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Non-perturbative effective action in gauge theories and quantum gravity

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Abstract

We use our recently developed algebraic methods for the calculation of the heat kernel on homogeneous bundles over symmetric spaces to evaluate the non-perturbative low-energy effective action in quantum general relativity and Yang–Mills gauge theory in curved space. We obtain an exact integral repesentation for the effective action that generates all terms in the standard asymptotic epxansion of the effective action without derivatives of the curvatures effectively summing up the whole infinite subseries of all quantum corrections with low momenta.

1 Introduction

One of the basic object in quantum field theory is the effective action (see [1, 9, 13, 16, 18]). It is a functional of the background fields that encodes, in principle, all the information of quantum field theory. It determines the full one-point propagator as well as all full vertex functions. Moreover, it gives

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the effective equations for the background fields, which makes it possible to study the back-reaction of quantum processes on the classical background.

One of the most powerful methods for the evaluation of the effective action is the heat kernel approach (see the books [9,15,16,18,19,21,22] and reviews [1, 7, 8, 10, 25]). Of course, the effective action (or the heat kernel) cannot be computed exactly. Therefore, various approximation schemes have been developed depending on the problem one is studying. First of all, there is the standard semi-classical expansion of the effective action in inverse powers of a (large) mass parameter of massive quantum fields, which corresponds to the short-time asymptotic expansion of the trace of the heat kernel in powers of the proper time. It describes such physical effects as polarization of vacuum of massive quantum fields by weak background fields. There has been tremendous progress in the explicit calculation of the coefficients of this asymptotic expansion over the last two decades (see [1,7,9,10,13,19,22,25]). However, the applicability of this approximation is rather limited — it does not apply to strong background fields and massless (or light) quantum fields. Therefore, there is a need for new non-perturbative approximation schemes.

Next, one is interested in scattering processes of energetic particles. Such processes are well described by the (essentially perturbative) high-energy approximation. The high-energy effective action can be computed in a sufficiently elaborated perturbation theory [1,8–10,13]. Although it is non-local, it is analytic in the background fields and, therefore, can be computed simply by expanding in powers of background fields (or their curvatures).

On the other hand, one is interested in studying the structure of the physical vacuum (the ground state) of the theory. Such problems are well described by the low-energy approximation. The low-energy effective action (or the heat kernel) is a local, but highly non-trivial (non-polynomial) functional of background fields and their curvatures, and, therefore, it cannot be computed in the usual perturbation theory. There are just a few very special cases, such as group manifolds, spheres, rank-one symmetric spaces and split-rank symmetric spaces when one can determine the spectrum of the Laplacian exactly and obtain closed formulas for the heat kernel in terms of the root vectors and their multiplicities [17, 21]. The complexity of the method crucially depends on the global structure of the symmetric space, most importantly its rank. Therefore, to study the low-energy effective action in the generic case one needs new essentially non-perturbative methods. The development of such methods for the calculation of the heat kernel was initiated in our papers [2, 4] for a gauge theory in flat space, which were then applied to study the vacuum structure of the Yang–Mills theory in [5, 7]. These ideas were first extended to scalar fields on curved manifolds in [3, 6] and finally to arbitrary twisted spin-tensor fields in [11, 12].

In the present paper we apply these methods to study the one-loop low-energy effective action in quantum general relativity and Yang–Mills theory in curved space with some twisted scalar and spinor fields. We consider a wide class of field theory models with the action

$$S = \int_M dx\, g^{1/2}\Bigg\{ \frac{1}{k^2}(R-2\Lambda) + \frac{1}{8e^2}\mathrm{tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \bar\psi\,[\gamma^\mu\nabla_\mu + M(\varphi)]\,\psi - \frac{1}{2}g^{\mu\nu}\nabla_\mu\bar\varphi\nabla_\nu\varphi - V(\varphi)\Bigg\}, \qquad (1.1)$$

where $g = |\det g_{\mu\nu}|$, $g_{\mu\nu}$ is a metric on the spacetime manifold M, R is the scalar curvature, $k^2 = 16\pi G$ is the Einstein coupling constant, G is the Newtonian gravitational constant, Λ is the cosmological constant, $\mathcal{F}_{\mu\nu}$ is the strength of the gauge fields $\mathcal{A}_\mu$ taking values in the adjoint representation of the Lie algebra of a compact simple gauge group G_{YM}, e is a coupling constant, φ and ψ are multiplets of real scalar fields and the Dirac spinor ones, which belong to some, in general, different representations of the gauge group, $M(\varphi)$ is a spinor mass matrix, $V(\varphi)$ is a potential for scalar fields, γ^μ are the Dirac matrices and ∇_μ is the covariant derivative in the corresponding representation.

