

E_8 gauge theory and gerbes in string theory

Hisham Sati^{1,2,3}

¹Department of Mathematics, Yale University, New Haven, CT 06520, USA

²Department of Pure Mathematics, University of Adelaide, Adelaide,
SA 5005, Australia

³The Erwin Schrödinger International, Institute for Mathematical Physics,
Boltzmannngasse 9, A-1090 Wien, Austria
hisham.sati@yale.edu

Abstract

The reduction of the E_8 gauge theory to ten dimensions leads to a loop group, which in relation to twisted K-theory has a Dixmier–Douady class identified with the Neveu–Schwarz H-field. We give an interpretation of the degree two part of the eta form by comparing the adiabatic limit of the eta invariant with the one loop term in type IIA. More generally, starting with a G -bundle, the comparison for manifolds with String structure identifies G with E_8 and the representation as the adjoint, due to an interesting appearance of the dual Coxeter number. This makes possible a description in terms of a generalized Wess-Zumino-Witten model at the critical level. We also discuss the relation to the index gerbe, the possibility of obtaining such bundles from loop space, and the symmetry breaking to finite-dimensional bundles. We discuss the implications of this and we give several proposals.

CONTENTS

1	Introduction	402
2	Review: phase of the M-theory partition function	404
3	Identification of the G -bundle and the eta forms	405
4	The four form as an index	407
4.1	The index gerbe	408
4.1.1	The zero-form component	410
4.1.2	The two-form component	410
5	The loop group description	411
5.1	The gerbe via loop space	412
5.2	Twist versus twisted, based versus unbased	412
5.3	The string class	414
5.4	The Higgs field and the reduction of the LE_8 to E_8	415
6	Bundles from loop space	416
6.1	Breaking the loop bundle to $U(n)$	416
6.2	Fourier decomposition	417
6.3	Relating X^{10} to LX^{10} : conditions for Fourier decomposition	418
6.4	Producing Fourier-decomposable loop bundles via deformation	419
6.5	The eta invariant of the horizontal Dirac operator	420
7	The generalized WZW description	421
7.1	The bosonic sector	422

E_8 GAUGE THEORY AND GERBES IN STRING THEORY	401
7.2 The fermionic sector	423
7.3 Coupling the supercharge to the vector potential	424
8 Further discussion and proposals	426
8.1 The relation to twisted K-theory	427
8.2 The sign involutions	427
8.3 The Ramond–Ramond (RR) fields	428
8.4 Higher order corrections	428
8.5 The quantization conditions	429
8.6 The role of E_8	429
8.7 The topology of spacetime	430
8.8 Spacetime as parametrizing families	430
8.9 Worldsheet versus spacetime	430
8.10 Adiabatic limits on the worldvolumes	430
8.11 The Higgs field and the topological membrane	431
8.12 The critical level	431
8.13 High energy and tensionless limits of strings	432
8.14 The noncommutative geometry of spacetime	433
8.15 The cosmological constant	433
8.16 Holography	434
Acknowledgments	434
References	435

1 Introduction

The form fields in M-theory and string theory play an important role in the characterization of the global structure of the theory. The study of their quantization conditions and partition functions has led to a wealth of topological and global analytic information about the objects and the fields of string theory. In particular, Diaconescu, Moore and Witten (DMW) [1] initiated the comparison of the partition function in M-theory, described by index theory of an E_8 bundle and a Rarita–Schwinger bundle, with the partition function in type IIA string theory described by K-theory.

We focus on the E_8 principal bundle [2] on the 11-dimensional spacetime of M-theory,

$$\begin{array}{ccc} E_8 & \rightarrow & P \\ & & \downarrow \\ S^1 & \rightarrow & Y^{11} \\ & & \downarrow \pi \\ & & X^{10} \end{array} \quad (1.1)$$

where Y^{11} in turn is a principal S^1 bundle over the ten-dimensional manifold X^{10} . Corresponding to P is an associated vector bundle V , both characterized by a degree four integral class a .

In [1], the NSNS B -field was switched off and so it was assumed that the M-theory C -field C_3 is a pullback from X^{10} . The implication is that the topological invariant, the phase $\Omega_M(C_3)$, depends only on a and not on C_3 , so that one writes $\Omega_M(a)$. In [3] the generalization of this to include the NSNS H -field was considered, generalizing also the case $[H] = 0$ [4], and corresponding to the situation when the bundles in M-theory are not lifted from the type IIA base. As in [3] our main focus will be the E_8 gauge theory because the Rarita–Schwinger bundle involves only natural bundles and such bundles are automatically lifted from the base of the S^1 -bundle.

In gauge theory, it has been known that periodic instantons of a gauge theory with structure group G on a space $Y^4 = S^1 \times X^3$ give rise to monopoles on X^3 with structure group the Kac–Moody group of G [5]. The situation in M-theory is analogous and so one expects that starting from an E_8 gauge theory on Y^{11} one gets an LE_8 bundle on X^{10} [6]. Indeed the computations at the classical level further confirm this [7, 8].

In [3] an expression was found for the phase of M-theory by using the adiabatic limit of the eta invariant.¹ The resulting expression was an integral

¹We note (with V. Mathai) that the analysis of the phase is where the nontrivial circle bundle matters in [3]. For the other parts of that paper we might as well have assumed a trivial circle bundle.

over the ten-dimensional base of the circle bundle, thus relating the M-theory data on the nontrivial circle bundle to the data of type IIA on X^{10} . However, that expression involved the eta forms, the higher degree analogs of the more familiar eta invariant, and the expression was not evaluated. There the desire was expressed to find an interpretation of the components of the eta form. It is the purpose of this note to propose such an interpretation for the first nontrivial eta forms, $\hat{\eta}^{(2)}$ of degree two. We do this by comparing the expression of the adiabatic limit with the one loop term in type IIA.

The two faces of the cohomology class in $H^3(X^{10}; \mathbb{Z})$, the Dixmier–Douady (DD) invariant, are utilized, the first being the obstruction to replacing the bundle LE_8 associated with a projective representation by a vector bundle which is the central extension \widehat{LE}_8 , and the second is that the DD-class describes a (stable) equivalence class of a gerbe² over X^{10} . This is used in order to relate the E_8 gauge theory to twisted K-theory. The former results in the loop group (LG) bundle upon reduction to ten dimensions and the latter can be interpreted as the K-theory of (bundle) gerbes [9]. For applications of bundle gerbes and DD-classes in families problems in quantum field theory (QFT) see [10, 11].

Starting with the principal E_8 bundle, we compare the adiabatic limit with the one loop term in type IIA string theory in ten dimensions. The observation is that for string manifolds, i.e., for those with $\lambda = p_1/2$ zero, the two-form component of the eta form is identified with the Neveu-Schwarz (NSNS) gerbe. What is interesting is the integer multiplying the generator. This leads to an interesting appearance of the dual Coxeter number of E_8 in front of the degree three generator. In fact the analysis can be made general and can be seen, in some sense, as a *discovery* of E_8 . Starting with a G -bundle and performing the dimensional reduction one gets a bundle of LG , the DD class of which is the obstruction to lifting to the bundle with structure group the central extension \widehat{LG} . From the above identification we can see that the level is -30 which, if we assume the adjoint representation, is the negative of the dual Coxeter number of E_8 , and so $G = E_8$. It is interesting that this identification works for manifolds with a String structure. This is in line with the proposals in [12–18] on the relevance of elliptic cohomology.

We distinguish between Dirac operators on the circle part and Dirac operators on the base part of the circle bundle. The study of the former uses

²We comment on the use of gerbes as geometric objects in our discussion. To describe higher degree fields one can also use differential characters. While they are closely related, what we have seen in the literature is that gerbes seem to be more adapted to analytical descriptions such as index theory — which we use in this note — whereas differential characters seem to be more useful in the description of the gauge fields in quantum (higher) gauge theories.

Mickelsson's construction [19]. This then leads us to suspect the possibility of having a Wess–Zumino–Witten (WZW) construction. Indeed we make the connection to such a construction, which suggests viewing spacetime as part of a generalized WZW model with E_8 as target. From the LE_8 point of view, the loop bundles coupled to the Dirac operator give contributions to the index. We also study the reduction of the LE_8 bundle down to finite-dimensional bundles using [20] and interpret the corresponding Higgs field à la [21]. An interesting example of the eta form is the index gerbe [22] related to the families index theorem [23]. We apply this to our problem and discuss the implications for string theory and M-theory. The latter is elaborated on in the last section.

2 Review: phase of the M-theory partition function

The topological part of the action that is used in the global M-theoretic considerations [1–3, 24] is the sum of the Chern–Simons term [25] and the one loop term [26, 27],

$$S_{11} = \frac{1}{6} \int_{Y^{11}} C_3 \wedge G_4 \wedge G_4 - \int_{Y^{11}} C_3 \wedge I_8, \quad (2.1)$$

where I_8 is the anomaly polynomial given in terms of the Pontrjagin classes of the tangent bundle of Y^{11} .

The above action was extended in [2] to a 12-dimensional manifold Z^{12} whose boundary is the original 11-manifold Y^{11} . This is possible because the relevant cobordism groups vanish. In 12 dimensions, the action can be written in terms of the index of the Dirac operator coupled to E_8 and the Rarita–Schwinger operator, i.e., a Dirac operator coupled to $TY^{11} - 3\mathcal{O}$ with \mathcal{O} a trivial line bundle,

$$S_{12} = \frac{1}{2} \text{Index}(D_{V(a)}) + \frac{1}{4} \text{Index}(D_{\text{R.S.}}), \quad (2.2)$$

where $V(a)$ is the vector bundle associated to the E_8 principal bundle with characteristic class a of dimension four. Then, using the Atiyah–Patodi–Singer index theorem with the appropriate boundary conditions, the above action in 11 dimensions can be written in terms of the (reduced) eta invariants, so that the resulting phase of the C -field is [2]

$$\Omega_M(C_3) = \exp \left[2\pi i \left(\frac{\bar{\eta}(D_{V(a)})}{2} + \frac{\bar{\eta}(D_{\text{R.S.}})}{4} \right) \right]. \quad (2.3)$$

The result is independent of the bounding manifold Z^{12} used.

