

# Wall-crossing of D4-branes using flow trees

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## Abstract

The moduli dependence of D4-branes on a Calabi–Yau manifold is studied using attractor flow trees, in the large volume limit of the Kähler cone. One of the moduli-dependent existence criteria of flow trees is the positivity of the flow parameters along its edges. It is shown that the sign of the flow parameters can be determined iteratively as function of the initial moduli, without explicit calculation of the flow of the moduli in the tree. Using this result, an indefinite quadratic form, which appears in the expression for the D4-D2-D0 BPS mass in the large volume limit, is proven to be positive definite for flow trees with 3 or less endpoints. The contribution of these flow trees to the BPS partition function is therefore convergent. From non-primitive wall-crossing is deduced that the  $S$ -duality invariant partition function must be a generating function of the rational invariants  $\bar{\Omega}(\Gamma) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m)}{m^2}$  instead of the integer invariants  $\Omega(\Gamma)$ .

## CONTENTS

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Wall-crossing and flow trees</b>	<b>5</b>
	<b>2.1</b> Kontsevich–Soibelman wall-crossing formula	<b>6</b>
	<b>2.2</b> Supergravity and flow trees	<b>9</b>
<b>3</b>	<b>D4-D2-D0 BPS-states</b>	<b>16</b>
	<b>3.1</b> BPS mass and stability	<b>18</b>
	<b>3.1.1</b> One endpoint	<b>21</b>
	<b>3.1.2</b> Two endpoints	<b>22</b>
	<b>3.1.3</b> Three endpoints	<b>28</b>
	<b>3.2</b> Non-primitive wall-crossing	<b>34</b>
<b>4</b>	<b>Summary and discussion</b>	<b>38</b>
	<b>Acknowledgments</b>	<b>39</b>
	<b>References</b>	<b>39</b>

## 1 Introduction

The BPS-spectrum of supersymmetric quantum field theories [25, 26, 45] and supergravity [11, 15, 46] depends in an intriguing way on the moduli of the theory. If moduli cross walls of marginal stability, BPS-states can combine or decay without violating physical conservation laws. As a consequence, the supersymmetric index  $\Omega(\Gamma; t)$  of BPS-states with charge  $\Gamma$ , is only locally constant and changes discontinuously as function of the moduli  $t$ . This is by no means an arbitrary process but happens according to a rather rigorous mechanism, whose implications are however not fully understood.

The moduli dependence of the supergravity BPS-spectrum appears as the possible decay or formation of multi-center solutions if the moduli are varied [11]. This has led to the conjecture that the moduli dependence of the

supergravity spectrum is captured by “attractor flow trees” [11, 15]. These trees are schematic (in some sense linearized) representations of supersymmetric solutions, which are much easier to analyse than the full supergravity solutions. Various results have been derived using the flow trees, such as the (semi-primitive) wall-crossing formula [15], and the derivation of BPS spectra [5, 10, 13, 15, 30].

The BPS-states of supergravity are represented in string theory as D-branes wrapped around cycles of a Calabi–Yau 3-fold  $X$ . From this point of view, one is interested in the BPS-spectrum of the D-branes, as function of the moduli of  $X$ . A fruitful interplay exists between stability of D-branes and stability in mathematics [20, 32, 34]. The BPS indices  $\Omega(\Gamma; t)$  are conjecturally equal to the rigorously defined Donaldson–Thomas invariants.

A central object in the study of BPS-states is the partition function, which is the generating function for the supersymmetric index  $\Omega(\Gamma; t)$  of BPS-states with charge  $\Gamma$ . The mixed ensemble is most natural for  $\mathcal{N} = 2$  supergravity [41], with the electric charges in the canonical ensemble and the magnetic charges in the microcanonical ensemble. Besides being the generating function of  $\Omega(\Gamma; t)$ , it is a useful object to test the validity on the microscopic level of duality groups. These are for  $\mathcal{N} = 2$  supergravity in 4 dimensions the  $S$ -duality group  $SL(2, \mathbb{Z})$  [6], and the electric–magnetic duality group  $Sp(2b_2 + 2, \mathbb{Z})$  (or a subgroup) [48]. Most desirable is a partition function which gives at any given point  $t$  in moduli space the BPS indices  $\Omega(\Gamma; t)$ , and which captures correctly the changes of the indices if the moduli are varied.

This is a rather difficult problem in general. However, one might construct the partition function using attractor flow trees from elementary building blocks, the black hole centers which cannot decay. Manschot [36] studied in this way the contribution to the partition function of a flow tree with 2 endpoints with D4–D2–D0 charge. The analysis was simplified by restricting to the large volume limit of a single complexified Kähler cone. It shows that a certain indefinite quadratic form, which appears in the expression for the BPS mass in this limit, is positive definite when evaluated for stable bound states of two constituents, or equivalently flow trees with 2 endpoints. This implies the convergence of the contribution to the partition function of these flow trees, which enumerates only the stable BPS-states at a point  $t$  in the moduli space. The generating function does not preserve  $S$ -duality, but can be made so by the addition of a “modular completion”, which (unexpectedly) also has the effect of changing it to a continuous function of the moduli. Continuity appeared in the literature before in the context of wall-crossing [25, 31].

The current paper extends the approach of Manschot [36] to flow trees with 3 endpoints. This solves various conceptual issues for a generalization to any number of endpoints. The larger flow trees complicate the analysis considerably, since the existence (or stability) conditions depend on the flow of the moduli throughout the tree, and are therefore only indirectly determined by the value  $t$  of the moduli at “infinity”. The most sensitive condition to variations of the moduli is the sign of the flow parameters along the edges of the tree. The flow parameter is a measure for the length of the edge, and therefore required to be positive for all edges of an existing flow tree. Fortunately, Subsection 2.2 derives an iterative expression in terms of  $t$  for this sign, without explicit computation of the flow of the moduli along the edges. Section 3 applies this result to BPS D4-branes, to proof that also for flow trees with 3 endpoints, an indefinite quadratic form is positive definite when restricted to stable flow trees, analogously to the case of 2 endpoints. This again ensures the convergence of the partition function. It is expected that this property continues to hold for flow trees with any number of endpoints.

To incorporate flow trees with equal charges for 2 of the 3 endpoints, one is required to use the semi-primitive wall-crossing formula. Section 3 argues that partition functions which capture non-primitive wall-crossing can only be compatible with  $S$ -duality, if it is a generating function of the rational invariants  $\bar{\Omega}(\Gamma; t) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m)}{m^2}$  [32] and *not* of the integer invariants  $\Omega(\Gamma; t)$ . The jumps of the indices in terms of  $\bar{\Omega}(\Gamma; t)$  are also more easily identified as contributions from flow trees than in terms of  $\Omega(\Gamma; t)$ . The contributions of the primitive and semi-primitive trees are shown to combine nicely into sums over certain lattices.

Unfortunately, the form of the stability condition for trees with 3 endpoints prevents an easy construction of the modular completion of its contribution to the partition function analogous to [36]. The compatibility of these flow trees with  $S$ -duality is thus not yet completely shown, but important prerequisites are satisfied. I hope to address this issue in future work.

I conclude the introduction with the outline of the paper. Section 2 reviews wall-crossing of BPS-states to render the paper self-contained. It reviews in particular the Kontsevich–Soibelman wall-crossing formula, wall-crossing in supergravity and the split attractor flow conjecture. It derives an expression for the sign of the flow parameters, without explicitly calculating the flow of the moduli throughout the tree. Section 3 applies the general discussion of Section 2 to D4-D2-D0 BPS-states. The main part of the section deals with the proof that the indefinite quadratic form is positive definite on the stable spectrum for  $N \leq 3$ . Subsection 3.2 comments on non-primitive wall-crossing, and why  $S$ -duality favours the rational invariants

$\bar{\Omega}(\Gamma; t)$ . Section 4 concludes with a short summary of the results and discussion.

## 2 Wall-crossing and flow trees

This section reviews briefly stability and wall-crossing of BPS-states in string theory compactified on a Calabi–Yau 3-fold  $X$  (more information can be found in the references). This compactification preserves  $\mathcal{N} = 2$  supersymmetry, such that the only massive BPS states preserve half of the supersymmetry. We will work in the Type IIA duality frame, where the electric–magnetic charges of supergravity correspond to D-branes wrapping even dimensional cycles of  $X$ . The charges are combined into a vector  $\Gamma = (P^0, P^a, Q_a, Q_0)^T$ , which is an element of a  $(2b_2 + 2)$ -dimensional symplectic lattice  $L$ , with symplectic inner product:

$$\langle \Gamma_1, \Gamma_2 \rangle = -P_1^0 Q_{0,2} + P_1 \cdot Q_2 - P_2 \cdot Q_1 + P_2^0 Q_{0,1}. \quad (2.1)$$

$\langle \Gamma_1, \Gamma_2 \rangle$  is often abbreviated to  $I_{12}$  in the following.

The  $\mathcal{N} = 2$  superalgebra contains a central element, the central charge  $Z : (L, C_X) \rightarrow \mathbb{C}$ , which associates to every  $\Gamma \in L$  and point of the moduli space  $t = B + iJ \in C_X$  (the complexified Kähler cone for Type IIA) a complex number  $Z(\Gamma, t) \in \mathbb{C} = \mathbb{R}^2$ . The mass  $M$  of a BPS-state is determined by the central charge:  $M = |Z(\Gamma, t)|$ . The (not complexified) Kähler cone is a  $b_2$ -dimensional cone which parametrizes the volumes of even dimensional cycles of  $X$ . The boundary of the cone corresponds to vanishing of the volume of 2-cycles. From the perspective of mirror symmetry, it is natural to consider the “extended Kähler moduli space” [2], which is the union of all Kähler cones of Calabi–Yaus which are birationally equivalent. These Calabi–Yaus are however not topologically equivalent, since continuation of the Kähler moduli beyond the boundary of the Kähler cone leads to flops of 2-cycles of  $X$ . Although flops do not lead to singular physics, we restrict our attention in this paper to  $C_X$ , corresponding to topologically equivalent Calabi–Yaus.

The index  $\Omega(\Gamma; t)$  is a measure for the number of BPS-states. It is defined by a weighted trace over the Hilbert space  $\mathcal{H}(\Gamma, t)$ :

$$\Omega(\Gamma; t) = \frac{1}{2} \text{Tr}_{\mathcal{H}(\Gamma, t)} (2J_3)^2 (-1)^{2J_3}, \quad (2.2)$$

where  $J_3$  is a generator of the rotation group  $\text{Spin}(3)$ . The sum over the Hilbert space shows that  $\Omega(\Gamma; t)$  are integers. An important property of the index is its independence of the string coupling constant  $g_s$  and the complex

structure moduli of  $X$  (in Type IIA). Therefore, the index can be determined and analyzed at finite  $g_s$  or in the limit  $g_s \rightarrow 0$  depending on which regime is better suited for the analysis. The first regime corresponds to four-dimensional supergravity, where many of the BPS-states appear as (possibly multi-centered) black holes. The limit  $g_s \rightarrow 0$  is the D-brane regime, where the BPS-states can often be related to mathematical objects.

As the notation suggests, the Hilbert space  $\mathcal{H}(\Gamma, t)$  depends on  $C_X$ . The indices  $\Omega(\Gamma; t)$  are only locally constant and may jump across codimension 1 hypersurfaces in the moduli space. These “walls of marginal stability” are determined by the alignment of central charges of the constituents  $Z(\Gamma_1, t)$  and  $Z(\Gamma_2, t)$  with  $\Gamma = \Gamma_1 + \Gamma_2$  (assuming that  $I_{12} \neq 0$ , otherwise the subspaces of the moduli space where the central charges align are walls of threshold stability), and divide the moduli space into chambers. Wall-crossing was first observed in 4 dimensions in supersymmetric gauge theory [45], and later in supergravity [11, 15].

## 2.1 Kontsevich–Soibelman wall-crossing formula

Supersymmetric D-brane configurations lend themselves well to more abstract descriptions like triangulated categories. Within this mathematical setting, Kontsevich and Soibelman [34] have proposed a formula, which captures changes of the invariants  $\Delta\Omega(\Gamma_1 + \Gamma_2; t)$  at a wall of marginal stability for generic  $\Gamma_1$  and  $\Gamma_2$ . This was an important open problem in physics, where the jumps of the indices were only known in restricted situations like semi-primitive charges [15] or Seiberg–Witten theory [22]. By now a lot of evidence exists for the validity of the KS-formula in generic BPS contexts [18, 19, 25, 26]. We briefly review the KS-formula here.

Kontsevich and Soibelman [34] introduces a Lie algebra with generator  $e_\Gamma$  for every charge  $\Gamma \in L$ . The commutation relations are given by

$$[e_{\Gamma_1}, e_{\Gamma_2}] = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle} \langle \Gamma_1, \Gamma_2 \rangle e_{\Gamma_1 + \Gamma_2}. \quad (2.3)$$

For every charge  $\Gamma$  an element  $T_\Gamma$  of the Lie group is defined by

$$T_\Gamma = \exp \left( - \sum_{n \geq 1} \frac{e_{n\Gamma}}{n^2} \right). \quad (2.4)$$

A sector in  $\mathbb{R}^2$  is defined as a region bounded by two rays whose starting point is at the origin. A sector is strict if the angle between the rays is less than  $180^\circ$ . A product  $A_V$  of elements  $T_\Gamma$  is associated to a strict sector

$V \in \mathbb{R}^2$ . The clockwise order of the central charges  $Z(\Gamma, t) \in V$  with  $\Gamma \in L$ , determines the order of the product:

$$A_V = \prod_{\Gamma \in L, Z(\Gamma, t) \in V}^{\curvearrowright} T_{\Gamma}^{\Omega(\Gamma; t)}. \quad (2.5)$$

If the moduli cross a wall of marginal stability, the order of the central charges changes and therefore likewise the order of the product. The claim of [34] is that the change of the  $\Omega(\Gamma; t)$  is precisely such that the product  $A_V$  does not change. The commutation relations of  $e_{\Gamma}$  thus determine the changes of indices if walls are crossed.

