g})$, schematically

$$\boxed{\mathsf{WZNW}_{\check{k}}({}^{\rm L}\mathfrak{g})} \quad \longleftarrow \quad \boxed{\mathsf{Toda}_b(\mathfrak{g})} \quad \longrightarrow \quad \boxed{\mathsf{WZNW}_k(\mathfrak{g})} \tag{9.3}$$

For each of the WZNW models there are two different limits one may consider, leading to diagrams such as

$$\begin{array}{ccccc} & & G\text{-Hitchin system} & & \\ & (\mathrm{A})_{\epsilon_2} \nearrow & & \nwarrow (\mathrm{B})_{\epsilon_1} & \\ \begin{array}{c}\mathfrak{g}\text{-Isomonodromic}\\ \text{deformations}\end{array} & & & & \begin{array}{c}\mathfrak{g}\text{-Beilinson-}\\ \text{Drinfeld system}\end{array} \\ & (\mathrm{C})_{\epsilon_1} \nwarrow & & \nearrow (\mathrm{D})_{\epsilon_2} & \\ & & G\text{-WZNW-model} & & \end{array} \tag{9.4}$$

and on the other hand

$$\begin{array}{ccccc} & & {}^LG\text{-Hitchin system} & & \\ & (\mathrm{A})_{\epsilon_1} \nearrow & & \nwarrow (\mathrm{B})_{\epsilon_2} & \\ \begin{array}{c}{}^{\rm L}\mathfrak{g}\text{-Isomonodromic}\\ \text{deformations}\end{array} & & & & \begin{array}{c}{}^{\rm L}\mathfrak{g}\text{-Beilinson–}\\ \text{Drinfeld system}\end{array} \\ & (\mathrm{C})_{\epsilon_2} \nwarrow & & \nearrow (\mathrm{D})_{\epsilon_1} & \\ & & {}^LG\text{-WZNW-model} & & \end{array} \tag{9.5}$$

This would again lead to two possible ways to describe the same limit in the conformal Toda theory $\mathsf{Toda}_b(\mathfrak{g})$. Extrapolating from case $\mathfrak{g} = \mathfrak{sl}_2$ we would expect that a good part of the geometric Langlands correspondence can be understood in this way.

In the $\mathfrak{sl}_2$-case we had discussed the relations between the $\mathfrak{sl}_2$-Toda (Liouville) theory and the quantization of the Teichmüller spaces. It seems worth pointing out that higher rank analogs of the quantum Teichmüller spaces have been defined in [20]. A relation between modular duality and Langlands duality that fits perfectly into the picture proposed above was pointed out in [20]. Proving the modular functor conjecture [20] for the higher quantized Teichmüller theories would be an important step towards the higher rank generalization of the quantum geometric Langlands correspondence.

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