We employ the geometric setup in [3]. The Riemannian metric on the circle bundle Y^{11} is $g_{Y^{11}} = \pi^*(g_{X^{10}}) + \pi^*(e^{2\phi/3})\mathcal{A} \otimes \mathcal{A}$, where $g_{X^{10}}$ is the Riemannian metric on X^{10} , $e^{2\phi/3}$ is the norm of the Killing vector along S^1 , which in this trivialization is given by ∂_θ , where θ is the coordinate on the circle, ϕ is the dilaton, i.e., a real function on X^{10} and \mathcal{A} is a connection one-form on the circle bundle Y^{11} . Note that the component of the curvature in the direction of the circle action is

$$R_{11} = e^{2\phi/3} = g_s^{2/3}. \tag{2.4}$$

Such a choice of Riemannian metric is compatible with the principal bundle structure in the sense that the given circle action acts as isometries on Y^{11} . Performing a rescaling to the above metric and using the identification (2.4), the desired metric ansatz leading to type IIA is

$$g_{Y^{11}} = g_s^{4/3} g_{S^1} + t g_s^{-2/3} g_{X^{10}} \tag{2.5}$$

in the limit $t \rightarrow \infty$ then $g_s \rightarrow 0$.³

Now let us consider the E_8 -coupled Dirac operator D on Y^{11} . Using the formalism of Bismut–Cheeger [28] (and Dai [29]) the adiabatic limit of the reduced eta invariants in the phase is [3]

$$\lim_{t \rightarrow \infty} \bar{\eta}(D_{V(a)}^t) = \int_{X^{10}} \hat{A}(\mathcal{R}^{X^{10}}) \wedge \hat{\eta}_{V(a)} + \Sigma, \tag{2.6}$$

where we write Σ collectively for the terms that include eta invariants of the Dirac operator on X^{10} coupled to the vector bundle $\ker D_{S^1}$ as well as for the dimensions of certain kernels. We do not record these terms as we will not need their explicit form in this note.

3 Identification of the G -bundle and the eta forms

We would like to see whether anything explicit can be said about the expression (2.6). In particular, we would like to understand whether a meaning can be given to the components of the eta forms $\hat{\eta}$ of the E_8 bundle. The general strategy that we follow is to try to identify as much as possible with terms that exist in type IIA string theory. However, we would also like to see whether starting from a G -bundle we can *discover* that the structure

³Note that we use the letter g to denote both the metric (with a subscript given by the symbol for a manifold) and the string coupling (with a subscript s). We hope that this will be clear from the context.

group is E_8 . This is what we would like to achieve. Let us assume a general vector bundle associated to a principal G -bundle with an unspecified structure group given by a Lie group G .

Since the adiabatic limit involves the \widehat{A} -genus of the tangent bundle of the base X^{10} , from the point of view of type IIA this implies that we should seek terms that contain such gravitational terms. Indeed there is the one-loop term which has a degree eight gravitational piece. Thus we look at this term

$$\int_{X^{10}} B_2 \wedge I_8, \tag{3.1}$$

which results from the reduction of the corresponding term in 11 dimensions and involves the B -field and the degree eight polynomial in the Pontrjagin classes of the tangent bundle of X^{10}

$$I_8 = \frac{1}{48}[p_2 - \lambda^2]. \tag{3.2}$$

It is obvious at this stage that (3.1) cannot be identified as it stands with the piece with the same degree for the gravitational term as in (2.6). However, this is possible provided that some assumptions hold. Note that while I_8 is not exactly \widehat{A}_8 , the two are related via (this is derived and used in [24])

$$I_8 = -30\widehat{A}_8 + \frac{1}{8}\lambda^2. \tag{3.3}$$

It is very interesting that if X^{10} is a *String* manifold, i.e., with $\lambda = 0$, then the comparison of (3.1) with (2.6) leads to a formula for the degree two component of the eta form

$$\widehat{\eta}^{(2)} = -30B_2 + d\alpha_1, \tag{3.4}$$

where α_1 is some one form. There are several points to be made at this stage. First, there is the extra factor $d\alpha_1$ that makes the identification of $\widehat{\eta}^{(2)}$ with the B -field only valid up to this exact term. Classically, we are interested in the action and consequently in the eta form. However, quantum-mechanically, what matters is the exponential of the action in the path integral, which in the case of M-theory gives (fractional powers of)

$$e^{2\pi i\eta}, \tag{3.5}$$

and in relating to type IIA we would be interested in the adiabatic limit of the function (3.5) rather than the adiabatic limit of η itself. As functions

on the circle can be viewed as closed one forms, this means that at the quantum level we are interested in the differential of the eta forms $d\hat{\eta}$. Via the identification we made above, this then means that we should be looking at the H -field rather than the B -field, which seems to be consistent with consideration from twisted K-theory in the general case. Considering then the differential of (3.4) removes the ambiguity coming from the exact term, and we thus have

$$d\hat{\eta}^{(2)} = -30H_3. \tag{3.6}$$

Thus in our context the two-form component of η is a connection on a gerbe. However, there is still a factor of 30 and a minus sign that need to be explained. In general, the factor multiplying the gerbe is related to the representation of the group used. In particular, for the adjoint representation that number would be the dual Coxeter number h^\vee corresponding to the spin gerbe. In our case, we can see that 30 is just the value of the dual Coxeter number for E_8 ⁴. Thus, from matching the topological terms in the action we are able to discover that the structure group involved in the original bundle in 11 (and 12) dimensions is E_8 , provided we specify the representation to be the adjoint representation — which seems to be the most natural choice for a gauge theory. Alternatively, if we assume that we start with an E_8 gauge theory, then the above procedure specifies the representation for that E_8 principal bundle giving the associated vector bundle in the adjoint representation. We will elaborate on this later in Section 7.

4 The four form as an index

In this section we would like to see whether the four-form G_4 can have interesting expressions in certain situations. In particular, we would like to see whether the fact that the topological action (2.1) is written as an index (2.2) in 12 dimensions is reflected in the topological part of the membrane action being also written as an index. We start by embedding the M2-brane in 11-dimensional spacetime Y^{11} . The understanding of the topology of both theories requires the extension to a “coboundary”, namely the membrane to a bounding four-manifold X^4 and M-theory to a bounding twelve-manifold Z^{12} . Of course one cannot just pull back an index,⁵ but we can assume

⁴Of course this is not unique. There are other choices: $A_{29} = \mathfrak{su}(30)$, $C_{29} = \mathfrak{sp}(58)$ and $D_{16} = \mathfrak{so}(32)$. Of the three the last seems the most relevant. In any case we leave this for future investigation.

⁵We cannot just pull back the value of the index but it is possible to pull back the index as a bundle.

that the vector bundles on Z^{12} get pulled back to X^4 . This means that the gauge part of the Atiyah–Singer index theorem contains the Chern character will be the same. We then have to look at the effect of the gravitational term. In comparing $\widehat{A}(Z^{12})$ and $\widehat{A}(X^4)$, it is obvious that they are in general different. However, they can give the same expression — when integrated — in a special case on which we focus. Assume that Z^{12} is decomposable into a product of two spaces, a four-dimensional space which we identify⁶ with the space X^4 cobounding the membrane, and an extra eight-dimensional piece N^8 . The index in 12 dimensions would then decompose as

$$\int_{Z^{12}} \widehat{A}(Z^{12}) \wedge \text{ch}(E) = \int_{X^4 \times N^8} \widehat{A}(X^4) \wedge \widehat{A}(N^8) \wedge \text{ch}(E), \quad (4.1)$$

where we use the multiplicative property of the \widehat{A} -genus $\widehat{A}(X_1 \times X_2) = \widehat{A}(X_1) \wedge \widehat{A}(X_2)$. We get the desired result if we further assume that N^8 has $\widehat{A}(N^8) = 1$. Such manifolds N^8 are called *Bott manifolds*, examples of which are manifolds of special holonomy. These manifolds occur naturally in compactifications of M-theory and string theory and so the situation that we described, although not completely general, is fairly generic in existing examples. Next we describe a different — but related — way of getting an index expression for the form field.

4.1 The index gerbe

In this section we consider a special kind of gerbe, namely the index gerbe [22, 23]. Two motivations for this are the fact that the general formula for $d\widehat{\eta}$ is given by the integral over the fiber of an index, and the embedding argument we gave in the preceding paragraph. A further motivation for considering this kind of gerbe is the following. We would like to understand the effect of the Dirac operator of the vertical tangent bundle on the eta invariant and consequently on the phase of the M-theory partition function. Furthermore, we would like to understand the local behavior, i.e., the behavior on local patches that cover the base, and how these patch together to form the global objects that appear in the eta invariant and in its adiabatic limit. If we concentrate on the behavior of the phase of the Dirac operator of the vertical tangent bundle on the local patches, then we are naturally led to the index gerbe.

⁶We think of the membrane as wrapping a subspace of spacetime.

The mathematical construction is given in [22], which we follow. For the M-theory circle bundle Y^{11} with a projection π to the type IIA base X^{10} , we consider the vertical tangent bundle⁷ $TS^1 = \ker \pi$ which can be viewed as a line bundle over X^{10} . We assume that TS^1 has a spin structure and so we can form the corresponding spinor bundle $S(S^1)$. We also have V , a complex vector bundle on Y^{11} with a compatible connection. For us, V is either the vector bundle associated to the E_8 vector bundle or the vector bundle $TY^{11} - 3\mathcal{O}$, i.e., the Rarita–Schwinger bundle. In this note we will concentrate on the first of the two bundles, and so we will use V to denote the E_8 bundle. We couple the spinors on the vertical bundle to the vector bundle V by forming $E = S(S^1) \otimes V$. Then π_*E is the infinite-dimensional vector bundle on X^{10} whose fiber over $x \in X^{10}$ is the space of sections $C^\infty(S^1_x; E|_{S^1_x})$. The base X^{10} acts as a parameterizing space for a family $D = \{D_x\}_{x \in X^{10}}$ of Dirac operators with D_x acting fiberwise on the space of sections over x .

The construction of the index gerbe is as follows [22]. We cover X^{10} by a set of charts $\{U_\alpha\}$ where α (and β, \dots) take values in an indexing set I . The Dirac operator D_α defined over a patch U_α will have a modulus and a phase, the latter being given by⁸

$$\frac{D_\alpha}{|D_\alpha|}. \tag{4.2}$$

This is a mod 2 quantity, i.e., it takes the values ± 1 . We are interested in the difference of phases on the overlap of patches. If $U_\alpha \cap U_\beta$ is nonempty then the eigenvalue of the operators

$$\frac{D_\alpha}{|D_\alpha|} - \frac{D_\beta}{|D_\beta|} \tag{4.3}$$

on the overlap can be 0, 2 or -2 .

The components of the differential of the eta form are given as the corresponding components of the integral over the circle (i.e., the fiber) of the Atiyah–Singer index formula for the Dirac operator on $S(S^1)$ coupled to the vector bundle V , i.e., for the coupled bundle E [22]. We are interested in the lowest two degrees of the eta form, namely the degree zero and degree two components.

⁷Note that we use this notation not to mean the tangent bundle of the circle itself.

⁸Assuming no zero modes. An alternative description can be found in [10].

4.1.1 The zero-form component

In this case the eta form is just half the Atiyah–Patodi–Singer eta invariant. For the Dirac operator D_α over a patch U_α , this is given by

$$\widehat{\eta}_\alpha^{(0)} = \frac{1}{2}\eta_\alpha, \tag{4.4}$$

so that the phase is simply

$$\exp[2\pi i \widehat{\eta}_\alpha^{(0)}]. \tag{4.5}$$

On a nonempty intersection $U_\alpha \cap U_\beta$ the difference

$$\widehat{\eta}_\beta^{(0)}|_{U_\alpha \cap U_\beta} - \widehat{\eta}_\alpha^{(0)}|_{U_\alpha \cap U_\beta} \tag{4.6}$$

is a \mathbb{Z} -valued function. If $f_\alpha : U_\alpha \rightarrow S^1$ is defined by (4.5) then on the nonzero overlap

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}, \tag{4.7}$$

so that these functions $\{f_\alpha\}_{\alpha \in I}$ piece together to form a function $f : X^{10} \rightarrow S^1$, such that the restriction of f to a patch U_α is f_α .