Note that the form of the wall-crossing formula also suggests that the invariants  $\bar{\Omega}(\Gamma; t)$ , defined by

$$\bar{\Omega}(\Gamma; t) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m; t)}{m^2}, \quad (2.6)$$

are convenient. These are valued in  $\mathbb{Q}$  and are conjecturally equal to the invariants which are the central topic in the work of Joyce et al. [31, 32]. The product formula (2.5) is in terms of these invariants more simply expressed using the elements  $R_{\Gamma}^{\bar{\Omega}(\Gamma; t)} = \exp(\bar{\Omega}(\Gamma; t) e_{\Gamma})$ . Equation (2.6) can be inverted with the Möbius inversion formula

$$\Omega(\Gamma; t) = \sum_{m|\Gamma} \frac{\bar{\Omega}(\Gamma/m; t)}{m^2} \mu(m), \quad (2.7)$$

with  $\Gamma$  primitive. The Möbius function  $\mu(n)$  is defined by:  $\mu(1) = 1$ ; if  $n > 0$  with prime decomposition  $n = p_1^{a_1} \dots p_k^{a_k}$ , then  $\mu(n) = (-1)^k$ , if  $a_i = 1$  for  $i = 1, \dots, k$ ; and  $\mu(n) = 0$  otherwise.

At a generic point of the walls, only the central charges of two non-parallel primitive charge vectors  $\Gamma_1$  and  $\Gamma_2 \in L$  align. We denote the chambers on either site of the wall by  $\mathcal{C}_A$  and  $\mathcal{C}_B$ . To determine the change of the BPS-indices between  $\mathcal{C}_A$  and  $\mathcal{C}_B$ , one can truncate the product (2.5) to the lattice generated by  $\Gamma_1$  and  $\Gamma_2$ . The product then becomes

$$\prod_{\frac{m}{n} \text{ decreasing}} T_{(m,n)}^{\Omega((m,n); t_A)} = \prod_{\frac{m}{n} \text{ increasing}} T_{(m,n)}^{\Omega((m,n); t_B)}, \quad (2.8)$$

where  $(m, n) = m\Gamma_1 + n\Gamma_2$ . Using the Baker–Campbell–Hausdorff formula

$$e^{tX} e^{tY} = e^{tY} e^{t^2[X, Y]} e^{\frac{1}{2}t^3(\text{ad } X)^2 Y} e^{\frac{1}{2}t^3(\text{ad } Y)^2 X} e^{-\frac{1}{4}t^4[X, [Y, [X, Y]]]} \dots e^{tX}, \quad (2.9)$$

with  $(\text{ad } X)Y = [X, Y]$  and  $t \in \mathbb{R}$ ,  $\Delta\Omega(m\Gamma_1 + n\Gamma_2; t)$  can be determined in principle. For  $(m, n) = (1, 1)$  one finds the well-known formula

$$\Delta\Omega(\Gamma; t) = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} \langle \Gamma_1, \Gamma_2 \rangle \Omega(\Gamma_1; t) \Omega(\Gamma_2; t), \quad (2.10)$$

where we assumed that  $\langle \Gamma_1, \Gamma_2 \rangle > 0$  and  $\text{Im}(Z(\Gamma_1)\bar{Z}(\Gamma_2)) > 0$  in  $\mathcal{C}_B$ ;  $\mathcal{C}_B$  is thus the stable chamber. A (product) formula is known for semi-primitive wall-crossing  $(m, n) = (1, n)$  from supergravity [15], which is consistent with equation (2.5). Equation (3.38) of Section 3 gives a similar formula, which is adapted for wall-crossing of D4-D2-D0 BPS-states in the large volume limit.

The first example of proper non-primitive wall-crossing is for  $(m, n) = (2, 2)$ . The KS-formula is now the only tool to compute the change in the index across a wall. To present the result, it is useful to use nested lists like  $((\Gamma_1, \Gamma_2), ((\Gamma_3, \Gamma_4), \Gamma_5))$ , which also play a large role in the discussion on flow trees in Subsection 2.2. We define the following numbers:

$$\bar{\Omega}(\Gamma_1, \Gamma_2; t) = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} \langle \Gamma_1, \Gamma_2 \rangle \bar{\Omega}(\Gamma_1; t) \bar{\Omega}(\Gamma_2; t), \quad (2.11)$$

which carries on to more complicated lists. For example the nested list  $((\Gamma_1, \Gamma_2), \Gamma_3)$  leads to:

$$\begin{aligned} \bar{\Omega}(((\Gamma_1, \Gamma_2), \Gamma_3); t) &= (-1)^{\langle \Gamma_1 + \Gamma_2, \Gamma_3 \rangle + \langle \Gamma_1, \Gamma_2 \rangle} \langle \Gamma_1 + \Gamma_2, \Gamma_3 \rangle \langle \Gamma_1, \Gamma_2 \rangle \\ &\quad \times \bar{\Omega}(\Gamma_1; t) \bar{\Omega}(\Gamma_2; t) \bar{\Omega}(\Gamma_3; t). \end{aligned} \quad (2.12)$$

The jump of the index  $\Delta\Omega(2\Gamma_1 + 2\Gamma_2; t)$  depends on the indices  $\Omega(a\Gamma_1 + b\Gamma_2; t_A)$  in  $\mathcal{C}_A$  with  $a, b \in [0, 2]$ . One finds using the KS-formula:

$$\begin{aligned} \Delta\bar{\Omega}(2\Gamma_1 + 2\Gamma_2; t_A) &= \bar{\Omega}(\Gamma_1, \Gamma_1 + 2\Gamma_2; t_A) + \bar{\Omega}(2\Gamma_1, 2\Gamma_2; t_A) + \bar{\Omega}((2\Gamma_1 + \Gamma_2), \Gamma_2; t_A) \\ &\quad + \frac{1}{2}\bar{\Omega}(\Gamma_1, (\Gamma_1, 2\Gamma_2); t_A) + \frac{1}{2}\bar{\Omega}(\Gamma_2, (\Gamma_2, 2\Gamma_1); t_A) \\ &\quad + \frac{1}{2}\bar{\Omega}((\Gamma_1, \Gamma_1 + \Gamma_2), \Gamma_2; t_A) + \frac{1}{2}\bar{\Omega}((\Gamma_2, \Gamma_2 + \Gamma_1), \Gamma_1; t_A) \\ &\quad + \frac{1}{4}\bar{\Omega}((\Gamma_2, (\Gamma_1, \Gamma_2)), \Gamma_1; t_A). \end{aligned} \quad (2.13)$$

We observe that the jump  $\Delta\bar{\Omega}(2\Gamma_1 + 2\Gamma_2)$  is packaged conveniently in terms of  $\bar{\Omega}$ 's and nested lists. Flow trees are also classified by nested lists, the terms in equation (2.13) are thus naturally identified with contributions of the corresponding flow trees. The KS-formula provides the non-trivial prefactors. Subsection 3.2 comments more on this.



## 2.2 Supergravity and flow trees

At finite string coupling  $g_s$  (such that the four-dimensional Newton constant  $G_4$  is finite), BPS-states correspond to solutions of the supergravity equations of motion which preserve half of the supersymmetry. These solutions often contain various black holes with macroscopic horizons. The (Kähler) moduli appear in supergravity as massless scalars. Their values at infinity are imposed as boundary conditions. They determine the value of the central charge, and therefore also the stability of bound states. The values of the moduli are generically not constant throughout a black hole solution, but “flow” to special values determined by the electric–magnetic charge of the black hole, due to the attractor mechanism [21]. A point of concern in the attractor mechanism is the possibility of multiple basins of attraction depending on the values of the moduli at infinity [40]. Denef [11] explains how this is related to the points in moduli space where the volume of a 2-cycle of  $X$  vanishes. This paper avoids these singularities by restricting the moduli to a single Kähler cone as explained in the introduction to this section.

The  $\mathcal{N} = 2$  supergravity Lagrangian admits the action of an  $Sp(2b_2 + 2, \mathbb{Z})$  duality group [48]. The relevant subgroup in the large volume limit are the translations  $\mathbb{Z}^{b_2}$  which act by

$$\mathbf{K}(k) = \begin{pmatrix} 1 & & & & & \\ & k^a & & & & \\ & & \mathbf{1} & & & \\ & \frac{1}{2}d_{abc}k^bk^c & & d_{abc}k^c & & \mathbf{1} \\ & \frac{1}{6}d_{abc}k^ck^bk^c & & \frac{1}{2}d_{abc}k^bk^c & & k^a & 1 \end{pmatrix}, \quad k \in \mathbb{Z}^{b_2}, \quad (2.14)$$

simultaneously on the charge  $\Gamma$  and the period vector  $\Pi = (1, t^a, \frac{1}{2}d_{abc}t^bt^c, \frac{1}{6}d_{abc}t^at^bt^c)^T$ . There is in addition an  $SL(2, \mathbb{Z})$  duality group [6] which can be related to the IIB  $S$ -duality group by a timelike T-duality or the c-map.  $S$ -duality acts by fractional linear transformations on  $\tau = C_0 + \frac{i\beta}{g_s}$ , and interchanges the  $B$ - and  $C$ -fields.

A brief review is now given about multi-center supergravity solutions, before discussing attractor flow trees. The general form of the metric of a BPS multi-center solution is [11]

$$ds^2 = -e^{2U} (dt + \omega)^2 + e^{-2U} d\vec{x}^2. \quad (2.15)$$

Since we consider asymptotically flat space-times  $\lim_{r \rightarrow \infty} U$ ,  $\omega = 1$ . The evolution of the Calabi–Yau periods in a single center solution is such that

$$2\text{Im} \left( e^{-U - i\alpha} Z(\Gamma', t) \right) = \sqrt{G_4} \frac{\langle \Gamma, \Gamma' \rangle}{r} + 2\text{Im} \left( e^{-i\alpha} Z(\Gamma', t) \right)_{r=\infty}, \quad (2.16)$$

for every charge  $\Gamma' \in L$ ;  $\alpha$  is the phase of  $Z(\Gamma, t)$  [11]. In principle one can solve for the evolution of the periods and moduli from this equation. The evolution is often described in terms of the flow parameter  $\rho = \sqrt{G_4}/2r$ .

More interesting for discussions about stability are solutions with more centers. Denef [11] shows that the distance between two centers in a 2-center solution is given by:

$$|x_1 - x_2| = \sqrt{G_4} \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{|Z(\Gamma_1 + \Gamma_2, t)|}{\text{Im}(Z(\Gamma_1, t)\bar{Z}(\Gamma_2, t))}, \quad (2.17)$$

where the moduli  $t$  are evaluated at  $r = \infty$ . The right-hand side can be positive or negative depending on the values of the moduli at infinity. A negative value indicates that the BPS-states do not exist at this point of the moduli space, or in other words that they are unstable. On the other hand, positivity does not imply stability, since it is not a sufficient condition for the existence of a full solution to the supergravity equations of motion. For example, solutions where the central charge vanishes at a regular point of the moduli space should be disregarded. If we assume that this does not happen, and the existence of the solution depends only on the sign of the right-hand side of equation (2.17), the contribution to the index of the 2-center solution as function of the moduli can be written as [14, 15, 36]:

$$\begin{aligned} & \frac{1}{2} \left( \text{sgn}(\text{Im}(Z(\Gamma_1, t)\bar{Z}(\Gamma_2, t))) + \text{sgn}(\langle \Gamma_1, \Gamma_2 \rangle) \right) \\ & \times (-1)^{\langle \Gamma_1, \Gamma_2 \rangle - 1} \langle \Gamma_1, \Gamma_2 \rangle \Omega(\Gamma_1)\Omega(\Gamma_2), \end{aligned} \quad (2.18)$$

with  $\text{sgn}(x)$  defined by

$$\text{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \quad (2.19)$$

Since close to the wall of marginal stability, the supergravity solution will always resemble a 2-center solution this is consistent with equation (2.10). Note that equation (2.18) gives a non-zero contribution at the wall.

Using that  $e^{-U} \rightarrow \sqrt{G_4}|Z(\Gamma, t)|/r$  for  $r \rightarrow 0$ , one finds from equation (2.16) that the attractor equations are equivalent to

$$\text{Im}(Z(\Gamma, t(\Gamma))\bar{Z}(\Gamma', t(\Gamma))) = -\langle \Gamma', \Gamma \rangle \quad (2.20)$$

for every  $\Gamma' \in L$ . One observes from this equation that if the moduli at infinity are fixed at the attractor point  $t(\Gamma)$ , the right-hand side of equation (2.17) can never be positive, and therefore 2-center solutions can not exist.

To understand all the implications of the supergravity viewpoint to BPS-stability, one needs to study solutions with more centers, which becomes quite complicated. Fortunately, the split attractor flow conjecture [11, 15] proposes a rather elegant framework for analyzing the stability of multi-center solutions as function of the background moduli. The conjecture has on the other hand not much bearing on those multi-center solutions, whose stability does not depend on the moduli. The mysterious scaling solutions lie in this class [15]. The conjecture does not distinguish such solutions from single center solutions. We briefly review the conjecture at this point, following [12, 15].

The central objects of the conjecture are the so-called “(attractor) flow trees”, which are simplified, schematic representations of supergravity solutions. An example of a flow tree is presented in figure 1. Its graph is a rooted tree (meaning a directed tree with all edges directed away from the root vertex, see e.g. [17]), and corresponds to a nested list of the total charge  $\Gamma$ .

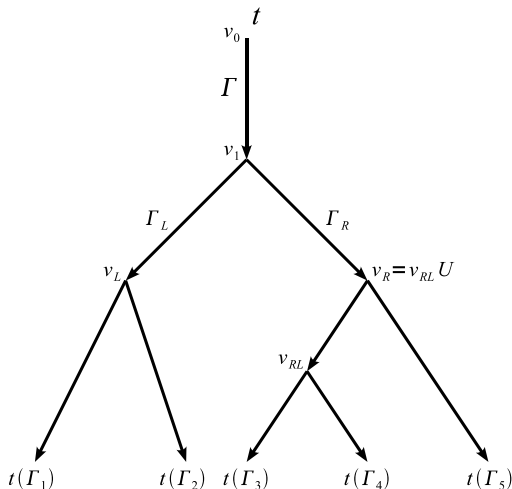


Figure 1: The attractor flow tree corresponding to  $((\Gamma_1, \Gamma_2), ((\Gamma_3, \Gamma_4), \Gamma_5))$ .

