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String orientations on simplicial homology manifolds

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Abstract

Simplicial homology manifolds are proposed as an interesting class of geometric objects, more general than topological manifolds but still quite tractable, in which questions about the microstructure of space-time can be naturally formulated. Their string orientations are classified by H^3 with coefficients in an extension of the usual group of D-brane charges, by cobordism classes of homology three-spheres with trivial Rokhlin invariant.

1 Notation and background

1.1 In homotopy-theoretic terms, a *string structure* on a smooth manifold M is a lift

$$\begin{array}{ccc} & & B\mathrm{O}\langle 8\rangle \\ & \nearrow & \downarrow \\ M & \longrightarrow & B\mathrm{O} \end{array}$$

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of the map classifying the stable tangent bundle of M; where the seven-connected cover $B\mathrm{O}\langle 8\rangle$ of $B\mathrm{O}$ is the fiber

$$B\mathrm{O}\langle 8\rangle \to B\mathrm{O} \to (B\mathrm{O})_{(7)}$$

of the map to its Postnikov approximation [28] having homotopy groups concentrated in degrees seven and below. Alternately:

$$(B\mathrm{O})\langle 8\rangle = B(\mathrm{O}\langle 7\rangle)$$

is the classifying space for the topological group

$$\mathrm{O}\langle 7\rangle \to \mathrm{O} \to \mathrm{O}_{(6)}$$

whose homotopy groups agree with those of O in degrees greater than or equal to seven.

A string structure can thus be interpreted as a "reduction" of the structure group of the stable tangent bundle of M, from O to $\mathrm{O}\langle 7\rangle$, just as a spin structure is similarly a "reduction" of that structure group to

$$\mathrm{Spin} = \mathrm{O}\langle 2\rangle \to \mathrm{O} \to \mathbb{Z}_2 \times H(\mathbb{Z}_2, 1).$$

An oriented manifold M admits a spin structure iff the Stiefel–Whitney class $w_2 = 0$; similarly, the existence of a string structure requires the vanishing of a version $p_1/2$ of the Pontrjagin class defined for spin manifolds. The set of spin structures on M admits a transitive free action of $H^1(M, \mathbb{Z}_2)$, and by essentially the same argument the set of string structures is an $H^3(X, \mathbb{Z})$-torsor. The associated twisted K-groups of X are natural repositories [7] for Ramond–Ramond D-brane charges.

The interest in string orientations comes from the quantum field theory, where they were recognized as necessary to define an analog of the Dirac operator on the space LM of free loops on M, but mathematical interest [1] in their properties goes back to the early 1970s [17]. There is a parallel interest in the representation of three-dimensional (3D) cohomology classes in local geometric terms, analogous [6] to the description of 2D classes by complex line bundles.

1.2 If G is a connected, simply connected simple Lie group, then

$$\pi_3(G) = \mathbb{Z}$$

by a classical theorem of Bott. A theorem of Kuiper says that the group $\mathrm{Gl}(\mathbb{H})$ of invertible bounded operators on an (infinite-dimensional) Hilbert space is contractible; the projective general linear group

$$\mathrm{PGl}(\mathbb{H}) = \mathrm{Gl}(\mathbb{H})/\mathbb{C}^\times \sim B\mathbb{T}$$

is therefore an Eilenberg–MacLane space of type $H(\mathbb{Z}, 2)$, from which it follows that the classifying space for $\mathrm{PGl}(\mathbb{H})$-bundles is an EM space of type $H(\mathbb{Z}, 3)$. The associated bundle

$$1 \to \mathrm{PGl}(\mathbb{H}) \to G\langle 4\rangle \to G \to 1$$

is an extension of topological groups [29]. The following construction is due to Kitchloo [19, appendix]:

A level one projective representation of the loop group of G on $\mathbb{H}$ defines a homomorphism to $\mathrm{PGl}(\mathbb{H})$, which pulls $\mathrm{Gl}(\mathbb{H})$ back to a version of the universal central extension of LG. Let $\mathcal{A}$ denote the topological Tits building of LG, modeled by the contractible space of connections on a trivial G-bundle over a circle; the pointed loops act freely on it, and the holonomy map makes it a principal ΩG bundle. The diagonal action of LG on

$$\mathcal{A} \times_{\Omega G} \mathrm{PGl}(\mathbb{H}) := G\langle 4\rangle$$

factors through an action of G on $G\langle 4\rangle$ lifting the action of G on itself by conjugation.