The differential of the eta form in this case is equal to the one-form part of the Atiyah–Singer formula [22]

$$\frac{1}{2\pi i} d \ln f = \left(\int_{S^1} \widehat{A}(R^{TS^1}) \wedge \text{ch}(F^V) \right)^{(1)} \in \Omega^1(X^{10}), \tag{4.8}$$

which when evaluated gives

$$\int_{S^1} c_1(F^V). \tag{4.9}$$

In our case of an E_8 bundle, the first Chern class of the E_8 bundle is zero because such bundles are characterized by a degree four class. This implies that f is constant, or more precisely that it is $\exp[2\pi i c]$ for a constant c .

4.1.2 The two-form component

In this case, what is important is differences of eta forms on double overlaps, and here one gets a gerbe [22]. The line bundle that enters the gerbe data

is built out of the two eigenspaces with nonzero eigenvalues, namely

$$L_{\alpha\beta} = \Lambda^{\max} P_+ \otimes \Lambda^{\max} P_-, \tag{4.10}$$

where P_{\pm} are the images of the orthogonal projections onto eigenspaces with eigenvalue ± 2 . These images are finite-dimensional vector bundles and so one can form the highest exterior powers. On the images, D_{α} is positive and D_{β} is negative on P_+ , and D_{α} is negative and D_{β} is positive on P_- . Furthermore, $L_{\alpha\beta}$ has a connection $\nabla_{\alpha\beta}$ induced from the connections $P_{\pm} \nabla P_{\pm}$, and with curvature $F_{\alpha\beta}$ an imaginary-valued two form on $U_{\alpha} \cap U_{\beta}$. In this case, the analog of (4.6) i.e., the difference of the two forms on the overlap is given by

$$\widehat{\eta}_{\beta}^{(2)}|_{U_{\alpha} \cap U_{\beta}} - \widehat{\eta}_{\alpha}^{(2)}|_{U_{\alpha} \cap U_{\beta}} = \frac{-1}{2\pi i} F_{\alpha\beta}. \tag{4.11}$$

The curvature of the gerbe with connection thus obtained is similarly given by the degree three part of the integral over S^1 of the Atiyah–Singer index formula

$$d\widehat{\eta}^{(2)} = \left(\int_{S^1} \widehat{A}(R^{TS^1}) \wedge \text{ch}(F^V) \right)^{(3)} \in \Omega^3(X^{10}). \tag{4.12}$$

Evaluating this expression gives

$$\int_{S^1} -\frac{\text{rank}(V)}{24} p_1(R^{TS^1}) + c_2(F^V), \tag{4.13}$$

which as a rational cohomology class lies in the image of integral classes $H^3(X^{10}; \mathbb{Z})$ in the rational cohomology $H^3(X^{10}; \mathbb{Q})$ as was proved in general in [22].

5 The loop group description

It has been proposed in [6] that the E_8 bundle in M-theory gives rise to an LE_8 bundle in type IIA on X^{10} . This was studied further in [3] where the corresponding classes of the bundles were identified. Starting from principal E_8 bundle over Y^{11} , the dimensional reduction of the M-theory to type IIA gives a LE_8 bundle P' in ten dimensions, characterized by the three-form $H_3 = \int_{S^1} G_4$. Due to the homotopy type of the Lie group E_8 , principal E_8 bundles over Y^{11} are classified by a class a in $H^4(Y^{11}, \mathbb{Z})$. Then the class on LE_8 is $u = \pi_* a \in H^3(X, \mathbb{Z})$, which was identified in [3] with the DD-class

$DD(LE_8)$.⁹ Over each point in the base, the space of sections is identified with the loop group LE_8 , because it can be viewed as maps from the M-theory circle to E_8 , since the bundle over the circle is trivial. Further, the obstruction to lifting the LE_8 bundle P to an $\widehat{LE_8}$ bundle \widehat{P} , covering P , is the DD-class. That is, such a lift is possible only when $H_3 = dB_2$ [3].

5.1 The gerbe via loop space

The isomorphism classes of gerbes on X^{10} form an abelian group \mathbb{G} with the product structure given by the product of gerbes. These isomorphism classes are classified by the characteristic class u for the gerbe, which is a map from \mathbb{G} to third integral cohomology. What is the relation of the gerbes on X^{10} to objects on the loop space? In general, there is a transgression map T that takes $\mathbb{G}(X^{10})$ to the isomorphism classes of line bundles $Line(LX^{10})$ on the loop space [30].

At the level of characteristic classes, there is a compatibility of transgressions, i.e., the transgression of the characteristic class $u(\mathbb{G})$ of a gerbe \mathbb{G} is the first Chern class of the transgression of the gerbe $T(\mathbb{G})$. The latter is a line bundle over the loop space so that $c_1(T(\mathbb{G})) \in H^2(LX^{10}; \mathbb{Z})$ is the first Chern class of this line bundle obtained by the transfer on cohomology

$$T : H^3(X^{10}; \mathbb{Z}) \longrightarrow H^2(LX^{10}; \mathbb{Z}), \quad (5.1)$$

and further, the transgression of the curvature of the gerbe, i.e., of the H -field, matches the curvature of the above line bundle over LX^{10} obtained by transgressing the gerbe [30].¹⁰ One can actually go one more step and relate the gerbe to the holonomy of a connection over the double-loop space of X^{10} by factoring the transgression above with the Bismut–Freed relation between the determinant line bundle on a space and the holonomy on the loop space [30].

5.2 Twist versus twisted, based versus unbased

The subgroup of based loops of E_8 is ΩE_8 , which is defined as the space of maps f from the circle to E_8 that preserve the identity, i.e., such that $f(1) = 1$. The relation between the group of based loops ΩE_8 and the

⁹We continue to take $c_1 = 0$ for the circle bundle as remarked in footnote 1.

¹⁰This is also done in the presence of a connection.

unbased ones LE_8 is $LE_8 = \Omega E_8 \rtimes E_8$, which can be seen from the split short exact sequence

$$\Omega E_8 \longrightarrow LE_8 \longrightarrow E_8. \tag{5.2}$$

The multiplication $E_8 \times \Omega E_8 \rightarrow LE_8$ is a diffeomorphism, with the inverse given by $LE_8 \rightarrow E_8 \times \Omega E_8$ which takes f to $(f(1), f(1)^{-1}f)$, and the two maps are smooth because of the differentiable structure on ΩE_8 . Using the inclusion $E_8 \hookrightarrow LE_8$, one can identify within the class of LE_8 -bundles those which come from E_8 -bundles. For example, within the class of associated vector bundles over X^{10} with fiber \mathbb{C}^{248} lie the vector bundles of the form $E \otimes LC$ with $E \rightarrow X^{10}$ a 248-dimensional vector bundle.

There is a similar sequence of classifying spaces corresponding to (5.2)

$$E_8 \longrightarrow EE_8 \times_{\text{conj}} E_8 \longrightarrow BE_8, \tag{5.3}$$

with the indicated conjugation action. Given a classifying map $X^{10} \rightarrow BE_8$ one can then pull back the E_8 -bundle¹¹ over X^{10} . This can be interpreted as a bundle over X^{10} with fiber $B\Omega E_8$. Thus, a section of this bundle defines a *twisted* principal ΩE_8 -bundle over X^{10} . A section of the $B\Omega E_8$ -bundle is also a map from X^{10} to $EE_8 \times_{\text{conj}} E_8 = BLE_8$ (with conjugation action) and thus classifies principal LE_8 -bundles.

Conversely, given a classifying map $X^{10} \rightarrow BLE_8 = EE_8 \times_{\text{conj}} E_8$, one can project down to BE_8 and pull back the E_8 -bundle as above. The original classifying map defines a section of this E_8 -bundle, and so a *twisted principal* ΩE_8 -bundle over X^{10} . Hence a *principal LE_8 -bundle can be interpreted as a principal ΩE_8 -bundle twisted by a principal E_8 -bundle*. This is a new angle on the construction in [3] and is related to the nonabelian gerbe construction in [31] for the case of the M5-brane.

The usual viewpoint on the relation between the RR and the NSNS fields is that the latter act as a twist to the former when described by cohomology or K-theory. The above bundle description, however, gives an alternative point of view where the NSNS fields seem to be the fields twisted by (part of) the RR fields. Furthermore, this provides some further justification — at least morally — for the proposal in [13] for treating the NSNS field H_3 and the RR field F_3 , in type IIB string theory, democratically, that is, untwist the NSNS twist and view both fields as untwisted elements of elliptic cohomology. In the current context, it is even more because the twist is done by the RR field F_4 , representing the E_8 bundle, and what is being

¹¹which is a bundle of groups and not a principal bundle.

twisted is the NSNS field H_3 , representing the LE_8 -bundle. The E_8 -bundle that defines the twisting is the pull-back of the *principal* E_8 -bundle from BE_8 . The $B\Omega E_8$ -bundle used above is the adjoint bundle of the principal E_8 -bundle.

5.3 The string class

In the case of the LG, the DD-class is in fact just the string class [21] which can be understood as an obstruction on the loop space of our spacetime [32, 33]. For physics purposes, it is desirable to work geometrically and, whenever possible, identify representatives of cohomology classes. Murrari and Stevenson [21] provided an explicit differential three-form representative of the de Rham image of the string class in real cohomology, which is defined using a connection and a Higgs field for the LG. Using this, the string class of our LE_8 bundle on X^{10} will be the integral over the circle of the Pontrjagin class of the corresponding E_8 bundle over $Y^{11} = S^1 \times X^{10}$.

For the LE_8 -bundle Q , the string class can be explicitly characterized as follows [21]. The Higgs field Φ is considered as the map from Q to the space of smooth sections $C^\infty([0, 2\pi], \mathfrak{e}_8)$ satisfying the transformation property

$$\Phi(pg) = \text{ad}(g^{-1})\Phi(p) + g^{-1}\partial_\theta g, \quad (5.4)$$

for $g \in LE_8$ and θ the coordinate on the circle. With A a connection on Q with curvature F , the string class of Q is represented in de Rham cohomology by the three form [21]

$$\frac{-1}{4\pi^2} \int_{S^1} \langle F, \nabla\Phi \rangle d\theta, \quad (5.5)$$

where $\nabla\Phi = \nabla_A\Phi - \partial_\theta A$. We notice that if the Higgs field is gauge-covariantly constant, i.e.,

$$\nabla_A\Phi = d\Phi + [A, \Phi] = 0, \quad (5.6)$$

then (5.5) becomes

$$\frac{1}{4\pi^2} \int_{S^1} \langle F, \partial_\theta A \rangle d\theta. \quad (5.7)$$

5.4 The Higgs field and the reduction of the LE_8 to E_8

In gauge theory, (spontaneous) symmetry breaking occurs if the structure group G of the principal bundle P over X is reducible to a closed subgroup K . This means that there is a principal subbundle of P with structure group K . The necessary and sufficient condition for such a reduction to occur is that the quotient bundle admits a global section. There is a one-to-one correspondence between these global sections Φ of the quotient bundle P/K over X and reduced subbundles $P^\Phi \subset P$. These sections Φ are treated physically as the Higgs fields corresponding to the symmetry breaking. This effect is a quantum effect which occurs when the Lagrangian is invariant under the symmetry group but the vacuum is not.