The nested list corresponding to figure 1 is  $((\Gamma_1, \Gamma_2), ((\Gamma_3, \Gamma_4), \Gamma_5))$ .<sup>1</sup> The vertices are all connected and have generically either one (the leaves) or three edges connect to it. The root vertex  $v_0$  (drawn at the top in figure 1) corresponds to the sphere at infinite radius in the supergravity solution, which surrounds the total charge  $\Gamma$ . The  $N$  bottom vertices (endpoints) represent black hole centers with charges  $\Gamma_i$ ,  $i = 1, \dots, N$  with  $\Gamma = \sum_{i=1}^N \Gamma_i$ . A tree with  $N$  bottom vertices has  $2N - 1$  edges and  $N - 1$  trivalent vertices. We denote the set of trivalent vertices by  $V$ , and the set of edges by  $E$ . The vertices, edges and charges can obviously be labeled by binary words, e.g.  $RLL$ .

It is useful to introduce some notation associated with a trivalent vertex  $v$ , for later recursive applications. A vertex which appears one vertex before  $v$  in the tree is denoted by  $vU$ . The edge between  $vU$  and  $v$  is denoted by  $e_v$ , and the charge along  $e_v$  by  $\Gamma_v$ . The charge splits at a trivalent vertex  $v$ :  $\Gamma_v = \Gamma_{vL} + \Gamma_{vR}$ ;  $\Gamma_{vL}$  goes off to the left and  $\Gamma_{vR}$  to the right.

Based on a nested list of charges, one can always construct the rooted tree. A flow tree is essentially an embedding of the rooted tree  $T$  in moduli space, which might or might not exist depending on the value  $t$  of the moduli at  $v_0$ . The flow of the moduli along an edge  $e \in E$  is given by the evolution of the periods for a single center black hole (2.16) with the corresponding charge  $\Gamma_e$ . An edge splits at a trivalent vertex  $v$  with moduli  $t_v$  into edges with charges  $\Gamma_{vL}$  and  $\Gamma_{vR}$ , only if  $t_v$  is at a wall of marginal stability for  $(\Gamma_{vL}, \Gamma_{vR})$ . If the moduli lie on the intersection of various walls of marginal stability, the valence of the vertices can increase accordingly. From equation (2.16), one deduces that the change of the flow parameter  $\Delta\rho_v = \rho_v - \rho_{vU}$  along  $e_v$  is:

$$\Delta\rho_v = \frac{\text{Im}(Z(\Gamma_{vL}, t_{vU}) \bar{Z}(\Gamma_{vR}, t_{vU}))}{\langle \Gamma_{vL}, \Gamma_{vR} \rangle |Z(\Gamma_{vL} + \Gamma_{vR}, t)|}. \quad (2.21)$$

The flows terminate at the bottom vertices, where they are at the corresponding attractor points  $t(\Gamma_i)$ .

A flow tree can now be defined more precisely. Given a choice  $t$  of moduli at  $v_0$ , a flow tree is a rooted tree  $T$ , which satisfies the following (stability) conditions [11]:

- A:  $\forall v \in V$  :  $\langle \Gamma_{vL}, \Gamma_{vR} \rangle \text{Im}(Z(\Gamma_{vL}, t_{vU}) \bar{Z}(\Gamma_{vR}, t_{vU})) > 0$ .
- B:  $\forall v \in V$  :  $Z(\Gamma_{vL}, t_v) \bar{Z}(\Gamma_{vR}, t_v) > 0$ .
- C: for  $i = 1, \dots, N$  : the attractor points  $t(\Gamma_i)$  do exist in the moduli space.

---

<sup>1</sup>For notational convenience, the  $\Gamma$ 's, comma's and outer parentheses are in the following omitted from the nested lists, thus  $((\Gamma_1, \Gamma_2), ((\Gamma_3, \Gamma_4), \Gamma_5)) \rightarrow (12)((34)5)$ .

Conditions A and B together imply that  $v$  lies at a wall of marginal stability. Condition A is also equivalent with the positivity of the flow parameter  $\Delta\rho_v$  (2.21) along  $e_v$ . Since it is a measure for the (inverse) length of the edge, the condition is an obvious necessary condition for the existence of a supergravity solution. After all this introductory material the attractor flow conjecture can be stated:

*Split attractor flow conjecture* [15]:

1. components of the moduli space of (four-dimensional) supergravity solutions with total charge  $\Gamma$  and values of the moduli at infinity  $t$ , are in 1 to 1 correspondence with flow trees starting with total charge  $\Gamma$  and moduli  $t$ ,
2. for fixed total charge  $\Gamma$  and moduli  $t$  only a finite number of flow trees exist. By 1. the Hilbert space of BPS-states factorizes into a direct sum of the corresponding flow trees.

This conjecture shows the potential of flow trees to describe the stability of BPS-states. It suggests an important role for the endpoints of the flow trees, since these BPS-objects are stable everywhere in the moduli space. As mentioned before, the endpoints do not necessarily correspond to a single center, due to the existence of scaling solutions [15]. However, the states corresponding to these endpoints cannot decay at any point in the moduli space. Following [8], we will call them “immortal” BPS-states. Since the index of an immortal object with charge  $\Gamma$  does not depend on  $t$ , we simply denote it by  $\Omega(\Gamma)$ . The immortal BPS-objects can thus be found by tuning the moduli to the corresponding attractor point. In agreement with this, only the  $N = 1$  tree exists if  $t = t(\Gamma)$ . A convenient aspect of the immortal BPS-objects is that more is known about their microscopic aspects, their degrees of freedom are typically those of a conformal field theory, which adds many symmetries to the problem.

Whether Condition A is satisfied for  $(T, t)$  is conveniently determined by a product formula:

**Condition A :**

$$S(T, t) = \prod_{v \in V} \frac{1}{2} \left( \text{sgn}(\text{Im}(Z(\Gamma_{vL}, t_{vU}) \bar{Z}(\Gamma_{vR}, t_{vU}))) + \text{sgn}(\langle \Gamma_{vL}, \Gamma_{vR} \rangle) \right) \neq 0. \quad (2.22)$$

The  $\frac{1}{2}$  appears in the definition of  $S(T, t)$  such that  $S(T, t)$  is  $\pm 1$  instead of  $\pm 2^{N-1}$  for flow trees. Similarly, Condition C can be reformulated as  $\prod_{i=1}^N \Omega(\Gamma_i) \neq 0$ . Thus, if one knows that Condition B is satisfied, the

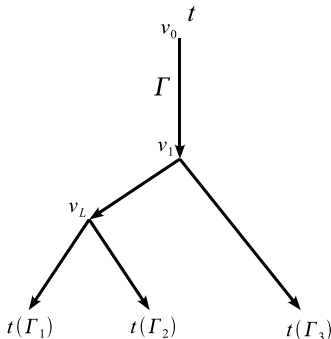


Figure 2: Flow tree  $T_{(12)3}$  corresponding to (12)3.

contribution of a flow tree to the index can be found essentially by iteration of equation (2.18). The product  $S(T, t)$  determines whether the tree corresponds to (stable) BPS-states, and the contribution of the flow tree to the index is given by the KS-formula. Some subtleties arise if multiple endpoints have equal charges; the next section will comment on this.

Much of the power of the split attractor flow conjecture lies in the possibility of recursive applications of arguments based on simple, elementary flow trees. The most elementary rooted tree is  $\lambda$ . However, verification of Condition A does not require determination of the flow of the moduli along its edges. This aspect becomes important for the rooted tree corresponding to (12)3, which is displayed in figure 2. We denote this flow tree by  $T_{(12)3}$ ; the closely related flow trees with the same total charge are  $T_{(23)1}$  and  $T_{(31)2}$ . Assuming that Condition B is satisfied, stability of the split at  $v_1$  is determined by  $\text{sgn}(I_{(1+2)3} \text{Im}(Z(\Gamma_1 + \Gamma_2, t)\bar{Z}(\Gamma_3, t)))$ , and similarly the stability of  $v_L$  by  $\text{sgn}(I_{12} \text{Im}(Z(\Gamma_1, t_1)\bar{Z}(\Gamma_2, t_1)))$ . One might think that the flow of the periods must be determined explicitly to determine  $\text{sgn}(I_{12} \text{Im}(Z(\Gamma_1, t_1)\bar{Z}(\Gamma_2, t_1)))$  in terms of  $t$ , but this follows fortunately more directly from equation (2.16). To see this, take first  $\Gamma' = \Gamma_3$  in equation (2.16), which shows that  $v_1$  corresponds to the flow parameter  $\rho_1$ :

$$\rho_1 = \frac{\text{Im}(Z(\Gamma_1 + \Gamma_2, t)\bar{Z}(\Gamma_3, t))}{\langle \Gamma_1 + \Gamma_2, \Gamma_3 \rangle |Z(\Gamma_1 + \Gamma_2 + \Gamma_3, t)|}.$$

If one now substitutes  $\rho_1$  for  $\rho = \sqrt{G_4}/2r$  and  $\Gamma' = \Gamma_1$  in equation (2.16), and uses that  $Z(\Gamma_1 + \Gamma_2, t_1) \parallel Z(\Gamma_3, t_1)$  and  $e^U > 0$ , one finds the desired

result

$$\begin{aligned} & \text{sgn} \left( \text{Im} \left( Z(\Gamma_1, t_1) \bar{Z}(\Gamma_2, t_1) \right) \right) \\ &= \text{sgn} \left( \frac{I_{(2+3)1}}{I_{(1+2)3}} \text{Im} \left( Z(\Gamma_1 + \Gamma_2, t) \bar{Z}(\Gamma_3, t) \right) + \text{Im} \left( Z(\Gamma_1, t) \bar{Z}(\Gamma_2 + \Gamma_3, t) \right) \right). \end{aligned} \quad (2.23)$$

A more symmetric way of writing this is

$$\begin{aligned} & \text{sgn} \left( \text{Im} \left( Z(\Gamma_1, t_1) \bar{Z}(\Gamma_2, t_1) \right) \right) \\ &= \text{sgn} \left( \sum_{\text{cyclic permutations of } ijk} \frac{I_{(i+j)k}}{I_{(1+2)3}} \text{Im} \left( Z(\Gamma_i, t) \bar{Z}(\Gamma_j, t) \right) \right), \end{aligned} \quad (2.24)$$

which makes more manifest that if  $\text{Im} \left( Z(\Gamma_1, t_1) \bar{Z}(\Gamma_2, t_1) \right) = 0$  all three central charges are aligned. It also shows that we have determined the stability at  $v_L$  of the two other trees  $T_{(23)1}$  and  $T_{(31)2}$ ; the only part that changes is  $I_{(1+2)3}$ . These expressions show that Condition A can be determined for any flow tree in terms of  $t$  in an algorithmic way. Note that  $T_{(12)3}$  can satisfy Condition A, while  $T_{(12)}$  does not if evaluated at  $t$ . See the discussion on page 28 and further for more details about this for D4-D2-D0 branes. If Condition B is satisfied and the splits of the charges are primitive, one can determine the contribution to the index from this flow tree:

$$\begin{aligned} \Omega((12)3; t) &= \frac{1}{4} (-1)^{I_{12} + I_{31} + I_{23}} I_{(1+2)3} I_{12} \Omega(\Gamma_1) \Omega(\Gamma_2) \Omega(\Gamma_3) \\ &\quad \times \left( \text{sgn} \left( \text{Im} \left( Z(\Gamma_1 + \Gamma_2, t) \bar{Z}(\Gamma_3, t) \right) \right) + \text{sgn} \left( I_{(1+2)3} \right) \right) \\ &\quad \times \left( \text{sgn} \left( \text{Im} \left( Z(\Gamma_1, t_1) \bar{Z}(\Gamma_2, t_1) \right) \right) + \text{sgn} \left( I_{12} \right) \right). \end{aligned} \quad (2.25)$$

The contribution of a tree with non-primitive splits has probably a very similar structure. The analysis of Subsections 2.1 and 3.2 suggests that the  $\Omega$ 's should be replaced by  $\bar{\Omega}$ 's and that a non-trivial overall factor might appear.

These generic and exact expressions are useful to make generic statements about attractor flow trees. A non-trivial question is for example whether the indices based on attractor flow trees only jump when walls of marginal stability for the total charge  $\Gamma$  are crossed, and not when something non-trivial happens for the subcharges at the relevant trivalent vertices. This is of course required by physical arguments, although not completely obvious for flow trees. Denef and Moore [15] shows that this is indeed the case in several concrete examples with  $N = 3$ . Using equation (2.24) one can show that for  $N = 3$ , the interplay between the three trees  $T_{(12)3}$ ,  $T_{(23)1}$  and  $T_{(31)2}$

is such that the index does not change when the stability of the splits at  $v_{L,R}$  changes. Equation (2.24) shows that  $\Omega((12)3; t)$  can jump, if

$$\text{sgn} \left( \sum_{\text{cyclic permutations of } ijk} I_{(i+j)k} \text{Im} (Z(\Gamma_i, t) \bar{Z}(\Gamma_j, t)) \right) \quad (2.26)$$

goes from  $\pm 1$  to  $\mp 1$  via 0. This is not necessarily a wall of marginal stability for  $\Gamma = \sum_{i=1}^3 \Gamma_i$ . However, the contributions to the index of the trees  $T_{(23)1}$  and  $T_{(31)2}$ , respectively  $\Omega((23)1; t)$  and  $\Omega((31)2; t)$ , are very similar to  $\Omega((12)3; t)$ . In particular, they also contain a factor (2.26) and will thus also jump when  $\Omega((12)3; t)$  does. To show that  $\Omega(\Gamma; t)$  does not jump, we have to show that the coefficient of the term (2.26) in  $\Omega((12)3; t) + \Omega((23)1; t) + \Omega((31)2; t)$  is zero, if (2.26) is zero. One can show that if (2.26) vanishes,  $I_{(1+2)3} \text{Im}(Z(\Gamma_1 + \Gamma_2, t) \bar{Z}(\Gamma_3, t))$  and the cyclic permutations have all the same sign; this is generically true in a neighborhood of the hypersurface where (2.26) is zero. Since  $\sum_{\text{cyclic permutations of } ijk} I_{(i+j)k} I_{ij} = 0$ , the coefficient of (2.26) thus vanishes. Note that it is very important here that the stability of the subtree is evaluated at  $v_1$  and not at  $v_0$ . This result for  $N = 3$  can be applied inductively. Thus the index determined by attractor flow trees does only jump when walls for the total charge are crossed.