Note that because $\mathrm{PGl}(\mathbb{H})$ is the group of automorphisms of $\mathrm{Gl}(\mathbb{H})$, the cohomology group

$$H^1(X, \mathrm{PGl}(\mathbb{H})) \cong H^1(X, H(\mathbb{Z}, 2)) \cong H^3(X, \mathbb{Z})$$

can be interpreted as a Brauer group of bundles of C^*-algebras (up to Morita equivalence) over X. A refinement of this argument [22] represents the Brauer group of bundles of *graded* C^*-algebras over X by the generalized Eilenberg–MacLane space

$$H(\mathbb{Z}_2, 1) \times H(\mathbb{Z}, 3),$$

which suggests interpreting $\mathrm{O}\langle 7\rangle = \mathrm{O}\langle 4\rangle$ in terms of a bundle of such algebras over (S)O.

2 Simplicial homology manifolds

2.1 I also want to thank Kitchloo for observing that in low dimensions, the map

$$\mathrm{PL/O} \to B\mathrm{O} \to B\mathrm{PL}$$

is almost an equivalence: the homotopy groups of the fiber are the Kervaire–Milnor groups of differentiable structures on spheres, which (aside from the smooth 4D Poincaré conjecture ...) are trivial below dimension seven. This implies that a string structure on a smooth manifold is the same as a smoothing of a topological manifold endowed with a lift

$$\begin{array}{ccc} & & B\mathrm{Top}\langle 8\rangle \\ & \nearrow & \downarrow \\ M & \longrightarrow & B\mathrm{Top} \end{array}$$

of the map classifying its tangent topological block bundle. The theorem

$$\mathrm{Top/PL} \sim H(\mathbb{Z}_2, 3)$$

of Kirby and Siebenman [25] seems also to point in this direction.

2.2 The classification of string structures on geometric objects more general than smooth manifolds is accessible nowadays, thanks to many researcher-years of deep work related to the Hauptvermutung, suggesting that questions like "Who ordered the differentiable structure" may not be out of reach. I will summarize some background from Ranicki's elegant account [23], but in some cases I'll use terminology from [16]:

Definition 2.1. A space X is a d-dimensional homology manifold iff for any $x \in X$,

$$H_*(X, X - \{x\}; \mathbb{Z}) \cong H_*(S^{d-1}; \mathbb{Z});$$

but a *simplicial* homology manifold is a simplicial complex K such that for any k-simplex $\sigma \in K$,

$$H_*(\mathrm{link}_K(\sigma); \mathbb{Z}) \cong H_*(S^{d-k-1}; \mathbb{Z}).$$

The polyhedron $|K|$ of K is a homology manifold iff K is a simplicial homology manifold. A manifold homology resolution $f : M \to X$ of a space X is a compact topological manifold M together with a surjective map f with acyclic point inverses.

The element $\varkappa_k(K) \in H^k(|K|, \Theta_{k-1})$ Poincaré dual to the cycle

$$\sum_{|\sigma|=d-k} [\mathrm{link}_K(\sigma)] \cdot \sigma \in H_{d-k}(|K|, \Theta_{k-1})$$

[30, pp. 63–65] with coefficients in the group of simplicial homology spheres (up to cobordism through PL homology cylinders) is trivial unless $k = 4$: for $\Theta := \Theta_3$ is the only nontrivial coefficient group. (It is *not* finitely generated [13, 14, 26].) There is a block bundle theory [15, 16] for simplicial homology manifolds, resulting in a fibration

$$B\mathrm{PL} \to B\mathrm{HL} \to H(\Theta, 4)$$

of classifying spaces.

Theorem 2.2 (cf. [8, 23 §5]). *A simplicial homology manifold K of dimension ≥ 5 admits a PL manifold homology resolution iff*

$$\varkappa_4(K) = 0.$$

The resolutions themselves are classified by maps to $H(\Theta, 3)$.