In the Kaluza–Klein reduction of gravity on S^1 , if one retains the non-zero Fourier modes then the resulting symmetry group on the base is a Kac–Moody extension of the Poincaré group [34]. Although this is a symmetry of the Lagrangian, it is not a symmetry of the vacuum, which in the absence of a cosmological constant is Minkowski space, and the surviving symmetry group is just the Poincaré group. For instance, the dilaton — the “size of the circle” — acts as a Goldstone boson associated with the spontaneous breakdown of global scale invariance. In (super)gravity, the massive modes are spin-2 particles. Likewise, in the gauge theory we expect the massive modes to correspond to massive gauge bosons.

Guided by the above discussion, we expect then that the symmetry in the LE_8 gauge theory will be broken if the vacuum does not respect that symmetry. It then seems reasonable to assume that the resulting group will be the finite-dimensional part, i.e., the Lie group E_8 , after truncating the Fourier modes coming from the loops.¹² Corresponding to the symmetry breaking

$$LE_8 \supset E_8 \tag{5.8}$$

there is a bundle reduction from Q to the subbundle Q^Φ with structure group E_8 . The Higgs field Φ will then be a section of the quotient bundle Q/E_8 , which is an ΩE_8 bundle. At the level of representations, the Higgs field will then take values in the corresponding $\Omega \mathfrak{e}_8$ bundle.¹³

From the point of view of the gauge theory on the S^1 fiber the space of gauge orbits is the classifying space $B\mathcal{G}_0 = \mathcal{A}/\mathcal{G}_0$, i.e., the quotient of the

¹²There are conditions for such a Fourier decomposition to occur. We will discuss this in the next section.

¹³We are assuming a particular situation. The general case is discussed in [21].

space of connections \mathcal{A} by the based gauge transformations \mathcal{G}_0 . The group of gauge transformations is just ΩE_8 and so the space of equivalence classes is just E_8 . So we see that from this point of view, modding out by ΩE_8 corresponds to removing redundant degrees of freedom of the gauge theory on the S^1 part of spacetime.

The discussions above on the symmetry breaking are also in line with the expectation from couplings and considerations of energy scales. Since the tension of a solitonic object is $1/\alpha'^2$ whereas that of a RR object (i.e., a D-brane) is $1/\alpha'$, then when the coupling is lowered the NSNS objects are more massive. This can also be seen from the complementary picture using the field strengths in the (effective) action.

6 Bundles from loop space

6.1 Breaking the loop bundle to $U(n)$

In [1], the explicit comparison between M-theory and K-theory was done by making use of the embedding $(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5 \subset E_8$. A natural question then is what happens when we start with the loop bundle of E_8 . We can think of this in two ways. First, we can “loop both sides”, i.e., get $\mathrm{LU}(n)$ bundles from the LE_8 bundle and in order to get finite-dimensional vector bundles we can break $\mathrm{LU}(n)$ to $U(n)$.¹⁴ Second, we can start with LE_8 and break it to E_8 , and then break E_8 to the unitary group à la DMW ($\mathrm{SU}(5)$ is sufficient in ten dimensions due to stability [1]). Note that the two ways are somewhat related because the classifying space for $\mathrm{LU}(n)$ is the same as the loop of the classifying space of $U(n)$, i.e., $\mathrm{BLU}(n) \cong \mathrm{LBU}(n)$. We have seen an outline of how the second scenario would work. In the rest of this section we consider the first scenario, and use the methods of [20] to get the decomposition

$$LE_8 \supset \mathrm{LU}(n) \supset U(n). \quad (6.1)$$

Consider *rank- n loop bundles* [20], which are infinite-dimensional bundles whose structure group is $\mathrm{LU}(n)$ and whose fibers are isomorphic to the loop space $L\mathbb{C}^n$. The classifying space for such a bundle is the loop space $\mathrm{LBU}(n)$, and so the bundle is classified by a map $f_E : X^{10} \rightarrow \mathrm{LBU}(n)$. Let $\mathrm{ev} : \mathrm{LBU}(n) \rightarrow \mathrm{BU}(n)$ be the evaluation map that evaluates a loop at

¹⁴The reason we are considering $U(n)$ instead of $\mathrm{SU}(n)$ will be explained in Section 6.3.

$1 \in S^1$. The underlying n -dimensional vector bundle $U(E) \rightarrow X^{10}$ is the bundle classified by the composition $\text{ev} \circ f_E : X^{10} \rightarrow \text{LBU}(n) \rightarrow \text{BU}(n)$.

We consider two classes of examples:¹⁵

1. Looping the bundle $E \rightarrow X^{10}$ leads to $LE \rightarrow LX^{10}$, which is classified by the map $f_E : LX^{10} \rightarrow \text{LBU}(n) \simeq \text{BLU}(n)$. If E is the tangent bundle TX^{10} , then LE is the tangent bundle TLX^{10} of LX^{10} .
2. Tensoring the bundle E fiberwise with LC . This corresponds to the map of classifying maps $[X^{10}, \text{BU}(n)] \rightarrow [X^{10}, \text{LBU}(n)]$ induced by the inclusion $\text{BU}(n) \rightarrow \text{LBU}(n)$ as the space of constant maps.

6.2 Fourier decomposition

We will start by relating the second class of examples in the previous subsection to the situation in DMW [1]. There, spinors on Y^{11} that transform as $e^{-ik\theta}$ under rotations of the circle were identified with the spinors on X^{10} with values in \mathcal{L}^k , where \mathcal{L} is the complex line bundle whose bundle of unit vectors is the M-theory S^1 -bundle. The coupling to the positive and negative chirality spin bundles S^+ and S^- , coming from the decomposition of the spin bundle on Y^{11} as $S = \pi^*(S^+) \oplus \pi^*(S^-)$, is $S^+ \otimes \mathcal{L}^k$ and $S^- \otimes \mathcal{L}^k$, respectively. Then, after including the coupling to the vector bundle E , one has the spinors coupled to the product bundles $E \otimes \mathcal{L}^k$. The loop description of this is given by the second example above, where we replace $E \otimes \mathcal{L}^k$ with $E \otimes LC$.

The loop bundles admit a (fiberwise) Fourier expansion analogous to that of \mathbb{C}^n -valued functions on the circle S^1 . For LC^n , this can be viewed as a map φ from LC^n to $\mathbb{C}[[z, z^{-1}]] \otimes \mathbb{C}^n$, the ring of formal power series in z and z^{-1} tensored with \mathbb{C}^n . One can further restrict to the “positive loops” $L_+ \mathbb{C}^n = \varphi^{-1}(\mathbb{C}[[z]] \otimes \mathbb{C}^n)$. These are the boundary values of the holomorphic maps from the two-disk to \mathbb{C}^n , $f : \mathbb{D}^2 \rightarrow \mathbb{C}^n$. In the current ten-dimensional setting, the disk is naturally interpreted as the fiber over type IIA with the bounding theory Z^{12} as the total space as in [4].

The Fourier decomposition of LC^n works as follows [20]. The group of positive loops $L_+ \mathbb{C}^n \subset LC^n$ has the interesting property of being invariant under multiplication by z , i.e., $zL_+ \mathbb{C}^n$ is a subset of LC^n with codimension n . The inclusion gives rise to a filtration

$$\dots \subset z^{-k} L_+ \mathbb{C}^n \subset z^{-(k+1)} L_+ \mathbb{C}^n \subset \dots \subset LC^n, \tag{6.2}$$

¹⁵The two classes are related, as we will see shortly.

where the union $\bigcup_k z^{-k}L_+\mathbb{C}^n$ is a dense subspace of $L\mathbb{C}^n$. This is the Fourier decomposition of $L\mathbb{C}^n$.

Analogously, a Fourier decomposition of rank n loop bundle $E \rightarrow X^{10}$ is a subbundle $E_+ \subset E$ such that $E = E_+ \oplus E_-$ with $E_+ = E_+^\perp$ and E_+ is invariant under multiplication by an element z in formal Laurent polynomials $\mathbb{C}[z, z^{-1}]$, $zE_+ \subseteq E_+$ of codimension n . The bundle theoretic analog of the Fourier decomposition of $L\mathbb{C}^n$ is the filtration

$$\dots \subset z^{-k}E_+ \subset z^{-(k+1)}E_+ \subset \dots \subset E_+, \tag{6.3}$$

whose union $\bigcup_k z^{-k}E_+$ is a fiberwise dense subbundle of E [20].

6.3 Relating X^{10} to LX^{10} : conditions for Fourier decomposition

The first of the two classes of examples in Section 6.1 can be examined with the use of rank- n loop bundles. For $E \rightarrow X^{10}$ an n -dimensional complex vector bundle (in our main case of interest, namely E_8 , we have $n = 248$) classified by the map f_E above, let $LE \rightarrow LX^{10}$ be the *induced* rank- n loop bundle over the loop space LX^{10} . The fiber of LE over $\gamma \in LX^{10}$ is the space of sections of the pull-back of E over the circle, $LE_\gamma = \Gamma_{S^1}(\gamma^*(E))$. For example, when $E = TX^{10}$, LTX^{10} is the infinite-dimensional tangent bundle of LX^{10} . The tangent space over γ is the space of vector fields living over γ .

Note that a Fourier decomposition is a much stronger condition than a polarization since the latter allows for some finite-dimensional ambiguity [20, 35]. The homotopy type of a map of based loop spaces $\Omega f_E : \Omega X^{10} \rightarrow \Omega BU(n) \simeq U(n)$ can be obtained by taking the holonomy map of a connection on E . If we require LE to have a Fourier decomposition, then the corresponding condition on E is that it must admit a homotopy flat connection [20]. Thus for our E_8 bundle to admit such connections would mean that the bundle is essentially trivial.

A rank- n loop bundle $\mathcal{E} \rightarrow X^{10}$ admits a Fourier decomposition if and only if the structure group of \mathcal{E} can be reduced to $U(n)$, viewed as the subgroup of constant loops in $LU(n)$ [20]. This can be rephrased in terms of disk bundles. Let $f : X^{10} \rightarrow \text{LBU}(n)$ classify a loop bundle $\mathcal{E} \rightarrow X^{10}$. Then \mathcal{E} admits a Fourier decomposition if and only if there is a lift of f to the space of maps $\text{Map}(\mathbb{D}^2, \text{BU}(n))$ from the two-disk \mathbb{D}^2 to the classifying space $\text{BU}(n)$. Again, for us, this \mathbb{D}^2 is the fiber of type IIA in the bounding theory Z^{12} of M-theory, in the spirit of [4]. Thus this means a 12-dimensional extension.