This derivation essentially ignored Condition B. More precisely put, it assumes that if one of the trees, say  $T_{(12)3}$ , exists as flow tree at some point in moduli space, it cannot be true that Condition B is not satisfied for  $T_{(23)1}$ , if Conditions A and C are satisfied (and similarly for  $T_{(31)2}$ ). To argue that this is correct, assume that this could be the case, and that at least one of the splits of  $T_{(23)1}$  is a wall of anti-marginal stability. If the moduli are then moved to the point where (2.26) vanishes,  $T_{(12)3}$  implies that the three central charges align for  $t_1$ , whereas  $T_{(23)1}$  implies that some will anti-align, which is a contradiction.

Another application of equation (2.23) is the analysis of walls of threshold stability, these are walls in moduli space where the central charges of say  $\Gamma_L$  and  $\Gamma_R$  get aligned, with  $\langle \Gamma_L, \Gamma_R \rangle = 0$ . For  $N = 3$ , this is for example  $\langle \Gamma_1 + \Gamma_2, \Gamma_3 \rangle = 0$  or a cyclic permutation. Specific examples of such cases are discussed in Ref. [5].

### 3 D4-D2-D0 BPS-states

This section applies the generic discussion of the previous section to D4-D2-D0 BPS-states. One of the aims is to construct a BPS partition function



which correctly captures the moduli dependence. The BPS partition function of  $\mathcal{N} = 2$  supergravity in the mixed ensemble [41] takes the following approximate form:

$$\mathcal{Z}(\tau, C, t) = \sum_{Q_A} \Omega(\Gamma; t) \exp\left(-2\pi \frac{\beta}{g_s} |Z(\Gamma, t)| + 2\pi i C^A Q_A\right),$$

where  $A = 0, \dots, b_2$ . We will use  $\beta/g_s = \tau_2$  and  $C^0 = \tau_1$  in the following. Part 2 of the split attractor flow conjecture suggests the decomposition of the partition function by rooted trees  $T$ :

$$\mathcal{Z}(\tau, C, t) = \sum_{T \in \mathcal{T}_P} \mathcal{Z}_T(\tau, C, t). \quad (3.1)$$

In contrast to the previous section, a rooted tree  $T$  in this sum corresponds to a nested list of magnetic charges  $P_i^A$  with the electric charge unspecified;  $\mathcal{T}_P$  is the total set of trees based on nested lists of magnetic charge vectors  $P^A$  with  $\sum_{i=1}^N P_i^A = P^A$ . The partition function enumerates all possible distributions of electric charge over the endpoints of these rooted trees, and determines as function of  $t$  whether they correspond to actual flow trees and contribute to the index. This section will always use trees in this sense. Thus  $T_{(11)}$  is a tree with equal magnetic charge vectors associated to the endpoints, which can still have a non-zero contribution to the index depending on the electric charges.

To proceed, we make two simplifications:

1.  $P^0 = 0$ , such that there is no netto D6-brane charge. The reason for this simplification is that the microscopic description is much better understood for immortal BPS-objects with  $P^0 = 0$  than for  $P^0 \neq 0$  by a lift to M-theory [35]. The near-horizon geometry of the resulting black string is  $\text{AdS}_3 \times S^2$  and the degrees of freedom combine to a two-dimensional  $\mathcal{N} = (4, 0)$  conformal field theory [39].
2.  $J \rightarrow \infty$ , which is the large volume limit of the Kähler moduli space. In this limit, quantum effects to the geometry do not play a role such that (relatively) basic geometric arguments generally suffice. The D-branes are well described in this limit as coherent sheaves on subspaces of  $X$ .

In the large volume limit the magnetic charge  $P$  (or equivalently the divisor wrapped by the D4-branes) must be positive, since it represents the support of a coherent sheaf. The BPS-states with  $P^0 = 0$ , which correspond to a single  $\text{AdS}_3$  throat in 5 dimensions (or equivalently M5-brane), appear in 4 dimensions as single centered or as multi-centered supergravity solutions.

In particular, BPS-states corresponding to the principal or polar terms in the partition function appear as bound states of D6 and anti-D6 branes [15]. When the moduli are varied such bound states might in principal decay. However this cannot happen in the large volume limit  $J \rightarrow \infty$ . Boer et al. [5] show that for  $t^a = \lim_{\lambda \rightarrow \infty} D^{ab}Q_b + i\lambda P^a$ , with  $D_{ab} = d_{abc}P^c$ , an uplift to 5 dimensions leads to only a single AdS<sub>3</sub> throat. Since in the limit  $\lambda \rightarrow \infty$  the dependence on  $\lambda$  disappears, this limit is closely related to the attractor point for D4-D2-D0 black holes, which is:  $t(\Gamma) = D^{ab}Q_b + i\sqrt{\hat{Q}_0/P^3}P^a$  ( $\hat{Q}_0$  is defined in the next subsection). These findings are consistent with the results in [36], where an analysis of the partition function showed that for  $t = \lim_{\lambda \rightarrow \infty} D^{ab}Q_b + i\lambda P^a$ ,  $\Omega(\Gamma; t)$  equals the CFT index.

Based on these considerations, one could state that the CFT states are those BPS-states in 4 dimensions, which cannot decay in the large volume limit. Since we will work exclusively in the large volume limit, we will use the word “immortal” for the objects which cannot decay in this limit and omit the  $t$ -dependence of the index:  $\Omega(\Gamma)$ . These immortal objects form of course a bigger class than the objects which are immortal in the whole moduli space. Note that different electric charges correspond to different attractor points:  $\Omega(\Gamma'; t(\Gamma))$  does not correspond to  $\Omega(\Gamma')$  generically.

### 3.1 BPS mass and stability

The form of the partition function shows that its convergence is essentially determined by properties of the mass  $|Z(\Gamma, t)|$  and of the indices  $\Omega(\Gamma; t)$ . The contribution to the partition function of a flow tree with a single endpoint is known to be convergent by CFT arguments. However, it is not evident that the contributions of flow trees with more endpoints always lead to convergent partition functions. This subsection proves that this is the case for flow trees with 1, 2 and 3 endpoints with D4-brane charge, which gives strong evidence that this will continue to hold for  $N > 3$ .

The central charge  $Z(\Gamma; t)$  is for  $J \rightarrow \infty$  given by

$$Z(\Gamma, t) = - \int_{\mathbf{X}} e^{-t} \wedge \Gamma.$$

The real and imaginary parts of  $Z(\Gamma, t)$  for D4-D2-D0 BPS-states are

$$\begin{aligned} \text{Re}(Z(\Gamma, t)) &= \frac{1}{2}P \cdot (J^2 - B^2) + Q \cdot B - Q_0, \\ \text{Im}(Z(\Gamma, t)) &= (Q - BP) \cdot J, \end{aligned} \tag{3.2}$$

where the triple intersection product  $d_{abc}$  is used to contract vectors. For  $P \cdot J^2 \gg |(Q - \frac{1}{2}B) \cdot B - Q_0|, |(Q - BP) \cdot J|$ , the mass takes the form:

$$|Z(\Gamma, t)| = \frac{1}{2}P \cdot J^2 + (Q - \frac{1}{2}BP) \cdot B - Q_0 + (Q - B)_+^2, \quad (3.3)$$

where terms of  $\mathcal{O}(J^{-2})$  are omitted. Note that at the attractor point  $t(\Gamma)$ ,  $J$  is never sufficiently large such that equation (3.3) is a valid approximation for  $|Z(\Gamma, t(\Gamma))|$ . The charges  $Q_a$  naturally take values in the lattice  $\Lambda^*$ , dual to  $\Lambda$  which has quadratic form  $D_{ab} = d_{abc}P^c$  and signature  $(1, b_2 - 1)$  by the Hodge index theorem [29].  $Q_+^2 = \frac{(Q \cdot J)^2}{P \cdot J^2}$  is the projection to a positive definite subspace of  $\Lambda \otimes \mathbb{R}$  parametrized by  $j = J/|J|$ . The positive definite combination  $2Q_+^2 - Q^2 = Q_+^2 - Q_-^2$  is called the majorant associated to  $j$ . Two expressions which are invariant under the action of  $\mathbf{K}(k)$  (2.14) are  $\hat{Q}_{\bar{0}} = -Q_0 + \frac{1}{2}Q^2$  and  $Q_a - d_{abc}B^bP^c$ .

Expression (3.3) is potentially problematic, since  $|Z(\Gamma, t)| - \frac{1}{2}P \cdot J^2$  is not obviously bounded below. This would therefore allow the possibility that addition of electric charge can result in a decrease of the mass, which is clearly unphysical. This would also have the direct consequence that if such states are part of the spectrum, the partition function (3.1) with the electric charges in the canonical ensemble is not convergent, independent of the growth of the index (except that it is non-zero).

To explain the problem more concretely, we consider a rooted tree with  $N$  endpoints, with (possibly non-primitive) charges  $\Gamma_i, i = 1, \dots, N$ . To every endpoint a lattice  $\Lambda_i$  with quadratic form  $D_i = d_{abc}P_i^c$  is associated. By a slight abuse of notation, we use  $P = (P_1, P_2, \dots, P_N) \in \Lambda_1 \oplus \Lambda_2 \oplus \dots \oplus \Lambda_N$  in addition to  $P = \sum_i^N P_i \in \Lambda$ ; and similarly for  $Q = (Q_1, Q_2, \dots, Q_N) \in \Lambda_1^* \oplus \Lambda_2^* \oplus \dots \oplus \Lambda_N^*$ . Using the duality invariant expressions one can write the mass as

$$\frac{1}{2}P \cdot J^2 + (Q - B)_+^2 + \sum_{i=1}^N \hat{Q}_{\bar{0},i} - \frac{1}{2}(Q_i - BP_i)_i^2. \quad (3.4)$$

The attractor endpoints only exist for  $\hat{Q}_{\bar{0},i} \geq -c_{R,i}/24 = -(P_i^3 + c_2(X) \cdot P_i)/24$ , where  $c_{R,i}$  are the CFT central charges of the endpoints [35]. The problem is thus reduced to the fact that the quadratic form  $(Q - B)_+^2 - \sum_{i=1}^N \frac{1}{2}(Q_i - BP_i)_i^2$  is indefinite with signature  $(Nb_2 - N + 1, N - 1)$ .

However, this section will show that it is positive definite if Condition A is satisfied:

$$\text{Condition A} \quad \implies \quad (Q - B)_+^2 - \sum_{i=1}^N \frac{1}{2} (Q_i - BP_i)_i^2 \geq 0, \quad (3.5)$$

thus it is in particular always positive definite for flow trees.

To this end, we start by taking a closer look at Condition A for these BPS-states. From equation (3.2) is clear that the central charge gets aligned along the positive real axis of the  $\mathbb{C}$ -plane for  $J \rightarrow \infty$ , the infinitesimal angle with the real axis can nevertheless vary, which leads to interesting wall-crossing phenomena. For a split  $(\Gamma_1, \Gamma_2)$ ,  $I_{12} \text{Im}(Z(\Gamma_1, t)\bar{Z}(\Gamma_2, t)) \geq 0$  becomes for  $J \rightarrow \infty$  and constituent charges  $\Gamma_1 = (0, P_1, Q_1, Q_{0,1})$  and  $\Gamma_2 = (0, P_2, Q_2, Q_{0,2})$ :

$$I_{12} (P_1 \cdot J^2 (Q_2 - BP_2) \cdot J - P_2 \cdot J^2 (Q_1 - BP_1) \cdot J) \leq 0, \quad (3.6)$$

where only the leading order in  $J$  is kept. Note that for this approximation no walls of marginal stability exist for Calabi–Yaus with  $b_2 = 1$ . The stability condition is invariant under rescalings of  $J$ :  $B + iJ \rightarrow B + i\lambda J$  with  $\lambda > 0$ . The space of variations of equation (3.7) due to  $J$  has therefore  $b_2 - 1$  dimensions, and is essentially a real projective space. Similarly, variations of  $B$  which are proportional to  $J$  do not change the stability condition. Thus the total space of stability conditions in the case of interest has real dimension  $2(b_2 - 1)$ . Since equation (3.6) is either  $\pm\infty$  or 0 for  $J \rightarrow \infty$ , we define a homogeneous function of degree 0:

$$\mathcal{I}(\Gamma_1, \Gamma_2; t) = \frac{P_1 \cdot J^2 (Q_2 - BP_2) \cdot J - P_2 \cdot J^2 (Q_1 - BP_1) \cdot J}{\sqrt{P_1 \cdot J^2 P_2 \cdot J^2 P \cdot J^2}}. \quad (3.7)$$

This has the special property that

$$\mathcal{I}(\Gamma_1, \Gamma_2; t)^2 = |Z(\Gamma_1, t)| + |Z(\Gamma_2, t)| - |Z(\Gamma, t)|.$$

Equation (3.6) is reminiscent of the stability condition for sheaves on surfaces, but already when subleading powers in  $J$  are taken into account, the equivalence between D-branes and coherent sheaves disappears [16]. Note that for  $P_2 = \vec{0}$ , the wall of marginal stability is given by  $Q_2 \cdot J = 0$ . In case  $P_2 = \vec{0}$ ,  $Q_2$  must be a positive vector in the large volume limit, since it represents the support of a coherent sheaf. Therefore,  $Q_2 \cdot J$  lies at the boundary of the Kähler cone, and such walls are not crossed, since we restrict ourselves to the Kähler cone. The assumption that the  $P_i$  are positive for

every endpoint, as was assumed in writing equation (3.4), is thus consistent with the restriction to this regime of the moduli space.

For a rooted tree, Condition A can be verified by the product  $S(T, t)$ , which can be determined iteratively using equation (2.23). To determine the contribution to the partition function of a rooted tree, also Conditions B and C on page 12 should be verified. The existence of the attractor point of all endpoints (Condition C) is determined by the CFT partition functions, the attractor point exists if  $\hat{Q}_{\bar{0},i} \geq -c_{R,i}/24$  (note again that for  $\hat{Q}_{\bar{0}} < 0$  multicenter solutions are required, but they cannot decay in the large volume limit). Finally, Condition B is essentially assumed by neglecting the lower orders in  $J$  to the stability condition:  $\text{Re}(Z(\Gamma, t)) \approx \frac{1}{2}P \cdot J^2 \gg 0$ . Alternatively, one can estimate the flow of the moduli as in [1], to see that in the very large volume limit the central charges will never be anti-parallel at the vertices.