2.3 Using this technology, the question motivating this note can be formulated as the problem of understanding the map

$$B(\mathrm{Top}\langle 7\rangle) = B(\mathrm{PL}\langle 7\rangle) \to B\mathrm{HL}.$$

Its fiber can be decomposed as

$$\mathrm{PL}/\mathrm{Top}\langle 7\rangle \to \mathrm{HL}/\mathrm{PL}\langle 7\rangle \to \mathrm{HL}/\mathrm{PL} = H(\Theta, 3);$$

the fibration

$$\mathrm{PL}/\mathrm{Top}\langle 7\rangle \to \mathrm{Top}/\mathrm{Top}\langle 7\rangle = H(\mathbb{Z}_2, 1) \times H(\mathbb{Z}, 3) \to \mathrm{Top}/\mathrm{PL} = H(\mathbb{Z}_2, 3)$$

shows that $\mathrm{PL}/\mathrm{Top}\langle 7\rangle$ is a three-stage Postnikov system, with homotopy group $\mathbb{Z}$ in degree three and $\mathbb{Z}_2$ in degrees one and two. The group $\pi_*(\mathrm{HL}/\mathrm{PL}\langle 7\rangle)$ is therefore $\mathbb{Z}_2$ in degree one and zero in degree two, while in degree three there is an exact sequence

$$0 \to \mathbb{Z} \to \pi_3(\mathrm{HL}/\mathrm{Top}\langle 7\rangle) \to \Theta \to \mathbb{Z}_2 \to 0.$$

The map on the right can be identified with Rokhlin's invariant

$$\Sigma \mapsto \rho(\Sigma) := \frac{1}{8} \text{ signature } (W) \text{ modulo } (2) : \Theta \to \mathbb{Z}_2$$

of a homology three-sphere Σ (where W is a spin four-manifold with $\partial W = \Sigma$). Let Θ_0 be the kernel of ρ.

Corollary 2.3. *When K is a smoothable ($\varkappa(K) = 0$) simplicial homology string manifold of dimension ≥5, PL manifold structures on a homology resolution are classified by elements of $H^3(|K|, \widetilde{\Theta})$, where*

$$\widetilde{\Theta} := \mathbb{Z} \oplus \Theta_0 \cong \pi_3(\mathrm{HL}/\mathrm{Top}\langle 7\rangle).$$

Proof. The exact sequence above splits, because the infinite cyclic group maps isomorphically to the third homotopy group of Top. This suggests, among other things, the existence of a combinatorial formula [3] for its Pontrjagin class. When that class vanishes, lifts of the classifying map from $|K|$ to BHL to a map from a homology resolution X are classified by maps to $\mathrm{HL}/\mathrm{Top}\langle 7\rangle$. □

2.4 The inclusion of the fiber in

$$H(\Theta, 3) \to B\mathrm{PL} \to B\mathrm{HL}$$

is trivial at odd primes: $B\mathrm{PL} \cong B\otimes$ [31], but K-theory is blind to Eilenberg–MacLane spaces $H(A, n)$ for $n > 2$. At the prime two, there are still open questions. In particular, it is not known if the Rokhlin homomorphism ρ splits: this is equivalent to the conjecture that all topological manifolds of dim > 4 are simplicial complexes.

Freed [11] identifies the classifying space of the Picard category of $\mathbb{Z}_2$-graded complex lines as a two-stage Postnikov system. Its associated infinite-loop spectrum

$$\Sigma^2 \tilde{I}\langle 0\rangle := \mathbb{L}_{\pm} \longrightarrow H\mathbb{Z}_2 \xrightarrow{\beta Sq^2} \Sigma^3 H\mathbb{Z}$$