Any loop bundle \mathcal{E} that has a Fourier decomposition has the following description [20]. \mathcal{E} has a Fourier decomposition if and only if it is isomorphic to $LC \otimes E$, where E is the underlying n -dimensional bundle over X^{10} . Thus this brings us back to the first class of examples and to the description of the Fourier decomposition in terms of $LC \otimes E$ explaining the mode expansions in [1].

6.4 Producing Fourier-decomposable loop bundles via deformation

There is a process that produces Fourier-decomposable bundles from loop space [20] which we now describe for completeness. Using the evaluation map $ev : LX^{10} \rightarrow X^{10}$, one can pull back bundles from the spacetime to loop space. The parallel transport operator induced by a connection on $E \rightarrow X^{10}$ can be interpreted as an automorphism, i.e., a gauge transformation, of the pull-back bundle $ev^*(E) \rightarrow LX^{10}$. Loop bundles LE can be deformed by such gauge transformations. Let $\mathcal{G}(ev^*(E))$ be the gauge group of bundle automorphisms of $ev^*(E)$. For X^{10} smooth and simply connected, there is a natural rank- n loop bundle

$$L^{\mathcal{G}}E \rightarrow \mathcal{G}(e^*(E)) \times LX^{10}, \tag{6.4}$$

satisfying interesting properties. For $t \in \mathcal{G}(e^*(E))$ a gauge parameter, let $L^t X^{10}$ denote the restriction of $L^{\mathcal{G}}E$ to $\{t\} \times LX^{10}$; then

- (1) for the identity gauge element, $id \in \mathcal{G}(e^*(E))$, $L^{id} X^{10} = LE \rightarrow LX^{10}$;
- (2) for t_{∇_E} , the parallel transport operator of a connection ∇_E on E , the bundle $L^{t_{\nabla_E}}E \rightarrow LX^{10}$ admits a natural isomorphism of loop bundles,

$$L^t E \cong C^\infty(S^1, \mathbb{C}) \otimes ev^*E, \tag{6.5}$$

and hence admits Fourier decomposition. Thus, starting from bundles E on spacetime X^{10} we can build Fourier decomposable bundles by going to loop space and performing a gauge transformation as above.

Can the loop bundle be reduced to groups other than $U(n)$? It turns out that the only compact subgroups of $LU(n)$ are conjugate to subgroups of $U(n)$ [36] and so $U(n)$ is the largest compact subgroup. This is appropriate and is in line with [1] as it leaves no ambiguity in getting unitary bundles.

We can also ask whether one could have started with an ΩE_8 bundle instead of an LE_8 bundle and performed the Fourier decomposition procedure on that bundle. In doing so one gets $\Omega U(n)$ in the intermediate step. However, there are no compact subgroups of $\Omega U(n)$ [36], and so one cannot connect to finite-dimensional bundles the same way.

6.5 The eta invariant of the horizontal Dirac operator

In Section 4.1 we related the eta invariant of the vertical tangent bundle to the index gerbe via the eta form that appeared in the adiabatic limit. Here we would like to briefly consider the part that is related to the spin bundle of X^{10} , i.e., the horizontal part. This gives the contributions to the action and the phase from the loop sector.

In DMW [1] the Atiyah–Patodi–Singer η -invariant was decomposed according to Fourier modes $e^{-ik\theta}$ of the circle as

$$\eta = \sum_{k \in \mathbb{Z}} \eta_k, \tag{6.6}$$

where η_k is the contribution from states that transform as $e^{-ik\theta}$ under rotation of the circle. From our discussion above, it is clear that the natural generalization of (6.6) is to consider coupling the Dirac operator to loop bundles LE , both for the Rarita–Schwinger and the E_8 parts.

We have seen how the vector bundles E can be replaced by the loop bundle LE in order to account for the looping. We have also seen how such loop bundles can then be Fourier decomposed resulting eventually in the breakdown to $E \otimes LC$. The first stage would give the general contribution from the “loop sector.” This connects nicely to [12, 14], where one of the ways of justifying the appearance of elliptic cohomology was to propose the source of this looping as being the Dirac operators coupled to the loop bundles. The second stage is obtained if one further wants to get the Fourier modes. Thus, the two-stage picture looks like¹⁶

$$\eta_{S \otimes E} \xrightarrow{\text{Looping}} \eta_{S \otimes LE} \xrightarrow{\text{Fourier}} \eta_{S \otimes E \otimes LC}. \tag{6.7}$$

¹⁶We could of course also loop the spin bundle itself. However, we preferred to keep the discussion in this section brief as we hope to revisit this elsewhere.

7 The generalized WZW description

In the discussion of the adiabatic limit, it was important to study the Dirac operator on the circle bundle. Had the circle bundle been trivial then we would not have had to analyze the eta invariants, since in that case a symmetry argument would show that the eta invariant contribution to the phase vanishes [1]. What we are interested in is the effect of the nontrivial M-theory circle, where no symmetry arguments can be used to extract the contribution of the eta invariant to the phase. Thus, essential in our discussion is the Dirac operator on the circle bundle, or more precisely, the Dirac operator on the “circle part”.¹⁷

The physical nature of the E_8 gauge theory in 11 dimensions is not understood. It does not seem to be a Yang–Mills theory — see the discussion in [24, 37]. In particular, we do not know the degrees of freedom of this theory.¹⁸ Nevertheless, the structure of the topological parts of the action seems to indicate that having an index of E_8 implies that we have a curvature F_2 of the bundle, and so the corresponding vector potentials must be present.¹⁹ With this assumption we can look at the Dirac operators coupled to these potentials. We can then form the space \mathcal{A} of \mathfrak{e}_8 -valued one forms on S^1 , where each point $A \in \mathcal{A}$ defines a Dirac operator D_A in the space \mathcal{H} of square-integrable spinors twisted by some representation of E_8 .

The principal loop group bundle gives rise via a representation to an associated vector bundle. The identification in Section 3 indicates that we are dealing with the adjoint representation. Thus we represent LE_8 on its Lie algebra $L\mathfrak{e}_8$ and consider the Hilbert space $L^2(S^1, \mathfrak{e}_8)$. We consider the complex Hilbert space \mathcal{H} that carries an irreducible unitary highest weight representation of the central extension \widehat{LE}_8 of the loop group LE_8 of level k . One has the Fock space as a product of a bosonic and a fermionic Fock space,

$$\mathcal{F} = \mathcal{F}_B^{(k,\lambda)} \otimes \mathcal{F}_F^{(h^\vee,\rho)}, \tag{7.1}$$

labeled, respectively, by λ and k , the weight of E_8 and the level, and by h^\vee , the dual Coxeter number and ρ , half the sum of the positive roots. The dual

¹⁷Of course, this is an oversimplification in terminology because the circle bundle is not a product.

¹⁸This is perhaps not surprising as we also do not know the degrees of freedom of M-theory itself.

¹⁹Of course alternatively, the way to describe this theory, if it exists, could be very different from the standard methods of differential geometry.

Coxeter number is given by the value of the quadratic Casimir of E_8 in the adjoint representations, i.e.,

$$-2h^\vee \delta^{ab} = \lambda^{acd} \lambda^b_{cd}, \tag{7.2}$$

which has the value 30 that we used in Section 3. Here the λ 's are the structure constants of E_8 . The Fock space is (7.1) with $k = -h^\vee$.

7.1 The bosonic sector

The fact that the level in our case is given by the negative of the dual Coxeter number implies that there is no bosonic Fock space and only \mathcal{F}_F occurs. The bosonic Fock space \mathcal{F}_B would correspond to the Sugawara currents. The appearance of the dual Coxeter number h^\vee implies that we are working at the *critical level* $k = -h^\vee$. The corresponding Kac–Moody symmetry is at the critical level and associated to it is a special “conformal field theory” which is not conventional because it does *not* have a stress tensor. It is in fact nonconformal.

Let $\{J^a\}$ be the basis for \mathfrak{e}_8 , and $\{J_a\}$ the dual basis with respect to the Killing form, normalized so that the length of the longest root is 2. The Sugawara–Segal current is

$$S(z) = \frac{1}{2} : J_a(z) J^a(Z) : \sum_{n \in \mathbb{Z}} S_n z^{-n-2}. \tag{7.3}$$

The commutation relations are

$$[S_n, J^a_m] = -(k + h^\vee) m J^a_{n+m}, \tag{7.4}$$

$$[S_n, S_m] = (k + h^\vee) \left((n - m) S_{n+m} + \frac{1}{12} k \cdot \text{dim} \mathfrak{e}_8 \cdot \delta_{n,-m} \right). \tag{7.5}$$

Away from the critical level, i.e., when $k \neq -h^\vee$, if the operators S_n are scaled to $L_n = (k + h^\vee)^{-1} S_n$ then (7.5) generates the Virasoro algebra with central charge $c_k = \frac{k \cdot \text{dim} \mathfrak{e}_8}{k + h^\vee}$. The replacement of S_n by L_n in (7.4) gives the action of the Virasoro algebra on $\widehat{L\mathfrak{e}_8}$,

$$[L_n, J^a_m] = -m J^a_{n+m}. \tag{7.6}$$

However, if $k = -h^\vee$ then the operators S_n commute with the affine algebra and commute among themselves

$$[S_n, J^a_m] = 0, \tag{7.7}$$

$$[S_n, S_m] = 0. \tag{7.8}$$

In particular, there is no usual conformal symmetry, and the second relation implies the absence of the energy-momentum tensor. If one was to mimic the construction for $k \neq -h^\vee$ then one would get that $L_n \rightarrow \infty$ and $c_{-h^\vee} \rightarrow \infty$, as well as the commuting algebra.

7.2 The fermionic sector

We have just seen that the usual Sugawara currents are absent. Thus from here on we focus on the fermionic sector,

$$\mathcal{F} = \mathcal{F}_F^{(h^\vee, \rho)}. \tag{7.9}$$

We follow [19] where this construction was made (for a different purpose). This (7.9) is the Fock space for the algebra of canonical anti-commutation relations (CAR) generated by the generators ψ_n^a , where $n \in \mathbb{Z}$ is a label for the momentum along the circle, and a belongs to an indexing set $1, 2, \dots, \dim E_8 = 248$, that satisfy the CAR

$$\{ \psi_n^a, \psi_m^b \} = 2\delta_{n,-m}\delta_{a,b}. \tag{7.10}$$

The Fock vacuum is characterized by the zero mode Clifford subalgebra of the CAR, and is a subspace of \mathcal{F}_F of dimension $2^{\dim E_8/2} = 2^{124}$. This vacuum subspace carries an irreducible representation of the Clifford algebra generated by the ψ_0^a 's. Any vector in the vacuum is annihilated by all ψ_n^a 's with $n < 0$.