The remaining part of this subsection will proof implication (3.5) for trees with 1, 2 and 3 endpoints, and comment briefly on  $N > 3$ . Also the contributions to the partition functions of these trees are discussed.

### 3.1.1 One endpoint

This case is trivial, since the potentially harmful term can be rewritten as

$$(Q - B)_+^2 - \frac{1}{2}(Q - B)^2 = \frac{1}{2}(Q - B)_+^2 - \frac{1}{2}(Q - B)_-^2, \quad (3.8)$$

which is positive definite on  $\Lambda$ . Before moving on to  $N = 2$ , a couple properties of the partition function for  $N = 1$  are reviewed. The partition function  $\mathcal{Z}_{T_1}(\tau, C, t)$  can be written in the following form:

$$\begin{aligned} \mathcal{Z}_{T_1}(\tau, C, t) = \sum_{Q_0, Q} \Omega(P, Q, Q_0) (-1)^{P \cdot Q} e^{(-\bar{\tau}(-Q_0 + Q^2/2) + \tau(Q - B)_+^2/2} \\ + \bar{\tau}(Q - B)_-^2/2 + C \cdot (Q - B/2)), \end{aligned}$$

where the leading term to the mass in (3.3) is omitted since it leads to a modular invariant overall factor. The lower bound of the mass together with the expected growth of the index imply that the series is convergent.

The CFT, which describes the degrees of freedom of immortal objects in the large volume limit, contains a spectral flow symmetry, which implies

that the indices  $\Omega(P, Q, Q_0)$  only depend on  $\hat{Q}_0 = -Q_0 + \frac{1}{2}Q^2$ , and the representative  $\mu$  of  $Q - \frac{1}{2}P^2$  in the coset  $\Lambda^*/\Lambda$  [4, 24]. This symmetry is also a well-known property of the dual supergravity in AdS<sub>3</sub> [5]. Modularity and spectral flow furthermore imply that the CFT elliptic genus can be decomposed in a theta function and a vector-valued modular form  $h_{P, Q - \frac{1}{2}P}(\tau)$  [4, 24]:

$$h_{P, Q - \frac{1}{2}P}(\tau) = \sum_{Q_0} \Omega(P, Q, Q_0) q^{-Q_0 + \frac{1}{2}Q^2}, \quad (3.9)$$

which satisfy the special property that  $h_{P, Q - \frac{1}{2}P}(\tau) = h_{P, Q - \frac{1}{2}P + k}(\tau)$  with  $k \in \Lambda$ . The definition (3.9) can be found in the existing literature, however Subsection 3.2 gives evidence for replacing the integer coefficients  $\Omega(P, Q, Q_0)$  by the rational coefficients  $\bar{\Omega}(P, Q, Q_0)$  for compatibility with  $S$ -duality.

### 3.1.2 Two endpoints

This case is dealt with by [36]. The potentially problematic term is in this case

$$(Q - B)_+^2 - \frac{1}{2}(Q_1 - B)_1^2 - \frac{1}{2}(Q_2 - B)_2^2. \quad (3.10)$$

To prove that this quantity is positive definite if  $S(T_{12}, t) \neq 0$  is satisfied, we can replace  $Q_i - BP_i$  by  $Q_i$  without loss of generality. We proceed by writing the quantities in equations (3.6) and (3.10) in terms of vectors in  $(\Lambda_1 \oplus \Lambda_2) \otimes \mathbb{R}$ , such that we can apply techniques of [27, 51]. Define the unit vectors  $\mathcal{J}_2$ ,  $\mathcal{P}_{12}$  and  $s_{12} \in (\Lambda_1 \oplus \Lambda_2) \otimes \mathbb{R}$  by

$$\begin{aligned} \mathcal{J}_2 &= \frac{(J, J)}{\sqrt{(P_1 + P_2) \cdot J^2}}, & \mathcal{P}_{12} &= \frac{(-P_2, P_1)}{\sqrt{(P_1 + P_2)P_1P_2}}, \\ s_{12} &= \frac{(-P_2 \cdot J^2 J, P_1 \cdot J^2 J)}{\sqrt{(P_1 + P_2) \cdot J^2 P_1 \cdot J^2 P_2 \cdot J^2}}. \end{aligned} \quad (3.11)$$

Innerproducts of these vectors with  $Q = (Q_1, Q_2)$  give the familiar quantities in  $S(T_{12})$ :  $\mathcal{P}_{12} \cdot Q = I_{12}/\sqrt{PP_1P_2}$  and  $s_{12} \cdot Q = \mathcal{I}(\Gamma_1, \Gamma_2, iJ)$ . These vectors satisfy:

**Proposition 3.1.**

$$s_{12} \cdot \mathcal{J}_2 = 0, \quad \mathcal{J}_2 \cdot \mathcal{P}_{12} = 0, \quad s_{12} \cdot \mathcal{P}_{12} \geq 1. \quad (3.12)$$

*Proof.* The first two identities follow trivially. It is straightforward to show that the third identity is positive. To show that it is  $\geq 1$ , notice that the lattice  $\Lambda_1 \oplus \Lambda_2$  has signature  $(2, 2b_2 - 2)$ . The three vectors  $\mathcal{J}_2$ ,  $\mathcal{P}_{12}$  and  $s_{12}$

---

<sup>2</sup>The shift by  $\frac{1}{2}P$  arises since  $Q$  is valued in the shifted lattice  $\Lambda^* + \frac{1}{2}P$  [23, 38].

are positive definite and since  $\mathcal{J}_2$  is orthogonal with  $s_{12}$  and  $\mathcal{P}_{12}$ , they span a lattice with signature  $(2, 1)$  if they are all linearly independent. Therefore,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & s_{12} \cdot \mathcal{P}_{12} \\ 0 & s_{12} \cdot \mathcal{P}_{12} & 1 \end{vmatrix} < 0,$$

which is equivalent to  $s_{12} \cdot \mathcal{P}_{12} \geq 1$ , where equality only holds if  $s_{12} = \mathcal{P}_{12}$ .  $\square$

In terms of these vectors, the claim becomes:

**Proposition 3.2.** *For  $Q = (Q_1, Q_2) \in \Lambda_1^* \oplus \Lambda_2^*$ ,  $\text{sgn}(s_{12} \cdot Q) - \text{sgn}(\mathcal{P}_{12} \cdot Q) \neq 0$  implies*

$$(Q_1)_1^2 + (Q_2)_2^2 - (Q \cdot \mathcal{J}_2)^2 < 0. \quad (3.13)$$

*Proof.* We can assume that  $\mathcal{P}_{12}$  and  $s_{12}$  are linearly independent, since otherwise  $\text{sgn}(s_{12} \cdot Q) - \text{sgn}(\mathcal{P}_{12} \cdot Q) = 0$ . Therefore,  $Q$ ,  $\mathcal{J}_2$ ,  $\mathcal{P}_{12}$  and  $s_{12}$  span generically a subspace of  $\Lambda_1 \oplus \Lambda_2$  with signature  $(2, 2)$ , or else  $Q$  is a linear combination of  $\mathcal{J}_2$ ,  $\mathcal{P}_{12}$  and  $s_{12}$ . Therefore,

$$\begin{vmatrix} Q^2 & Q \cdot \mathcal{J}_2 & Q \cdot \mathcal{P}_{12} & Q \cdot s_{12} \\ Q \cdot \mathcal{J}_2 & 1 & 0 & 0 \\ Q \cdot \mathcal{P}_{12} & 0 & 1 & s_{12} \cdot \mathcal{P}_{12} \\ Q \cdot s_{12} & 0 & s_{12} \cdot \mathcal{P}_{12} & 1 \end{vmatrix} \geq 0,$$

which is equivalent to

$$Q^2 - (Q \cdot \mathcal{J}_2)^2 \leq \frac{(Q \cdot \mathcal{P}_{12})^2 + (Q \cdot s_{12})^2 - 2Q \cdot \mathcal{P}_{12} Q \cdot s_{12} s_{12} \cdot \mathcal{P}_{12}}{1 - (s_{12} \cdot \mathcal{P}_{12})^2}.$$

Since  $\text{sgn}(s_{12} \cdot Q) - \text{sgn}(\mathcal{P}_{12} \cdot Q) \neq 0$  implies  $Q \cdot \mathcal{P}_{12} Q \cdot s_{12} \leq 0$ , the proposition follows.  $\square$

Before we continue with  $N = 3$ , we elaborate a bit more on the contribution of  $N = 2$  flow trees to the partition function. To construct the partition function, first the contribution of the flow tree to the index must be determined. We assume here that the magnetic vectors are primitive, such that the primitive wall-crossing formula can be used. Subsection 3.2 comments on the implications of non-primitive wall-crossing for the partition function.

Since the D0-brane charges  $Q_{0,i}$  do not appear in the stability condition, the derivation of the jump becomes somewhat more complicated. To

determine the change between two adjacent chambers  $\mathcal{C}_A$  and  $\mathcal{C}_B$ , the spectrum can be truncated to states with charges  $\Gamma_1 = (P_1, Q_1, Q_{0,1})$ ,  $\Gamma_2 = (P_2, Q_2, Q_{0,2})$  and  $\Gamma = (P, Q, Q_0)$  with  $(P_1, Q_1) + (P_2, Q_2) = (P, Q)$ . Here the  $(P_i, Q_i)$  are kept fixed, but the  $Q_{0,i}$  are not since the wall is independent of  $Q_{0(i)}$ . Equation (2.5) can thus be truncated to

$$\prod_{Q_{0,1}} T_{\Gamma_1}^{\Omega(\Gamma_1)} \prod_{Q_0} T_{\Gamma}^{\Omega(\Gamma; t_A)} \prod_{Q_{0,2}} T_{\Gamma_2}^{\Omega(\Gamma_2)} = \prod_{Q_{0,2}} T_{\Gamma_2}^{\Omega(\Gamma_2)} \prod_{Q_0} T_{\Gamma}^{\Omega(\Gamma; t_B)} \prod_{Q_{0,1}} T_{\Gamma_1}^{\Omega(\Gamma_1)}. \quad (3.14)$$

The Lie algebra elements  $e_{\Gamma}$  are central. Using the Baker–Campbell–Hausdorff formula for this algebra  $e^X e^Y = e^Y e^{[X,Y]} e^X$ , one can derive that the change in the index across the wall is:

$$\begin{aligned} \Delta\Omega(\Gamma; t_A \rightarrow t_B) &= (-1)^{P_1 \cdot Q_2 - P_2 \cdot Q_1 - 1} (P_1 \cdot Q_2 - P_2 \cdot Q_1) \\ &\quad \times \sum_{Q_{0,1} + Q_{0,2} = Q_0} \Omega(\Gamma_1; t_A) \Omega(\Gamma_2; t_B). \end{aligned} \quad (3.15)$$

This change of the index was assumed in [36], but not derived from the KS-formula.

Since equation (3.15) gives the jump of the index towards the stable chamber, the contribution  $\Omega_{T_{12}}(\Gamma; t)$  of  $T_{12}$  to the total index, is given by equation (3.15) with the moduli at the right hand side at the corresponding attractor points. One finds for the generating function

$$\begin{aligned} h_{T_{12}, Q - \frac{1}{2}P}(\tau; t) &= \sum_{Q_0} \Omega_{T_{12}}(\Gamma; t) q^{-Q_0 + \frac{1}{2}Q^2} \\ &= \sum_{Q_1 + Q_2 = Q} \frac{1}{2} (\text{sgn}(\mathcal{I}(\Gamma_1, \Gamma_2; t)) - \text{sgn}(I_{12})) (-1)^{P_1 \cdot Q_2 - P_2 \cdot Q_1} \\ &\quad \times (P_1 \cdot Q_2 - P_2 \cdot Q_1) q^{\frac{1}{2}Q^2 - \frac{1}{2}(Q_1)_1^2 - \frac{1}{2}(Q_2)_2^2} \\ &\quad \times h_{P_1, \mu_1}(\tau) h_{P_2, \mu_2}(\tau), \end{aligned}$$

where  $Q^2$  and  $(Q_i)_i^2$  are the quadratic forms based on  $P$  and  $P_i$  respectively.  $h_{T_{12}, Q - \frac{1}{2}P}(\tau; t)$  is not a vector-valued modular form; however Manschot [36] continues by showing that summing over the D2-brane charges, leads to the partition function

$$\mathcal{Z}_{T_{12}}(\tau, C, t) = \sum_{(\mu_1, \mu_2) \in \Lambda_1^* / \Lambda_1 \oplus \Lambda_2^* / \Lambda_2} \overline{h_{P_1, \mu_1}(\tau)} \overline{h_{P_2, \mu_2}(\tau)} \Psi_{(\mu_1, \mu_2)}(\tau, C, B), \quad (3.16)$$



with

$$\begin{aligned} \Psi_{(\mu_1, \mu_2)}(\tau, C, B) = & \sum_{\substack{Q_1 \in \Lambda_1 + \mu_1 + P_1/2 \\ Q_2 \in \Lambda_2 + \mu_2 + P_2/2}} S(T_{12}, t) I_{12} (-1)^{P_1 \cdot Q_1 + P_2 \cdot Q_2 - 1} \\ & \times e \left( \tau(Q - B)_+^2 / 2 + \bar{\tau} \left( \sum_{i=1,2} (Q_i - B)_i^2 - (Q - B)_+^2 \right) / 2 \right. \\ & \left. + C \cdot (Q - B/2) \right). \end{aligned} \quad (3.17)$$

$\Psi_{(\mu_1, \mu_2)}(\tau, C, B)$  determines which charge combinations are stable and which are not. It does not transform as a theta function, but using techniques of indefinite theta functions [51], one can complete it to a function  $\Psi_{(\mu_1, \mu_2)}^*(\tau, C, B)$  which does transform as a theta function with weight  $(\frac{1}{2}, b_2 + \frac{1}{2})$ . We therefore call  $\Psi_{(\mu_1, \mu_2)}(\tau, C, B)$  a mock Siegel theta function. Using the completed function,  $\mathcal{Z}_{T_{12}}(\tau, C, t)$  transforms precisely as  $\mathcal{Z}_{T_{1+2}}(\tau, C, t)$  (with  $T_{1+2}$  the  $N = 1$  flow tree with magnetic charge  $P_1 + P_2$ ). An intriguing phenomenon of the modular completion is that it replaces the discontinuity of the partition function across walls by a continuous transition. One could say that the discontinuous invariants  $\Omega(\Gamma; t)$  are replaced by functions  $\Omega(\Gamma; t, \tau_2)$  of  $t$  and  $\tau_2$ , which approach the original invariants in the limit  $\tau_2 \rightarrow \infty$ . If this structure is valid in general, taking the limit and crossing a wall between  $\mathcal{C}_A$  and  $\mathcal{C}_B$ , leads to the following commutative diagram:

$$\begin{array}{ccc} \Omega(\Gamma; t_A, \tau_2) & \xrightarrow{t_A \rightarrow t_B} & \Omega(\Gamma; t_B, \tau_2) \\ \downarrow \tau_2 \rightarrow \infty & & \downarrow \tau_2 \rightarrow \infty \\ \Omega(\Gamma; t_A) & \xrightarrow{\text{KS}} & \Omega(\Gamma; t_B) \end{array}$$

For a better understanding of the way  $\Psi_{(\mu_1, \mu_2)}(\tau, C, B)$  determines, which states are stable and which not, we explain briefly the concept of indefinite theta functions.