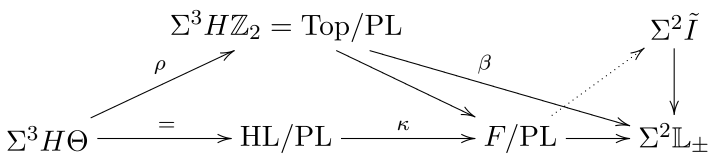
of the double suspension of the Anderson dual $\tilde{I}$ [20, Appendix B, 20] of the sphere spectrum, which is characterized by a short exact sequence

$$0 \to \mathrm{Ext}(E_{*-1}, \mathbb{Z}) \to \tilde{I}^*(E) \to \mathrm{Hom}(E_*, \mathbb{Z}) \to 0,$$

associated to a spectrum E. This same small Postnikov system appears as the base

$$F/\mathrm{PL} \cong \Sigma^{4*} H\mathbb{Z} \times \Sigma^{4*+2} H\mathbb{Z}_2 \times \Sigma^2 \mathbb{L}_\pm \ (* > 1),$$

of the (two-localization) of the infinite-loop space classifying piecewise-linear structures on a Poincaré-duality space [21, Chapter VII]. These observations can then be assembled into a diagram



of spectra.

On the other hand, Anderson's exact sequence implies $\tilde{I}^2(H\Theta_0) = 0$, so (the nonzero invariant defined by Rokhlin's invariant) $\rho \in \tilde{I}^1(H\mathbb{Z}_2)$ in the exact sequence

$$\cdots \to \tilde{I}^2(H\Theta_0) \to \tilde{I}^1(H\mathbb{Z}_2) \to \tilde{I}^1(H\Theta) \to \dots ;$$

maps to (the non-zero invariant defined by) $\kappa \in \tilde{I}^1(H\Theta)$: which defines a homomorphism from $B\Theta$ to $\mathbb{L}_\pm$, interpretable as a topological field theory mapping the category with one object, and 3D homology cylinders as morphisms, to the category of $\mathbb{Z}_2$-graded complex lines.

This can be regarded as a lift of Rokhlin's invariant, regarded as a topological field theory taking values in the Picard category of real lines. It suggests the interest of super-Chern–Simons theories [10, 12, §9] defined on simplicial homology spin manifolds.

3 Über die Hypothesen, welche zu Grunde der Physik liegen

Following [6, §VII], it is tempting to interpret the elements of $\mathbb{Z}$ in $\widetilde{\Theta}$ as topological twists "in the large" (or at infinity), and elements of Θ_0 as twists "in the small". Physics has a tradition of concern (cf. eg Wheeler) with the possibility that the microstructure of the Universe might in some way be nonEuclidean. This seems legitimate: experiments in *very* high-energy physics probe the topology of space-time at very short distance, and it is conceivable that at very fine scales physical space might be described by some kind of quantum foam model [24], possibly involving ensembles with varying topology.

These ideas have a big literature, but interested researchers seem not to be very aware of the long history of interest in analogous questions among topologists. In particular, the extended homology cobordism group $\widetilde{\Theta}$ seems to capture rather precisely the idea that the space-time "bubbles" in which very-short-distance interactions occur—I'm thinking of the way Feynman diagrams are often drawn—might be bounded by non-standard spheres.

If physics starts by hypothesizing the existence of a simplicial homology manifold structure[1] on some (say, 10D or 11D) space-time K, then the vanishing of w_2 decides the existence of a spin structure and the vanishing of $\varkappa(K)$ decides the existence of a PL resolution X. When $p_1/2 = 0$, resolutions of $|K|$ admit string structures; this is quite like the standard situation. However, the possible *twists* of that string resolution (related to B-fields [27], Vafa's discrete torsion [2, §1.6] and perhaps more generally to D-brane charges [5, §4.4]) lie in $H^3(|K|, \widetilde{\Theta})$, which is much bigger than the usual group of gerbes over $|K|$. There may even be 'experimental' evidence for the physical relevance of twisting by homology three-spheres, in that deep results about the structure of such manifolds are derived by scattering Yang–Mills bosons off them: i.e., from Donaldson theory [9].

As for exotic homology manifolds [4], hypotheses non fingo: in part because they, unlike simplicial homology manifolds, seem to lack the clocks and measuring rods that play the role of rulers and compasses in classical geometry. This is probably a lack of imagination on my part; a deeper concern is that major questions in 4D geometric topology remain open. Here I want only to make the point that simplicial homology manifolds are understood well enough to test against the models of contemporary physics.

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[1] conceptually similar to a causal structure

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