The generators J_n^a of the loop algebra act on the Fock space \mathcal{F}_F and they satisfy the commutation relations (CRs)

$$[J_n^a, J_m^b] = -\lambda_{abc}J_{n+m}^c - \frac{h^\vee}{4}n\delta_{n,-m}\delta_{a,b}, \tag{7.11}$$

where λ_{abc} are the structure constants on the Lie group E_8 . Explicitly, the loop generators are given as bilinears in the oscillator generators

$$J_n^a = -\frac{1}{4}\lambda_{abc}\psi_{n+m}^b\psi_{-m}^c, \tag{7.12}$$

where normal ordering is not needed because the structure constants are totally antisymmetric. It is also understood that the RHS of this expression

involves the sum over the contracted indices. The fermionic Hamiltonian is given by

$$H_F = -\frac{1}{4}n : \psi_n^a \psi_{-n}^a : + 2h^\vee \cdot \frac{\dim E_8}{24} \quad (7.13)$$

with the reality condition $(\psi_n^a)^* = \psi_{-n}^a$. The second term is the zero mode sector corresponding to the classical case. Corresponding to the Hamiltonian is its square root, the supercharge Q , which satisfies $Q^2 = H_F$, and which is defined by

$$\begin{aligned} Q &= -\frac{i}{12} \lambda_{abc} \psi_n^a \psi_m^b \psi_{-m-n}^c \\ &= \frac{i}{3} \psi_n^a J_{-n}^a. \end{aligned} \quad (7.14)$$

It is interesting to see what one gets when one restricts to the zero momentum mode sector. For this, the generators ψ_n^a become the 248-dimensional Euclidean gamma matrices $\psi_0^a = \gamma^a$ for the Lie group E_8 . In this case, the supercharge Q becomes (part of) Kostant's cubic Dirac operator

$$\mathcal{K} = -\frac{i}{12} \lambda_{abc} \gamma^a \gamma^b \gamma^c. \quad (7.15)$$

Since the structure constants are totally antisymmetric, it follows that the product of gamma matrices is totally antisymmetric, i.e., we can replace the product of gamma matrices in (7.15) by the antisymmetrized product γ^{abc} .

Since the dimension of the group manifold E_8 is even, we can define the chirality operator ψ_0^{249} , the analog of γ^5 in four dimensions, and use it to get a grading operator

$$\Gamma = (-1)^F \psi_0^{249}, \quad (7.16)$$

where F is the fermion number operator and so $(-1)^F$ is the supersymmetry index. This gives, for $n \neq 0$,

$$\psi_n^a F + F \psi_n^a = \frac{n}{|n|} \psi_n^a. \quad (7.17)$$

7.3 Coupling the supercharge to the vector potential

One can couple the supercharge operator to the vector potential A on the circle with values in the Lie algebra \mathfrak{e}_8 , to form the family of operators

parameterized by A [19]

$$Q_A = Q - \frac{h^\vee}{4} \psi_n^a A_{-n}^a, \tag{7.18}$$

where A_{-n}^a are the Fourier components of A satisfying $(A_n^a)^* = -A_{-n}^a$. For a loop $g \in LE_8$, the corresponding lift \hat{g} to the central extension \widehat{LE}_8 acts by conjugation on Q_A resulting in a g -gauge transformation on A ,

$$\hat{g}^{-1} Q_A \hat{g} = Q_{A^g}, \tag{7.19}$$

where $A^g = g^{-1}(A + d)g$.

The operator Q_A has a kernel that lies in the conjugacy class, so that Q_A is not invertible on that set [19]. This occurs at $\frac{\lambda+\rho}{k+h^\vee} \in \mathfrak{h}^*$ in the dual Cartan subalgebra, which can also be viewed to be in the Cartan algebra \mathfrak{h} using the Cartan–Killing form. Our case is $k = -h^\vee$ and so this would imply that this set is infinite. However the situation is delicate.²⁰ In general there is a continuum of conjugacy classes, which are given for $SU(2)$ for example, by any latitude circle. The model however, picks out particular quantized conjugacy classes. In the classical limit a choice is λ/k , which can be seen as a limit of the general formula $\frac{\lambda+\rho}{k+h^\vee}$ (cf. [38]). Since k can be viewed as a measure of energy for the model then this latter equation is in some sense a strong coupling version of the classical formula. This can be seen for example by series expansion with λ/k the lowest order term. One way to see that k is a sort of momentum cutoff is by using the Peter–Weyl theorem, which says that $L^2(E_8)$ can be written as a sum over all representation labels γ of the direct sum of the representation space R_γ and its dual R_γ^*

$$L^2(E_8) = \widehat{\bigoplus_\gamma R_\gamma \otimes R_\gamma^*}. \tag{7.20}$$

In this context, the sum should be taken over $\gamma \leq k$ which gives the level k the interpretation of a momentum cut-off.

The family problem can be described by [19]

$$\begin{array}{ccc} \text{Fred}_* & \longleftarrow & \mathcal{A} \\ & & \downarrow \pi' \\ & & E_8 \end{array} \tag{7.21}$$

where the top arrow takes a connection $A \in \mathcal{A}$ to the corresponding supercharge Q_A in the space of self-adjoint Fredholm operators, and where the

²⁰We thank Christoph Schweigert for an explanation on this.

whole structure is over E_8 . The vertical arrow relates E_8 to \mathcal{A} via taking the holonomy of A , and the projection down to E_8 , denoted by π' , is not always possible. The DD-class for π is given by [19]

$$\text{DD}(\pi') = k \cdot \omega = -h^\vee \cdot \omega, \quad (7.22)$$

where ω is the canonical integral generator of the cohomology $H^3(E_8; \mathbb{Z}) = \mathbb{Z}$ of E_8 .

At this stage the physics is occurring over the group manifold E_8 itself and not over the spacetime. Since LE_8 bundles over the ten-dimensional spacetime are completely characterized by the DD-class, then this means that we can pull back from E_8 to spacetime and get our gerbes in spacetime to be coming from the basic gerbe on E_8 . Since LE_8 bundles have a classifying space $E_8 \times BE_8$, then there is certainly a map to E_8 viewed as part of the product. The discussion suggests a generalized WZW model, i.e., with the two-dimensional worldsheet being replaced by the ten-dimensional spacetime.

8 Further discussion and proposals

We have identified the first two nonzero degrees of the eta forms by comparing the expression for the adiabatic limit of the eta invariant, representing the phase of the C -field in M-theory, with that of the topological term in type IIA string theory. The comparison gives essentially that $\widehat{\eta}^{(2)}$ is the B -field. The construction works for String manifolds, i.e., for manifolds with vanishing String class $\lambda = 0$. The appearance of the dual Coxeter number in (3.4) can be used in two different ways. First, starting from a G -bundle one discovers that G should be E_8 under the natural assumption that the representation is the adjoint. Alternatively, one can start with a specified E_8 gauge theory and using h^\vee identify the relevant representation as being the adjoint representation.

Further, we are able to use a generalized WZW construction to utilize the appearance of h^\vee , via the families index theorem and the twisted K-theory for E_8 . Mapping to our families problem with Dirac operator on the M-theory circle parameterized by points in the type IIA spacetime X^{10} , suggests the possibility of a generalized WZW model where the spacetime is embedded in E_8 . We also discuss the appearance of the infinite dimensional loop bundles, and their contribution to the partition function and the phase. To connect to finite-dimensional bundles the condition of Fourier decomposition arises, which puts severe restrictions on the bundle. We discuss the symmetry breaking problem in general for $LE_8 \rightarrow E_8$ giving a Higgs field with values in ΩE_8 . Corresponding to the infinite symmetry is an infinite number

of generators, and when the symmetry is broken the generators are absent. This is another way to explain why in the construction only the fermionic Fock space was seen and the Sugawara bosonic currents were absent.

8.1 The relation to twisted K-theory

Starting with a positive energy representation of LE_8 on a Hilbert space \mathcal{H} (at a fixed level) one can take the principal LE_8 bundle over E_8 and make it into a $PU(\mathcal{H})$ bundle using the projective representation. Using the construction in [9], twisted K-theory classes over E_8 can be thought of as equivariant maps f from P to Fred_* , where P is the principal $PU(\mathcal{H})$ bundle over E_8 with a given DD-invariant $\omega \in H^3(E_8; \mathbb{Z})$, and Fred_* is the space of Fredholm operators, and the equivariance condition is $f(pg) = g^{-1}f(p)g$ for $g \in PU(\mathcal{H})$. In our case, the principal bundle P is obtained by embedding the loop group LE_8 inside $PU(\mathcal{H})$ through the projective representation of LE_8 [19]. We have seen the relation to the twisted K-theory of the Lie group E_8 . However, we are ultimately interested in the twisted K-theory of spacetime. Is there a way to get the twisted K-theory of spacetime starting from the twisted K-theory of E_8 ? The mathematical answer to this question is positive if there is a map from X^{10} to E_8 through which we can pull back the K-theory group. Physically, this again suggests a generalized WZW sigma model $X^{10} \hookrightarrow E_8 (= B\Omega E_8)$. Alternatively, if there is a map to E_8 — a section of the bundle — one can use a pullback of the generator of the three-class in E_8 as the resulting H -flux in spacetime X , and the pullback of $H^3(E_8)$ will give a subgroup of $H^3(X)$ isomorphic to \mathbb{Z} . This requires further investigation.

8.2 The sign involutions

Note that the identification of the two-form piece of the eta form included a minus sign in the prefactor. There are several sign reversals that work nicely together. Reversing the sign of the Coxeter number amounts to using the dual representation, i.e., lowest weight in place of highest weight representation. For the gerbe itself, the reversal of sign operation $B \mapsto -B$ has the following interpretation. At the level of twisted Chern characters, this amounts to interchanging a bundle E with its complex conjugate \overline{E} , i.e.,

$$\text{ch}_H(E) \leftrightarrow \text{ch}_{-H}(\overline{E}). \tag{8.1}$$

From 11-dimensional supergravity, we have the gravitino supersymmetry rule which gives the generalized spinor equation. Upon dimensional

reduction to ten dimensions the connection, built out of contractions of G_4 with the 11-dimensional gamma matrices, decomposes into two connections, one for each of the positive and negative chirality spinor bundles S^\pm , induced by the “generalized connection” $\nabla \pm H$ of the tangent bundle, where ∇ is the Levi–Civita connection on the tangent bundle of X^{10} and H is the one-form built out of the NSNS field H_3 . The choice of sign corresponds to a choice of orientation, and so the reversal of sign corresponds to the reversal of orientation of spacetime. So the three ingredients: the dual Coxeter number, the twisted bundle, and the twisted work together. This means for instance that taking a representation of highest weight using a twist H for a bundle E on the spacetime X^{10} with a given orientation is equivalent to taking a lowest weight representation using a twist $-H$ for the conjugate bundle \bar{E} (coming from reversing the orientation).