### *Indefinite theta function*

An indefinite theta function sums over part of an indefinite lattice, which belongs either to the positive or negative definite part of the lattice. Typically such sums do not transform as modular forms, but can be made so in special cases by the addition of a non-holomorphic term [51]. The idea is most easily explained by considering a lattice  $\Lambda$  with signature  $(1, b_2 - 1)$  [27, 51].

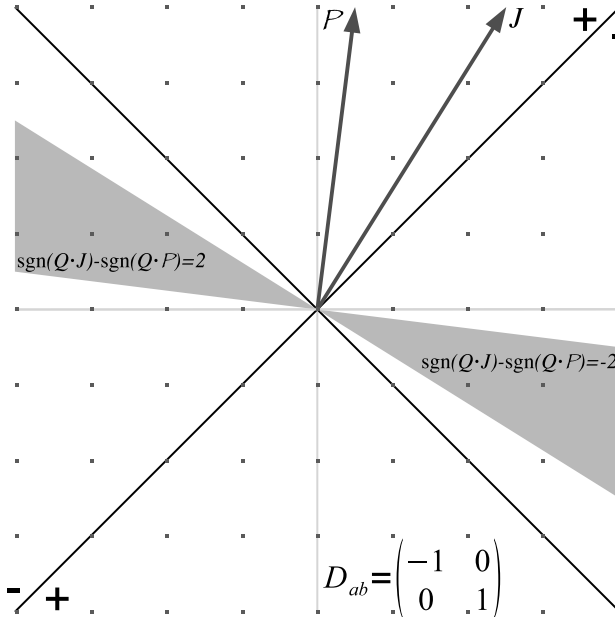


Figure 3: An indefinite lattice; the lattice points inside the green region contribute to the theta function defined in the text.

Given two positive vectors  $J, \mathcal{P} \in \Lambda$  with  $J \cdot \mathcal{P} > 0$ , one can prove that the condition  $\frac{1}{2}(\text{sgn}(J \cdot Q) - \text{sgn}(\mathcal{P} \cdot Q)) \neq 0$  implies that  $Q^2 < 0$ . This proof is completely analogous to the proof of Proposition 3.2; just omit the term with  $Q \cdot \mathcal{J}_2$  and identify  $\mathcal{P}, J$  with  $\mathcal{P}_{12}$  and  $s_{12}$ . Figure 3 displays the lattice points for which the condition is satisfied for a two-dimensional lattice with quadratic form  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  (which is incidentally the intersection form of 2-cycles on  $\mathbb{C}\mathbb{P}^2$  blown up at a point). The green region in the figure contains the lattice points for which the condition is satisfied. This region changes when  $J$  and/or  $\mathcal{P}$  are varied. (From the point of view of wall-crossing, we think of  $\mathcal{P}$  as fixed and  $J$  as variable.)

The indefinite theta function is defined as the sum over all lattice points, satisfying the condition:

$$\theta_\mu(\bar{\tau}, z) = \sum_{k \in \Lambda} \frac{1}{2}(\text{sgn}(J \cdot Q) - \text{sgn}(\mathcal{P} \cdot Q)) \bar{q}^{k^2/2} y^k, \quad (3.18)$$

which is convergent. Its Fourier coefficients are locally constant as function of  $J$ , but can change if the boundary of the green region passes a lattice point. These indefinite theta functions do not have the nice modular properties which holomorphic theta functions or Siegel theta functions are known

to have. However, the indefinite theta function can be completed to a function with the familiar modular properties, by replacing  $\text{sgn}(x)$  in (3.18) by  $E(x\sqrt{\tau_2})$  with  $E(z) = 2 \int_0^z e^{-\pi u^2} du$  [51]. Note that the discontinuous function  $\text{sgn}(Q \cdot J)$  as function of  $J$  is replaced now by a continuous function. Moreover,  $E(x\sqrt{\tau_2})$  approaches  $\text{sgn}(x)$  for  $\tau_2 \rightarrow \infty$ , the “thickness of the step” is of order of  $\sqrt{2/\tau_2}$ .

The function  $\Psi_{(\mu_1, \mu_2)}(\tau, C, B)$  is very similar to the function (3.18). An important difference is that the boundary of the positive definite cone depends on the moduli by  $Q \cdot \mathcal{J}_2$  in equation (3.13). Another difference is that  $\Psi_{(\mu_1, \mu_2)}(\tau, C, B)$  contains the factor  $P_1 \cdot Q_2 - P_2 \cdot Q_1$  multiplying the exponential, which leads to a more complicated modular completion.

### *Entropy enigma*

One can easily compare the relative magnitude of the contribution to the index of flow trees with  $N = 1$  and 2 using the partition function (3.16). A special class is formed by flow trees with  $N > 1$  whose index exceeds the index of the flow tree with  $N = 1$ , the so called entropy enigmas. We consider here entropy enigmas in the Cardy regime of the CFT where  $\hat{Q}_0 \gg P^3$ . Andriyash and Moore [1] showed earlier the existence of entropy enigmas for D4-D2-D0 branes for weak topological string coupling  $g_{\text{top}} \sim \sqrt{\hat{Q}_0/P^3}$ . The entropy of the single center is in the Cardy regime:

$$\pi \sqrt{\frac{2}{3}(P^3 + c_2 \cdot P) (Q_{\bar{0}} + \frac{1}{2}Q^2)}. \quad (3.19)$$

Application of the Cardy formula to equation (3.16) shows that the condition for enigmatic  $N = 2$  flow trees is:

$$(P^3 + c_2 \cdot P) (Q_{\bar{0}} + \frac{1}{2}Q^2) < (P_1^3 + P_2^3 + c_2 \cdot P) (Q_{\bar{0}} + \frac{1}{2}(Q_1)_1^2 + \frac{1}{2}(Q_2)_2^2). \quad (3.20)$$

Note that the right-hand side also captures the entropy due to distributing the total D0-brane charge in different ways between the two endpoints, otherwise one should just add up the entropy of both endpoints.

Charges  $\Gamma_1$  and  $\Gamma_2$ , which satisfy this relation, are not hard to find. To this end, write  $Q$  as  $\mu - P/2 + k$  with  $\mu \in \Lambda^*/\Lambda$  and  $k \in \Lambda$ . Choose  $Q$  such that  $k^2 = P_1 \cdot k^2 + P_2 \cdot k^2 = 0$ . Therefore,  $P_1 \cdot k^2 = -P_2 \cdot k^2$ . Without loss of generality we can assume that  $P_1 \cdot k^2 \geq 0$ . Taking  $Q_2 = 0$  leads now to an enigmatic configuration for sufficiently large  $k$ . It is not difficult to see that this can very well happen for strong topological string coupling  $g_{\text{top}} \sim \sqrt{\hat{Q}_0/P^3} \gg 1$ . Substituting this choice of charges into the stability

condition shows that there exist regions in the moduli space where such bound states are stable. These enigmas show that one has to be careful by estimating the magnitude of the total index by the CFT index away from the attractor point.

### 3.1.3 Three endpoints

This subsection discusses flow trees with three endpoints with D4-D2-D0 charges. We will prove that also in this case the claim (3.5) is true, such that the partition function for flow trees with  $N = 3$  is convergent. The total lattice is now a sum of three lattices:  $\Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$ . The case  $N = 3$  is qualitatively different from  $N = 2$ , since the flow of the moduli needs to be taken into account. What we want to prove is:

$$S(T_{(12)3}, t) \neq 0 \quad \implies \quad (Q - B)_+^2 - \sum_{i=1}^3 \frac{1}{2}(Q_i - BP_i)_i^2 \geq 0, \quad (3.21)$$

with  $S(T_{(12)3}, t)$  given by equations (2.22) and (2.23).

The requirement that the stability of the subtree (12) is determined in terms of  $t_1$  instead of  $t$  has the consequence that the stability condition is not directly related to a determinant like equation (3.14). Therefore, we will reduce  $S(T_{(12)3}, t) \neq 0$  to special cases where an argument based on a determinant can be used. To this end, define for generic flow trees the “unphysical” condition:

**Condition U :**

$$U(T, t) = \prod_{v \in V} \frac{1}{2} (\text{sgn}(\langle \Gamma_{vL}, \Gamma_{vR} \rangle) - \text{sgn}(\mathcal{I}(\Gamma_{vL}, \Gamma_{vR}, t))) \neq 0.$$

Note that the non-vanishing of  $U(T, t)$  is determined here by the stability of all splits at  $v \in V$  in terms of  $t$ . If stability would be based on this condition, the jumps of the index might appear at other points in the moduli space than the walls of marginal stability for the total charge. It is however a useful condition since:

**Proposition 3.3.**

$$U(T, t) \neq 0 \quad \implies \quad (Q - B)_+^2 - \sum_{i=1}^N \frac{1}{2}(Q_i - BP_i)_i^2 \geq 0, \quad (3.22)$$

*Proof.* It is again sufficient to prove the proposition for  $B = 0$ . The vectors defined in equation (3.11), are easily generalized to vectors for vertex 1 in

the tree  $T$ :  $1 \rightarrow 1L$ , and  $2 \rightarrow 1R$ . In terms of these vectors, Condition **U** becomes:

$$U(T, t) = \prod_{v \in V} \frac{1}{2} (\text{sgn}(\mathcal{P}_{vLR} \cdot (Q_{vL}, Q_{vR})) - \text{sgn}(s_{vLR} \cdot (Q_{vL}, Q_{vR}))) \neq 0. \quad (3.23)$$

We will use induction to arrive at the desired result. The proposition is true for  $N = 2$  by Proposition 3.2. For general  $N > 2$ , the attractor flow tree can be seen as a combination of two trees  $T_{1L}$  and  $T_{1R}$ , which merge at vertex 1. We index the endpoints of  $T_{1L}$  and  $T_{1R}$  respectively by  $i = 1, 2, \dots, k$  and  $i = k + 1, \dots, N$ , such that the left-hand side of the inequality in equation (3.22) is equal to

$$\frac{(Q_{1L} \cdot J)^2}{P_{1L} \cdot J^2} - \sum_{i=1}^k (Q_i)_i^2 + \frac{(Q_{1R} \cdot J)^2}{P_{1R} \cdot J^2} - \sum_{i=k+1}^N (Q_i)_i^2 - (s_{1LR} \cdot (Q_{1L}, Q_{1R}))^2. \quad (3.24)$$

The product  $U(T, t)$  factorizes as

$$U(T, t) = \frac{1}{2} (\text{sgn}(\mathcal{P}_{1LR} \cdot (Q_{1L}, Q_{1R})) - \text{sgn}(s_{1LR} \cdot (Q_{1L}, Q_{1R}))) S(T_L, t) S(T_R, t). \quad (3.25)$$

By the induction hypothesis, the sum of the first two terms is positive if  $S(T_L, t)$  is non-zero, and the similarly the sum of the second two if  $S(T_R, t)$  is non-zero. Therefore one can argue analogously to the proof of Proposition 3.2 that  $(Q_1, Q_2, \dots, Q_N)$ ,  $\mathcal{J}_2$ ,  $\mathcal{P}_{1LR}$  and  $s_{1LR}$  span a space of signature  $(2, 2)$  in  $\Lambda_1 \oplus \Lambda_2 \oplus \dots \oplus \Lambda_N$ . Equation (3.24) is therefore negative if  $U(T) \neq 0$ .  $\square$

For a tree with  $N = 3$ ,  $S(T_{(12)3}, t) \neq 0$  implies in most cases that  $U(T, t) \neq 0$ , with  $T$  one of the three trees with  $N = 3$ . Specifically,  $S(T_{(12)3}, t) \neq 0$  together with

$$I_{12} (I_{2(31)} P_1 \cdot J^2 + I_{(23)1} P_2 \cdot J^2) \leq 0, \quad (3.26)$$

implies  $U(T_{12}, t) \neq 0$ , and consequently  $U(T_{(12)3}, t) \neq 0$ . To analyze the remaining cases, we divide them into three classes:

$$\begin{aligned} \text{I} & : I_{12} I_{31} > 0 \quad \text{and} \quad I_{12} I_{23} < 0, \\ \text{II} & : I_{12} I_{31} < 0 \quad \text{and} \quad I_{12} I_{23} > 0, \\ \text{III} & : I_{12} I_{31} > 0 \quad \text{and} \quad I_{12} I_{23} > 0. \end{aligned} \quad (3.27)$$

To prove the positivity for these classes, we only need to be concerned with those trees for which  $S(T_{12}, t) = 0$  and  $S(T_{(12)3}, t) \neq 0$ . Then it is possible to show that **I** implies  $U(T_{2(31)}, t) \neq 0$ ; and similarly that **II** implies  $U(T_{1(23)}, t) \neq 0$ . Class **III** cannot be reduced to  $U(T, t) \neq 0$  for some  $T$ , and the proof requires a little more work.

Let  $P = P_1 + P_2 + P_3$  and define the following unit vectors:

$$\begin{aligned} \mathcal{P}_{12} &= \frac{(-P_2, P_1, 0)}{\sqrt{(P_1 + P_2)P_1P_2}}, & \mathcal{P}_{23} &= \frac{(0, -P_3, P_2)}{\sqrt{(P_2 + P_3)P_2P_3}}, \\ \mathcal{P}_{31} &= \frac{(P_3, 0, -P_1)}{\sqrt{(P_1 + P_3)P_1P_3}}, & \mathcal{P}_{(12)3} &= \frac{(-P_3, -P_3, P_1 + P_2)}{\sqrt{P(P_1 + P_2)P_3}}, \\ s_{(12)3} &= \frac{(-P_3 \cdot J^2 J, -P_3 \cdot J^2 J, (P_1 + P_2) \cdot J^2 J)}{\sqrt{P \cdot J^2 (P_1 + P_2) \cdot J^2 P_3 \cdot J^2}}, \\ \mathcal{J}_3 &= \frac{(J, J, J)}{\sqrt{P \cdot J^2}}. \end{aligned} \quad (3.28)$$

Analogously to Proposition 3.1, one can show various useful relations between these vectors. The innerproduct of  $\mathcal{J}_3$  with any other vector in (3.28) vanishes. Furthermore,

$$\mathcal{P}_{12} \cdot s_{(12)3} = \mathcal{P}_{12} \cdot \mathcal{P}_{(12)3} = 0, \quad s_{(12)3} \cdot \mathcal{P}_{(12)3} > 1. \quad (3.29)$$

**Proposition 3.4.** *Let  $Q = (Q_1, Q_2, Q_3) \in \Lambda_1^* \oplus \Lambda_2^* \oplus \Lambda_3^*$ . If the following conditions are satisfied*

$$\begin{aligned} a) & \quad (s_{(12)3} \cdot Q) (\mathcal{P}_{(12)3} \cdot Q) \geq 0, \\ b) & \quad (\mathcal{P}_{12} \cdot Q) (\mathcal{P}_{31} \cdot Q) \geq 0, \\ c) & \quad (\mathcal{P}_{12} \cdot Q) (\mathcal{P}_{23} \cdot Q) \geq 0, \end{aligned} \quad (3.30)$$

then

$$\sum_{i=1}^3 (Q_i)_i^2 - (Q \cdot \mathcal{J}_3)^2 < 0. \quad (3.31)$$

Condition a) is equivalent to the stability condition for the two center split  $(1 + 2)3$ ; Conditions b) and c) are equivalent to Condition **III** in equation (3.27).

*Proof.* We start by showing an implication of condition (a) in (3.30). The positive definite subspace of  $\Lambda$  is spanned by the orthonormal basis given by

$\mathcal{J}$ ,  $\mathcal{P}_{12}$  and  $\mathcal{P}_{(12)3}$ . Consequently, the vectors  $Q$ ,  $s_{(12)3}$ ,  $\mathcal{J}$ ,  $\mathcal{P}_{12}$  and  $\mathcal{P}_{(12)3}$  span generically a space of signature  $(3, 2)$ . Therefore,

$$\begin{vmatrix} Q^2 & Q \cdot \mathcal{J}_3 & Q \cdot \mathcal{P}_{12} & Q \cdot s_{(12)3} & Q \cdot \mathcal{P}_{(12)3} \\ Q \cdot \mathcal{J}_3 & 1 & 0 & 0 & 0 \\ Q \cdot \mathcal{P}_{12} & 0 & 1 & 0 & 0 \\ Q \cdot s_{(12)3} & 0 & 0 & 1 & \mathcal{P}_{(12)3} \cdot s_{(12)3} \\ Q \cdot \mathcal{P}_{(12)3} & 0 & 0 & \mathcal{P}_{(12)3} \cdot s_{(12)3} & 1 \end{vmatrix} > 0.$$

From this determinant follows that

$$Q^2 - (Q \cdot \mathcal{J}_3)^2 - (Q \cdot \mathcal{P}_{12})^2 < 0, \quad (3.32)$$

if condition (a) in (3.30) is satisfied. Therefore  $Q$ ,  $\mathcal{J}_3$  and  $\mathcal{P}_{12}$  span in this case a space with signature  $(2, 1)$ . We want to show that conditions (b) and (c) imply that “ $-(Q \cdot \mathcal{P}_{12})^2$ ” can be omitted from the inequality. To this end, we choose to complement the set of three vectors  $Q$ ,  $\mathcal{J}_3$  and  $\mathcal{P}_{12}$  by

$$\mathcal{P}_{23\perp} = \mathcal{P}_{23} - (\mathcal{P}_{23} \cdot \mathcal{P}_{(12)3}) \mathcal{P}_{(12)3}, \quad (3.33)$$

which is the component of  $\mathcal{P}_{23}$  orthogonal to  $\mathcal{P}_{(12)3}$ . As a result,  $Q$ ,  $\mathcal{J}_3$ ,  $\mathcal{P}_{12}$  and  $\mathcal{P}_{23\perp}$  span a space of signature  $(2, 2)$ . Since  $\mathcal{P}_{12}$  and  $\mathcal{P}_{23\perp}$  are both orthogonal to  $\mathcal{J}_3$  and  $\mathcal{P}_{(12)3}$ , they span a space of signature  $(1, 1)$ . Conditions b) and c) imply that  $(\mathcal{P}_{12} \cdot Q)(\mathcal{P}_{23\perp} \cdot Q) > 0$ , since

$$\mathcal{P}_{23\perp} = \frac{1}{PP_3(P_1 + P_2)} \left( PP_1P_3\mathcal{P}_{23} + PP_2P_3 \sqrt{\frac{P_1P_3(P_1 + P_3)}{(P_2 + P_3)P_2P_3}} \mathcal{P}_{31} \right). \quad (3.34)$$

This also shows that  $\mathcal{P}_{12} \cdot \mathcal{P}_{23\perp} < 0$ . Using these relations together with the argument of the sign of the determinant:

$$\begin{vmatrix} Q^2 & Q \cdot \mathcal{J}_3 & Q \cdot \mathcal{P}_{12} & Q \cdot \mathcal{P}_{23\perp} \\ Q \cdot \mathcal{J}_3 & 1 & 0 & 0 \\ Q \cdot \mathcal{P}_{12} & 0 & 1 & \mathcal{P}_{12} \cdot \mathcal{P}_{23\perp} \\ Q \cdot \mathcal{P}_{23\perp} & 0 & \mathcal{P}_{12} \cdot \mathcal{P}_{23\perp} & \mathcal{P}_{23\perp}^2 \end{vmatrix} > 0,$$

one obtains the desired result

$$Q^2 - (Q \cdot \mathcal{J}_3)^2 < 0. \quad (3.35)$$

□

This proof gives more confidence that positivity can be proven for any  $N$ . It is conceivable that for any  $N$ ,  $S(T, t) \neq 0$  can be reduced for most  $T$  to  $U(T') \neq 0$  for several  $T'$ , and that in the remaining cases it can be proved as well. An obstacle for an easy inductive proof, analogous to the one for  $U(T, t)$ , is the fact that stability of subtrees at  $v_0$  is not ensured by stability at  $v_1$ . The quadratic form for  $T_{12}$  is not even positive definite for  $S(T_{(12)3}) \neq 0$ .

Proposition 3.4 implies that the lattice sum

$$\begin{aligned} & \Psi_{(\mu_1, \mu_2, \mu_3)}(\tau, C, B) \\ &= \sum_{\substack{Q_1 \in \Lambda_1 + \mu_1 + P_1/2 \\ Q_2 \in \Lambda_2 + \mu_2 + P_2/2 \\ Q_3 \in \Lambda_3 + \mu_3 + P_3/2}} S(T_{(12)3}, t) I_{(12)3} I_{12} (-1)^{P_1 \cdot Q_1 + P_2 \cdot Q_2 + P_3 \cdot Q_3} \\ & \times e \left( \tau(Q - B)_+^2 / 2 + \bar{\tau} \left( \sum_{i=1}^3 (Q_i - B)_i^2 - (Q - B)_+^2 \right) \right) / 2 \\ & + C \cdot (Q - B/2) \Big), \end{aligned} \tag{3.36}$$

is convergent. Analogously to the discussion in Subsection 3.1.2, this object does not transform as a modular form. Since it is a lattice sum it is not unlikely that a modular completion exists for this sum as for  $N = 2$ . This is also expected from  $S$ -duality. However, due to the complexity of  $S(T_{(12)3}, t)$ , this does not seem as easy as straightforward. If  $S(T_{(12)3}, t)$  is replaced by  $U(T_{(12)3}, t)$  one can iterate the procedure in [36]. We will not attempt to find the modular completion of equation (3.36), but leave this for future research.

Nevertheless, we can now write down the contribution of flow trees with three endpoints to the partition function:

$$\begin{aligned} \mathcal{Z}_{T_{(12)3}}(\tau, C, t) &= \sum_{(\mu_1, \mu_2, \mu_3) \in \Lambda_1^* / \Lambda_1 \oplus \Lambda_2^* / \Lambda_2 \oplus \Lambda_3^* / \Lambda_3} \overline{h_{P_1, \mu_1}(\tau)} \overline{h_{P_2, \mu_2}(\tau)} \overline{h_{P_3, \mu_3}(\tau)} \\ & \times \Psi_{(\mu_1, \mu_2, \mu_3)}(\tau, C, B). \end{aligned} \tag{3.37}$$

The other topologies of the tree can similarly be taken into account. If the  $P_i$  are primitive and different, the partition functions for  $N = 1, 2$  and 3 capture correctly the total jumps of the indices across walls. We would also like to include the case when the  $P_i$  are possibly equal. In that case one must use the semi-primitive wall-crossing formula, we will come back to this point in Subsection 3.2.



Table 1: Non-zero intersection numbers of Calabi–Yaus with  $b_2 = 2$  [7] and  $b_2 = 3, 4$  [33].

$b_2$	2	3	4
$d_{abc}$	$d_{111} = 8,$ $d_{112} = 4$	$d_{111} = 8, d_{112} = 2,$ $d_{113} = 2, d_{123} = 1$	$d_{112} = 4, d_{113} = 2, d_{122} = 4,$ $d_{123} = 2, d_{124} = 2, d_{134} = 1,$ $d_{224} = 2, d_{234} = 1$

*Numerical experiments*

Besides the analytical proof of the claim, it is instructive to carry out numerical experiments to answer questions like: what portion of the set of rooted trees is a flow tree for given  $t$ ? or what is the overlap between Conditions A and U. I have done numerical experiments with three Calabi–Yaus, with  $b_2 = 2, 3$  and 4. The Calabi–Yau with  $b_2 = 2$  is discussed in more detail in [7], and  $b_2 = 3, 4$  in [33]. The only relevant data for our purpose are the triple intersection numbers, which are listed in Table 1.

Many different tables with combinations of statistical data can be generated. I suffice here by giving Table 2, which lists the number configurations with  $S(T_{(12)3}, t) \neq 0$ , the number for which  $U(T_{(12)3}, t) \neq 0$ , and the number of configurations which lie in both classes. A C++ code has searched  $10^9$  configurations per Calabi–Yau, using a random number generator. The random number generator chose its values for the moduli and the charges in the following domains:  $J^a \in [1, \dots, 12]$ ,  $P^a \in [1, \dots, 10]$ ,  $Q_a \in [-20, \dots, 20]$ . The variation of the quantities in the table between different runs of  $10^9$  configurations is  $< 0.05\%$ . Clearly, the physical condition  $S(T_{(12)3}, t) \neq 0$  is less often satisfied than the condition  $U(T_{(12)3}, t) \neq 0$ , although it is not a subset of it. One can also read off from the table, that for all three Calabi–Yaus the ratio of the number of charge combinations with  $T_{(12)3}$  stable, but  $T_{12}$  unstable in terms of  $t$  ( $S(T_{12}, t) = 0$ ), is between 6 and 7%. It would be

Table 2: Number of trees in a search of  $10^9$  trees  $T_{(12)3}$ , for which  $S(T_{(12)3}, t) \neq 0$ ,  $U(T_{(12)3}, t) \neq 0$  and the number of trees which satisfy both conditions.

$b_2$	$S(T_{(12)3}, t) \neq 0$	$U(T_{(12)3}, t) \neq 0$	$S(T_{(12)3}, t) \neq 0$ $\cap U(T_{(12)3}, t) \neq 0$
2	18147241	29465018	17016426
3	22255909	35817183	20750877
4	23264713	37135142	21654091

interesting to better understand the dependence on Calabi–Yau, moduli or charges of these and other ratios, and derive them analytically.