8.3 The Ramond–Ramond (RR) fields

One might wonder whether the RR fields can be obtained from the index gerbe and identified with the higher eta forms. While this would be nice, it is unfortunately not the case. In principle, from the index gerbe one can get even degrees on X^{10} provided one replaces the M-theory circle bundle with the bounding disk bundle, with the corresponding expression given by replacing S^1 in the Atiyah–Singer index formula giving the degrees $2k$ in the same way that (4.12) gives degree three. These are actually Deligne classes [22]. However, it is obvious from the index formula that it contains the \widehat{A} -genus of the vertical tangent bundle and not the (square-root) of the \widehat{A} -genus of TX^{10} . Had we been able to get the RR fields this way (in cohomology) then we would have obtained a derivation of the cosmological constant, i.e., the degree zero component of the RR field, as the degree zero component of the index formula. We now go back to the original situation with an odd-dimensional fiber. In the special case when the vertical tangent bundle is trivial the degree zero component would be

$$\int_{\mathbb{D}^2} c_1(V). \quad (8.2)$$

For E_8 , this vanishes, but can be non-zero for the Rarita–Schwinger bundle.

8.4 Higher order corrections

The strategy we followed in the identification was to use the topological terms that exist in type IIA string theory. The rest of the terms that are

not identified will either have a different interpretation or that new terms in the type IIA action have to be added to them to get a topological description for the result. In particular, we only focused on the E_8 part in M-theory and on the one-loop term in type IIA. The absence of the Chern–Simons term in our consideration of type IIA may be viewed as setting F_4 to zero. It would be interesting to pursue this further. Higher terms should be explained in terms of higher order corrections. However, at the moment these do not seem to be written nicely in terms of characteristic classes.²¹ On the positive side, we expect the symmetry appearing to turn out to be very useful in giving a handle on these higher order (gravitational) corrections, e.g., by constraining their structure.

8.5 The quantization conditions

Note that the index gerbe ended up being essentially the H -field. Chasing this back to M-theory the standard way we know that this comes from integrating G_4 over the circle. This means that the index gerbe would come from G_4 written as an index, and having the same expression (4.12) without the integral. Alternatively, if one uses the putative index formula in [15, 16] then G_4 would have a shift coming from \widehat{A}_4 leading to $G_4 - \lambda/24$. It is interesting that requiring this to be an integral class implies that λ is divisible by 24, a condition for orientability with respect to topological modular forms (TMF). Note that this uses the definition in [15, 16] for the zeroth component of the character to be one, which in comparison can be seen to indicate the abelian nature of G_4 .

8.6 The role of E_8

We have started from a G -bundle in 11 dimensions and checked whether one can *discover* E_8 . Indeed if we specify the representation to be the adjoint representation, which is the natural choice for a gauge theory, then E_8 is specified via its dual Coxeter number. Alternatively, if we choose to start from an E_8 bundle in 11 dimensions then we can discover that the representation for the loop group in ten dimensions is the adjoint. We believe that this gives more evidence for the role of E_8 in M-theory, beyond just being a model for $K(\mathbb{Z}, 3)$ in low dimensions (compare [24, 39]). Another role of E_8 is suggested in Section 7.3 where the discussion indicates that spacetime may be used as the pre-image of a generalized sigma model with E_8 as target.

²¹We thank B. Pioline for a comment on this.

8.7 The topology of spacetime

In Section 3 we saw that the comparison of the adiabatic limit of the eta invariant containing the eta forms with the one-loop term in type IIA string theory worked by imposing the String condition (cf. [32,33]) $\lambda = p_1/2 = 0$ on our manifold X^{10} . This means that these manifolds are of special importance when considering the global aspects of M-theory and string theory. This is in line with the proposals in [12–18]. This should not be surprising since, after all, it is the topological one-loop term that led to the identification of the level as being the critical one.

8.8 Spacetime as parametrizing families

One interpretation of the viewpoint in the analysis of the Dirac operators in [3] is that the ten-dimensional spacetime X^{10} acted as a parametrizing family for the vertical Dirac operator, i.e., the operator on the M-theory circle part. In view of the idea in this paper this suggests a family version of the modularity problem. This would encode the effects of the extra dimensions (over ten) in a systematic way.

8.9 Worldsheet versus spacetime

In most of the comments in the other paragraphs the cited works on the critical level referred to that of the worldsheet conformal field theory. However what we have here is a CFT-like structure arising from spacetime, and more precisely from the extra directions over the ten-dimensional base. Is there any relation between the worldsheet on one side and the M-theory circle and the F-theory elliptic curve on the other? Indeed, it has been proposed in [14] and further explained and elaborated in [18] that such a correlation exists. Via this latter identification then what we are considering is a “*conformal field theory (CFT) structure parametrized by type II spacetime*”.

8.10 Adiabatic limits on the worldvolumes

The discussion in this note suggests that the structures on the spacetimes and on the worldvolumes are analogous and are correlated. More precisely, this implies that for the circle bundle Y^{11} in the M-theory spacetime over the type IIA base X^{10} there goes with it a corresponding circle bundle M^3 for the membrane over the string worldsheet base Σ_g . The worldsheet with the B-field in the action is the base of a \mathbb{D}^2 -bundle giving the total

four-dimensional membrane cobounding theory with topological action the integral of G_4 . Further, the discussion on the four form being an index (Section 4) suggests that there is an adiabatic limit taken on the worldvolumes that corresponds to the one in the spacetime targets. Given the circle bundle $S^1 \rightarrow M^3 \rightarrow \Sigma_g$, with the metric

$$g_{M^3} = tg_{\Sigma_g} + A \otimes A, \tag{8.3}$$

the adiabatic limit for the vertical tangent bundle gives

$$\lim_{t \rightarrow \infty} \bar{\eta}(D^t) = \int_{\Sigma_g} \widehat{A}(\Sigma_g) \wedge \widehat{\eta}. \tag{8.4}$$

For dimensional reasons, $\widehat{A}(\Sigma_g) = 1$ and so we are left with only the integral of the degree two eta form $\widehat{\eta}^{(2)}$, which by our spacetime arguments is just $-30B_2$. From the point of view of two dimensions this would then give some special role for theories in the large volume limit.

8.11 The Higgs field and the topological membrane

Topological BF-theory for a G bundle on a Riemann surface Σ_g (see e.g., [40] for a review) is characterized by the flatness condition $F_A = 0$ as the extrema of the action. The moduli space for such solutions is $\mathcal{M}_F(\Sigma_g, G)$ and the corresponding field theory has the partition function

$$Z(\Sigma_g) = \int D\phi DA \exp\left(\frac{1}{4\pi^2} \int_{\Sigma_g} \text{Tr } i\phi F_A\right), \tag{8.5}$$

where ϕ is the adjoint-valued field. In our case, we propose that the corresponding theory will have the action (5.5) as the starting point where the role of ϕ is played by Φ (in fact $\nabla\Phi$ since we have a form degree shift). We interpret this as giving the topological part of the membrane action where we have a partition function analogous to (8.5).

8.12 The critical level

The supergravity description of string theory corresponds to the large tension limit, i.e., when the string coupling α' tends to 0. A consistent truncation is given by the massless modes which are the fields of the effective theory. We have seen in (7.1) that the theory we found is a peculiar theory at the critical level, i.e., k equals the negative of the dual Coxeter number $h^\vee = 30$.

This theory is not conformal in the sense that there is no energy-momentum tensor. The Segal–Sugawara current is commutative. While there is no Virasoro symmetry one can still have the affine Lie symmetry. Theories with level k equal to the dual Coxeter number are very special and they play an important role in the geometric Langlands program since they provide a natural way of constructing Hecke eigensheaves [41]. The extensive construction of [42] does not make use of the critical level, nor of loop groups.²² Thus our proposals can be seen as bringing in new elements in the connection between the physics and the mathematics, albeit through a different approach. The Relation (7.7) implies that the operators S_n belong to the center of the enveloping algebra of $\widehat{\mathcal{L}\mathfrak{e}_8}$. This also implies that the algebra of infinitesimal diffeomorphisms of the punctured disk acting on $\widehat{\mathcal{L}\mathfrak{e}_8}$ cannot be realized as an internal symmetry of the space of states at $k = -h^\vee$. What is playing the role of the curve in this case is the extra curve in ten dimensions leading to F-theory as in [14, 18]. This suggests the current context as a potential setting for the Langlands program for ten-dimensional spacetime. We further note that from the 11-dimensional point of view, the infinite volume limit for the ten-dimensional base — which is desirable for the sigma model (cf. [43]) — is just the adiabatic limit of the circle bundle, the main setting for this note.

8.13 High energy and tensionless limits of strings

What is the physical implication of the critical level? The answer is that it provides a window for high energy regimes. In [44] it was argued that the critical level corresponds to the tensionless limit of string theory. This is the limit where a huge new symmetry emerges due to the dramatic increase in the number of zero-norm states. Classically, the tensionless limit arises when $k \rightarrow \infty$ but quantum mechanically one has to include the shift by h^\vee . One thus expects that the shifted level to be a measure of coupling. Indeed, the WZW analysis leads to the identification of the level as an inverse coupling constant, namely²³ [44, 45]

$$\alpha' = \frac{1}{k + h^\vee}. \quad (8.8)$$

²²I thank Anton Kapustin for a comment on this.

²³The analysis in [45] is done for noncompact cosets. We remove a relative minus sign (cf. entry 2 above). For $k = -h^\vee < 0$, the vacuum is given by

$$J_n^a |0\rangle = 0 \quad \text{for } n > 0, \quad (8.6)$$

$$\psi_n^a |0\rangle = 0 \quad \text{for } n > 0. \quad (8.7)$$

If (8.6) is used then (8.7) is satisfied (also works for both having $n < 0$). Unitarity is fixed if the representation is flipped from positive energy to negative energy because we get the desired sign in the commutators.

Thus the critical level seems to be the appropriate setting for studying very high energy properties of string theory where the huge symmetry is still unbroken. The discussion in this note suggests likewise that large spacetime symmetries can be detected and should be analyzed at this level.

8.14 The noncommutative geometry of spacetime

For the WZW model, the classical level, i.e., when $k \rightarrow \infty$, corresponds to classical geometry. As the level is brought back from infinity to smaller and smaller values, the classical geometry of spacetime starts undergoing quantum deformations that introduce noncommutativity to the coordinates so that the spacetime becomes noncommutative (see [46]). The commutativity keeps increasing until the level reaches the critical value at which stage the the geometry becomes singular and the noncommutativity becomes infinite [45]. In our case of course the B -field was essential in deriving the critical level. The point is that in this limit the discussion of noncommutativity should be nonperturbative as opposed to the usual perturbative deformation approach that is done at finite values of the noncommutativity, i.e., spacetime is intrinsically noncommutative at the quantum level as opposed to being deformed to be so.

8.15 The cosmological constant

Recall again that, in the WZW model, the classical limit corresponds to taking the level to infinity, $k \rightarrow \infty$. From a geometric point of view, taking the classical limit means going to the large volume (or “radius”) limit, because the more the curvature increases the more quantum effects we have and the deeper we go into the quantum regime. One way of measuring this is through the cosmological constant Λ_g , whose value can also be a measure of curvature. Large values of Λ_g correspond to high curvature and hence to strong coupling. From this we see that in type IIA string theory the cosmological constant, which is the zeroth component F_0 of the RR field F , if related to the central extension of LE_8 , as suggested in [6], then such a relationship should be more of an inverse relationship rather than the linear relation $F_0 = k$ that was proposed in [6]. This is because for the same space — the ten-dimensional type IIA manifold — having $k \rightarrow \infty$ would then mean $F_0 \rightarrow \infty$ which gives strong coupling, in contrast to AdS/CFT where one is comparing the coupling of the bulk to the coupling on the boundary related by strong/weak duality. However, if *some* form of a one-to-one relationship exists between the cosmological constant and the central extension, then our discussions in this note would specify the cosmological constant because the theory singled out a particular level, namely the critical level. It seems

reasonable (e.g., from coset model considerations [44]) to expect $F_0 = T/k$, where T is the string tension.

8.16 Holography

The appearance of the E_8 Kac–Moody symmetries suggests the possibility that type IIA string theory have a description in terms of a quantum field theory. One can also go further to ask whether there is a holography in which type IIA is the codimension one theory. Indeed if this is the case then we can suggest the cobounding theory of type IIA, i.e., the 11-dimensional “theory” — let us name it M_A — on M^{11} whose boundary is type IIA. This is the result of viewing the two-disk bundle \mathbb{D}^2 over type IIA in two different ways. Starting from the 12-dimensional theory on Z^{12} we can take its boundary Y^{11} to arrive at M-theory and then take the S^1 -reduction to arrive at type IIA, or alternatively take the S^1 reduction of Z^{12} to get the “theory” M_A and then take the boundary to get to type IIA string theory. The topological terms would then be

$$\frac{1}{6} \int_{M^{11}} H_3 \wedge F_4 \wedge F_4 - I_8 \wedge F_4. \quad (8.9)$$

Further, this can also be seen as a more intrinsic definition of the differentials of the eta forms $d\hat{\eta}$.

Obviously there is a lot of work to be done. We hope to gain a better understanding and to report more in the near future.

Acknowledgments

I would like to thank Alan Carey, Jarah Evslin, Varghese Mathai, Jouko Mickelsson, Tony Pantev, Christoph Schweigert, and Bai-Ling Wang for useful discussions and explanation, and Arthur Greenspoon for suggestions on improving the presentation. I especially thank Jouko Mickelsson for very useful comments and suggestions on the manuscript, especially on Section 7. I also thank Ralph Cohen for explaining the main result of [20]. I acknowledge the hospitality of IHES, DIMACS and the Department of Mathematics at Rutgers, MSRI and the organizers of the program “New Topological Structures in Physics” where part of this work was carried out. This research is supported by an ESI Junior Research Fellowship associated with the program “Gerbes, Groupoids, and Quantum Field Theory.”

References

- [1] E. Diaconescu, G. Moore and E. Witten, *E_8 gauge theory, and a derivation of K -theory from M -theory*, Adv. Theor. Math. Phys. **6** (2003), 1031, [arXiv:hep-th/0005090](#).
- [2] E. Witten, *On flux quantization in M -theory and the effective action*, J. Geom. Phys. **22** (1997), 1–13, [arXiv:hep-th/9609122](#).
- [3] V. Mathai and H. Sati, *Some relations between twisted K -theory and E_8 gauge theory*, J. High Energy Phys. **0403** (2004), 016, [arXiv:hep-th/0312033](#).
- [4] G. Moore and N. Saulina, *T -duality, and the K -theoretic partition function of Type IIA superstring theory*, Nucl. Phys. **B670** (2003), 27, [arXiv:hep-th/0206092](#).
- [5] H. Garland and M. K. Murray, *Kac-Moody monopoles and periodic instantons*, Comm. Math. Phys. **120** (1988), 335.
- [6] A. Adams and J. Evslin, *The loop group of E_8 and K -Theory from 11d*, J. High Energy Phys. **02** (2003), 029, [arXiv:hep-th/0203218](#).
- [7] J. Evslin, *From E_8 to F via T* , J. High Energy Phys. **0408** (2004), 021, [arXiv:hep-th/0311235](#).
- [8] A. Bergman and U. Varadarajan, *Loop groups, Kaluza–Klein reduction and M -theory*, J. High Energy Phys. **0506** (2005), 043, [arXiv:hep-th/0406218](#).
- [9] P. Bouwknegt, A. Carey, V. Mathai, M. Murray and D. Stevenson, *Twisted K -theory and K -theory of bundle gerbes*, Comm. Math. Phys. **228** (2002), 17, [arXiv:hep-th/0106194](#).
- [10] A. Carey, J. Mickelsson and M. Murray, *Index theory, gerbes, and Hamiltonian quantization*, Comm. Math. Phys. **183** (1997), 707, [arXiv:hep-th/9511151](#).
- [11] A. Carey, J. Mickelsson and M. Murray, *Bundle gerbes applied to quantum field theory*, Rev. Math. Phys. **12** (2000), 65, [arXiv:hep-th/9711133](#).
- [12] I. Kriz and H. Sati, *M theory, type IIA superstrings, and elliptic cohomology*, Adv. Theor. Math. Phys. **8** (2004), 345, [arXiv:hep-th/0404013](#).
- [13] I. Kriz and H. Sati, *Type IIB string theory, S -duality and generalized cohomology*, Nucl. Phys. **B715** (2005), 639, [arXiv:hep-th/0410293](#).
- [14] I. Kriz and H. Sati, *Type II string theory and modularity*, J. High Energy Phys. **08** (2005), 038, [arXiv:hep-th/0501060](#).
- [15] H. Sati, *M -theory and characteristic classes*, J. High Energy Phys. **0508** (2005), 020, [arXiv:hep-th/0501245](#).

- [16] H. Sati, *Flux quantization and the M-theoretic characters*, Nucl. Phys. **B727** (2005), 461, [arXiv:hep-th/0507106](#).
- [17] H. Sati, *Duality symmetry and the form-fields in M-theory*, J. High Energy Phys. **0606** (2006), 062, [arXiv:hep-th/0509046](#).
- [18] H. Sati, *The Elliptic curves in gauge theory, string theory, and cohomology*, J. High Energy Phys. **0603** (2006), 096, [arXiv:hep-th/0511087](#).
- [19] J. Mickelsson, *Gerbes, (twisted) K-theory, and the supersymmetric WZW model*, in “Infinite Dimensional Groups and Manifolds”, ed. T. Wurzbacher, IRMA Lect. Math. Theor. Phys., **5**, de Gruyter, Berlin, 2004, [arXiv:hep-th/0206139](#).
- [20] R. L. Cohen and A. Stacey, *Fourier decompositions of loop bundles*, in “Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory”, eds. P. Goerss and S. Priddy, Contemp. Math., **346**, AMS, Providence, RI, 2004, 85–95, [arXiv:math.AT/0210351](#).
- [21] M. Murray and D. Stevenson, *Higgs fields, bundle gerbes and string structures*, Comm. Math. Phys. **243** (2003), 541, [arXiv:math.DG/0106179](#).
- [22] J. Lott, *Higher-degree analogs of the determinant line bundle*, Comm. Math. Phys. **230** (2002), 41, [arXiv:math.DG/0106177](#).
- [23] A. L. Carey and B.-L. Wang, *On the relationship of gerbes to the odd families index theorem*, [arXiv:math.DG/0407243](#).
- [24] E. Diaconescu, D. Freed and G. Moore, *The M-theory 3-form and E_8 gauge theory*, [arXiv:hep-th/0312069](#).
- [25] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in eleven dimensions*, Phys. Lett. **B76** (1978), 409–412.
- [26] C. Vafa and E. Witten, *A one-loop test of string duality*, Nucl. Phys. **B447** (1995), 261, [arXiv:hep-th/9505053](#).
- [27] M. J. Duff, J. T. Liu and R. Minasian, *Eleven dimensional origin of string/string duality: A one loop test*, Nucl. Phys. **B452** (1995), 261, [arXiv:hep-th/9506126](#).
- [28] J. Bismut and J. Cheeger, *η -invariants and their adiabatic limits*, J. Amer. Math. Soc. **2** (1989), 1, 33.
- [29] X. Dai, *APS boundary conditions, eta invariants and adiabatic limits*, Trans. Amer. Math. Soc. **354** (2002), 107.
- [30] U. Bunke, *Transgression of the index gerbe*, Manuscripta Math. **109** (2002), 263, [arXiv:math.DG/0109052](#).
- [31] P. Aschieri and B. Jurco, *Gerbes, M5-brane anomalies and E_8 gauge theory*, J. High Energy Phys. **0410** (2004), 068, [arXiv:hep-th/0409200](#).

- [32] T. Killingback, *World-sheet anomalies and loop geometry*, Nuclear Phys. **B288**, (1987), 578.
- [33] R. Coquereaux and K. Pilch, *String structures on loop bundles*, Comm. Math. Phys. **120** (1989), 353.
- [34] L. Dolan and M. J. Duff, *Kac–Moody symmetries of Kaluza–Klein theories*, Phys. Rev. Lett. **52** (1984), 14.
- [35] A. Pressley and G. Segal, *Loop groups*, Oxford University Press, New York, 1986.
- [36] A. Stacey, *Finite-dimensional subbundles of loop bundles*, Pacific J. Math. **219** (2005), 187.
- [37] J. Evslin and H. Sati, *SUSY vs E_8 gauge theory in 11 dimensions*, J. High Energy Phys. **0305** (2003), 048, [arXiv:hep-th/0210090](#).
- [38] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert, *The geometry of WZW branes*, J. Geom. Phys. **34** (2000), 162, [arXiv:hep-th/9909030](#).
- [39] G. Moore, *Anomalies, Gauss laws, and Page charges in M-theory*, Comptes Rendus Phys. **6** (2005), 251, [arXiv:hep-th/0409158](#).
- [40] G. Thompson, *1992 Trieste lectures on topological gauge theory and Yang–Mills theory*, in “Trieste Summer School on High Energy Physics and Cosmology 1992”, eds. E. Gava et al., World Scientific, 1993, [arXiv:hep-th/9305120](#).
- [41] E. Frenkel, *Lectures on the Langlands program and conformal field theory*, [arXiv:hep-th/0512172](#).
- [42] A. Kapustin and E. Witten, *Electric–magnetic duality and the geometric langlands program*, [arXiv:hep-th/0604151](#).
- [43] E. Frenkel and A. Losev, *Mirror symmetry in two steps: A–I–B*, [arXiv:hep-th/0505131](#).
- [44] U. Lindström and M. Zabzine, *Tensionless strings, WZW models at critical level and massless higher spin fields*, Phys. Lett. **B584** (2004), 178, [arXiv:hep-th/0305098](#).
- [45] I. Bakas and C. Sourdis, *On the tensionless limit of gauged WZW models*, J. High Energy Phys. **0406** (2004), 049, [arXiv:hep-th/0403165](#).
- [46] J. Fröhlich and K. Gawędzki, *Conformal field theory and geometry of strings*, “CRM Proc. Lect. Notes”, **7**, AMS, Providence, RI, 1994, [arXiv:hep-th/9310187](#).