### 3.2 Non-primitive wall-crossing

This last subsection discusses some aspects of non-primitive wall-crossing. Denef and Moore [15] presents a formula for the jumps of the index, for semi-primitive wall-crossing  $\Gamma \rightarrow N\Gamma_1 + \Gamma_2$ , which is known to be compatible with the KS-formula. For the application to D4-D2-D0 BPS-states in the large volume limit, where the walls are independent of  $Q_{0,(i)}$ , a wall-crossing formula with an additional parameter for the D0-brane charge is desired. This formula can be derived from the KS-formula similar to [9]. We take the constituent charges to be  $\Gamma_1 = (N\gamma_1, Q_{0,1})$  and  $\Gamma_2 = (\gamma_2, Q_{0,2})$ , with  $\gamma_1 = (P_1, Q_1)$  and  $\gamma_2 = (P_2, Q_2)$ , respectively. One finds for the generating series of the indices

$$\begin{aligned} & \sum_{N=0}^{\infty} \sum_{Q_0} \Delta\Omega((N\gamma_1 + \gamma_2, Q_0); t) u^N v^{Q_0} \\ &= \sum_{Q_{0,2}}^{\infty} \Omega((\gamma_2, Q_{0,2})) v^{Q_{0,2}} \\ & \quad \times \prod_{k=1}^{\infty} \prod_{Q_{0,1}} \left( 1 - (-1)^{I_{12}k} u^k v^{Q_{0,1}} \right)^{I_{12}k \Omega((k\gamma_1, Q_{0,1}))}. \end{aligned} \quad (3.38)$$

The  $\Delta\Omega(\Gamma; t)$  are the contributions to the index in a stable chamber for  $T_{12}$  with  $I_{12} > 0$ . For  $N = 1$  one obtains our previous result (2.10). One finds for  $N = 2$ :

$$\begin{aligned} \Delta\Omega(\Gamma; t) = & - \sum_{Q_{0,1}+Q_{0,2}=Q_0} 2I_{12} \Omega((2\gamma_1, Q_{0,1}); t) \Omega((\gamma_2, Q_{0,2}); t) \\ & + \sum_{\substack{Q_{0,1}+Q_{0,2}+Q_{0,3}=Q_0 \\ Q_{0,1} \neq Q_{0,3}}} I_{12}^2 \Omega((\gamma_1, Q_{0,1}); t) \\ & \times \Omega((\gamma_1, Q_{0,3}); t) \Omega((\gamma_2, Q_{0,2}); t) \\ & + \sum_{2Q_{0,1}+Q_{0,2}=Q_0} \frac{1}{2} I_{12} \Omega((\gamma_1, Q_{0,1}); t) \Omega((\gamma_2, Q_{0,2}); t) \\ & \times (I_{12} \Omega((\gamma_1, Q_{0,1}); t) - 1). \end{aligned} \quad (3.39)$$

This expression raises a puzzle. The discussion of [36] (see the review on page 22 and further), suggests that a prerequisite for  $S$ -duality invariance of

the generating function of  $\Delta\Omega(\Gamma; t)$ , is that it can be expressed in terms of products of vector-valued modular forms of  $SL(2, \mathbb{Z})$ . However, the “ $-1$ ” in the last line makes that a factor  $h_{P_1, \mu_1}(2\tau)$  would appear in the current case, which is not a vector-valued modular form of  $SL(2, \mathbb{Z})$  but of the congruence subgroup  $\Gamma_0(4)$ . The resolution to this puzzle is that the correct definition of  $h_{P, \mu}(\tau)$  is *not* as generating function of  $\Omega(\Gamma)$  but instead of  $\bar{\Omega}(\Gamma) = \sum_{m|\Gamma} \frac{1}{m^2} \Omega(\Gamma/m; t)$ . Requiring that the newly defined  $h_{P, \mu}(\tau)$  transform as an  $SL(2, \mathbb{Z})$  vector-valued modular form is compatible with semi-primitive wall-crossing. To this end, redefine  $h_{P, Q - \frac{1}{2}P}(\tau)$ :

$$h_{P, Q - \frac{1}{2}P}(\tau) = \sum_{Q_0} \bar{\Omega}((P, Q, Q_0)) q^{Q_0 + \frac{1}{2}Q^2}. \quad (3.40)$$

The generating function of  $\Omega(\Gamma)$  transforms only under a congruence subgroup  $\Gamma_0(M)$ , with  $M$  a product of primes  $p$ :  $M = \prod_{p^{\alpha_p}|P} p^{\alpha_p}$ , for total magnetic charge  $P$ . For  $N = 2$ , it is  $h_{2P_1, 2\mu_1}(\tau) - \frac{1}{4}h_{P_1, \mu_1}(2\tau)$  which has an expansion with integer coefficients, but does not transform well under  $SL(2, \mathbb{Z})$ .

Using this new definition, the contribution to the generating function of  $\sum_{Q_0} \Delta\Omega(\Gamma; t) q^{-Q_0 + \frac{1}{2}Q^2}$  in a stable chamber is:

$$\begin{aligned} & \sum_{2Q_1 + Q_2 = Q} q^{\frac{1}{2}Q^2 - (Q_1)_1^2 - \frac{1}{2}(Q_2)_2^2} \\ & \times \left( \frac{1}{2} I_{12}^2 h_{P_1, \mu_1}^2(\tau) h_{P_2, \mu_2}(\tau) - 2 I_{12} h_{2P_1, 2\mu_1}(\tau) h_{P_2, \mu_2}(\tau) \right). \end{aligned}$$

The two terms can be identified as contributions of the trees  $T_{(12)_1}$  and  $T_{(2,1)_2}$ .<sup>3</sup>  $T_{(12)_1}$  should be considered as a special (degenerate) case of  $T_{(12)_3}$ . We also observe that modularity of the complete partition function, requires that the  $T_{(12)_1}$ -contribution should combine with a mock Siegel theta function of the lattice  $\Lambda_1 \oplus \Lambda_1 \oplus \Lambda_2$ , whereas the  $T_{(2,1)_2}$ -contribution should combine with a mock Siegel theta function of  $\Lambda_{2,1} \oplus \Lambda_2$  (where  $\Lambda_{2,1}$  has quadratic form  $2d_{abc}P_1^c$ ). Therefore, to show the compatibility of semi-primitive wall-crossing with modularity, one is forced to understand the extended flow trees, which we studied before. If we insert the products  $S(T_{(12)_1}, t)$  (which

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<sup>3</sup>The tree  $T_{(2,1)_2}$  has two endpoints, one with magnetic charge  $2P_1$  and one with  $P_2$ .

is  $-\frac{1}{2}$  or 0) and  $S(T_{(2,1,2)}, t)$ , and add the contributions of  $T_{(2,1,2)}$  with primitive charges, we find

$$\begin{aligned} & \sum_{Q_1+Q_2+Q_3=Q} q^{\frac{1}{2}Q^2 - \frac{1}{2}(Q_1)_1^2 - \frac{1}{2}(Q_3)_1^2 - \frac{1}{2}(Q_2)_2^2} S(T_{(12)1}, t) I_{(12)1} \\ & \times I_{12} h_{P_1, \mu_1}(\tau) h_{P_1, \mu_3}(\tau) h_{P_2, \mu_2}(\tau) \\ & - \sum_{Q_1+Q_2=Q} q^{\frac{1}{2}Q^2 - \frac{1}{2}(Q_1)_{2,1}^2 - \frac{1}{2}(Q_2)_2^2} S(T_{(2,12)}, t) 2I_{12} h_{2P_1, 2\mu_1}(\tau) h_{P_2, \mu_2}(\tau). \end{aligned} \tag{3.41}$$

The sum over  $Q$  will give the correct mock Siegel theta functions (3.17) and (3.36); the positivity condition of Subsection 3.1 implies the convergence of the series. Note that for  $P$  primitive, the semi-primitive wall-crossing formula for  $N = 2$  is precisely such that modularity and integrality are compatible. This also suggests more generally, that the contribution to the partition function from a rooted tree, based on a nested list of magnetic charge, preserves  $S$ -duality. One will find products of vector-valued modular forms corresponding to the different endpoints.

More evidence for the claim that  $\bar{\Omega}(\Gamma)$  are the correct invariants in the context of  $S$ -duality, can be found from the partition functions of  $\mathcal{N} = 4$  Yang-Mills on a surface [47], which are closely related to D4-brane partition functions on a divisor of a Calabi–Yau. These partition functions are generating functions of the Euler number  $\chi(\mathcal{M})$  of the instanton moduli space  $\mathcal{M}$ , which are related to the DT-invariants by  $\Omega(\Gamma; t) = (-1)^{\dim \mathcal{M}(\Gamma)} \chi(\mathcal{M}(\Gamma))$  [16]. Yoshioka has calculated in Refs. [49, 50] the partition function for  $U(2)$  Yang-Mills (rank 2 sheaves) on  $\mathbb{C}\mathbb{P}^2$ . The two partition functions for sheaves of rank 2 with  $c_1 = 0 \pmod{2}$  and  $1 \pmod{2}$  are given by

$$h_{2,0}(\tau) = -\frac{f_{2,0}(\tau)}{\eta(\tau)^6}, \quad h_{2,1}(\tau) = \frac{f_{2,1}(\tau)}{\eta(\tau)^6}, \tag{3.42}$$

where  $f_{2,i}(\tau)$  are the generating functions of the class numbers  $H(n)$ :

$$f_{2,0}(\tau) = \sum_{n=0}^{\infty} 3H(4n)q^n, \quad f_{2,1}(\tau) = \sum_{n=1}^{\infty} 3H(4n-1)q^{n-\frac{1}{4}}. \tag{3.43}$$

$(h_{2,0}(\tau), h_{2,1}(\tau))$  transforms as a vector-valued modular form of weight  $-\frac{3}{2}$ .<sup>4</sup> However, the coefficients of  $h_{2,0}(\tau)$  are not integers. To obtain integers, one needs to subtract the contribution of multiple  $U(1)$  instantons  $\frac{1}{4} \frac{1}{\eta(2\tau)^3}$ ;

<sup>4</sup>The vector  $(f_{2,0}(\tau), f_{2,1}(\tau))$  is actually a mock modular form; a modular completion must be added for proper transformation properties under  $SL(2, \mathbb{Z})$  [47].

the resulting vector transforms only under  $\Gamma_0(2)$ . The  $--$ sign in (3.42) is crucial and follows from the factor  $(-1)^{\dim \mathcal{M}(\Gamma)}$ . Similar results are known for  $K3$  [37, 47].

Equation (3.41) suggests that the contribution of flow trees to the index is most conveniently expressed in terms of  $\bar{\Omega}(\Gamma)$ . This continues to be true for semi-primitive wall-crossing with a larger multiplicity of  $\Gamma_1$  and non-primitive wall-crossing in general. Consider for example wall-crossing for  $(2\Gamma_1, 2\Gamma_2)$ . Equation (2.13) expresses  $\Delta\bar{\Omega}(2\Gamma_1 + 2\Gamma_2; t)$  as a sum of terms indexed by nested lists which can be attributed to different flow trees. It is not difficult to see that this is a generic property of the jumps given by the KS-formula. The non-trivial information provided by the KS-formula are the prefactors of the contributions. Nested lists and flow trees are clearly useful tools for enumerating invariants subject to wall-crossing.

Of course, the integer invariants  $\Omega(\Gamma)$  are useful too. For example, we have seen that the semi-primitive wall-crossing formula is a nice product formula in terms of them. This has a geometric interpretation in terms of halos ( $N$  centers of  $\Gamma_1$  placed on an equal distance around a center with charge  $\Gamma_2$ ), and correctly accounts for the bose/fermi statistics [15].

One might wonder why  $S$ -duality and integrality of the invariants are not compatible although they are both well motivated from physics. A pragmatic reason is that modularity seems to require that the jumps of the indices can be written in terms of products of invariants, such that the sum of the arguments of the invariants equals the total charge. Such an identification is possible for  $\bar{\Omega}(\Gamma; t)$  but not for  $\Omega(\Gamma; t)$ .

Another physical motivation for the rational invariants are IIB D-brane instantons. The IIA BPS-states can be mapped to IIB instantons by a timelike T-duality, which suggests that the instanton numbers are equal to the BPS-invariants  $\Omega(\Gamma; t)$ . The invariants  $\bar{\Omega}(\Gamma; t)$  appear for instantons in their measure [42], the sum over  $m|\Gamma$  incorporates the contributions of multiple instantons. This sum appears for D1-D(-1) instantons in fact after a Poisson resummation of a manifestly  $S$ -duality invariant sum (analogous to Poincaré series) [43, 44]. The relation between  $\Omega(\Gamma; t)$  and  $\bar{\Omega}(\Gamma; t)$  is analogous to Gromov–Witten invariants of  $m$ -fold covers of worldsheet instantons  $\bar{n}_{Q,g} = \sum_m \frac{n_{Q/m,g}}{m^3}$ , where  $n_{Q,g}$  are also expected to be integers [3]. The rational invariants raise the question about the status of the MSW CFT for non-primitive magnetic charges  $P$ . If this is a proper CFT, the modular invariant partition function must have integer coefficients. However, since the BPS-object is not protected by conservation laws against decomposition into smaller objects, the degrees of freedom might not combine to a proper conformal field theory.

## 4 Summary and discussion

The previous sections discussed the KS wall-crossing formula and flow trees, and applied these to D4-D2-D0 black holes. Two new results that are generally applicable to BPS wall-crossing using flow trees are:

- The sign of the flow parameter along every edge can be determined iteratively in terms of the initial moduli  $t$ , without explicit calculation of the flow throughout the tree.
- It is demonstrated that  $\Delta\bar{\Omega}(\Gamma; t)$  as derived from the KS-formula, can be decomposed into certain combinations of rational invariants  $\bar{\Omega}(\Gamma, t)$  classified by nested lists, which also classify the flow trees. This suggests that the contribution to the index of a flow tree is conveniently expressed in terms of the rational invariants.

The discussion on wall-crossing for D4-D2-D0 black holes is restricted to the large volume limit of a single Kähler cone. The following results are obtained:

- For  $N \leq 3$  is proven that the indefinite quadratic form  $(Q - B)_+^2 - \sum_{i=1}^N (Q_i - B)_i^2$  is positive definite for flow trees, since it is implied by the positivity of flow parameters in the tree. This result is expected to be true for any  $N$ , which would imply that the BPS partition function in the mixed ensemble is convergent.
- The contribution to the partition function of flow trees with three endpoints is constructed, including the case where two endpoints have equal charge. The contribution of trees with non-primitive and primitive charges nicely combine to products of vector-valued modular forms, and mock Siegel theta functions.
- The  $S$ -duality invariant partition function is a generating function of the rational invariants  $\bar{\Omega}(\Gamma, t)$ . It is conceivable that the contributions to the partition function of trees with prescribed magnetic charges preserve  $S$ -duality.

Various aspects of wall-crossing for D4-D2-D0 BPS-states remain to be better understood. A major aspect which was not addressed here, is the modular completion of the mock Siegel theta function for  $N = 3$ . This prevented a confirmation of  $S$ -duality by the supergravity partition function in this paper, although it is shown that important prerequisites are satisfied. The main obstacles are 1) the signature of the indefinite quadratic form is  $(2, 3b_2 - 2)$ , and 2) the complexity of the flow tree condition  $S(T_{(12)3}, t) \neq 0$ . The mathematical literature only reports on indefinite theta functions and their modular completions for signature  $(1, n - 1)$ . Another aspect which

deserves a better understanding is the physical interpretation and derivation of the modular completion, it might be related to perturbative contributions. Contributions to the partition function of flow trees with  $N > 3$  are also left for future research.

This paper made various restrictions on the charges and the region of moduli space; I hope to address in future research non-zero D6-brane charge, to include finite volume effects and to cross walls between Kähler cones. Another interesting direction is to understand better the condition  $S(T_{(12)3}, t) \neq 0$  from a more mathematical perspective, now it can be determined so easily in terms of  $t$ . An interesting application in this context might be wall-crossing for sheaves on surfaces as in [28].

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