Our goal is to compute the one-loop effective action for this model assuming a covariantly constant background, that is, a background metric with covariantly constant curvature, a background gauge field with the covariantly constant strength tensor and also some covariantly constant background scalar fields.

This paper is organized as follows. In Section 2 we describe briefly the construction of the one-loop effective action in gauge field theories. In Section 3, we describe the heat kernel method for the calculation of functional determinants of elliptic partial differential operators of Laplace type. In Section 4, we describe the low-energy approximation and derive some of its consequences, in particular, we present the results for the heat trace of our earlier paper [12]. In Sections 5–7 we apply these results to evaluate the effective action in general relativity, the Yang–Mills theory and also for the matter (scalar and the spinor) fields. In conclusion, we summarize our results.

2 Effective action in gauge field theories

We describe briefly the construction of the one-loop effective action in gauge field theories. Let M be a globally hyperbolic spacetime manifold with a (pseudo)-Riemannian metric. Let $\mathcal{V}$ and $\mathcal{G}$ be two fiber bundles over M such that $\dim \mathcal{G} < \dim \mathcal{V}$. Let both bundles V and G be equipped with some Hermitian positive-definite metrics and with the corresponding natural L^2 scalar products $(\ ,\ )_{\mathcal{V}}$ and $(\ ,\ )_{\mathcal{G}}$.

The sections of the bundle $\mathcal{V}$ are quantum (gauge) fields. The dynamics of the quantum fields is described by the action $S : C^\infty(\mathcal{V}) \to \mathbb{R}$. At the linearized level it is described by the second-order differential operator, $P : C^\infty(T\mathcal{V}) \to C^\infty(T\mathcal{V})$ defined by

$$(h, Ph)_V = \frac{d^2}{d\varepsilon^2} S(\varphi + \varepsilon h)\Big|_{\varepsilon=0} . \tag{2.1}$$

If this operator is non-degenerate then the one-loop effective action is determined by its determinant [18]

$$\Gamma_{(1)} = \sigma \frac{i}{2} \log \mathrm{Det}\,(-P), \tag{2.2}$$

where $\sigma = +1$ for bosonic fields and $\sigma = -1$ for fermionic fields. In the following we consider the bosonic theory.

In gauge theory the operator P is degenerate. This means that the action has some invariant flows which define a first-order differential operator $N : C^\infty(T\mathcal{G}) \to C^\infty(T\mathcal{V})$. Let $\bar{N} : C^\infty(T\mathcal{V}) \to C^\infty(T\mathcal{G})$ be the first-order differential operator such that for any $\xi \in C^\infty(\mathcal{G})$, $h \in C^\infty(T\mathcal{V})$

$$(\bar{N}h, \xi)_{\mathcal{G}} = (h, N\xi)_{\mathcal{V}}, \tag{2.3}$$

and $F : C^\infty(T\mathcal{G}) \to C^\infty(T\mathcal{G})$ be the operator defined by

$$F = \bar{N}N. \tag{2.4}$$

Finally, let $L : C^\infty(T\mathcal{V}) \to C^\infty(T\mathcal{V})$ be the second-order differential operator defined by

$$L = -P - N\bar{N}. \tag{2.5}$$

We consider only the case when the gauge generators are linearly independent. This means that the rank of the leading symbol of the operator N equals the dimension of the bundle $\mathcal{G}$. We also assume that the leading symbols of the generators N are complete in that they generate all zero-modes of the leading symbol of the operator P. Then the leading symbols of the operators L and F are non-degenerate and the one-loop effective action has the form [9, 18]

$$\Gamma_{(1)} = \frac{i}{2}\Big(\log \mathrm{Det}\, L - 2\log \mathrm{Det}\, F\Big). \tag{2.6}$$

Strictly speaking, one should include the contribution, $\log \mathrm{Det}\,\gamma$, of the determinant of the gauge group metric γ. However, in the cases of our primary interest (general relativity and Yang–Mills theory) the gauge group metric γ is a zero-order differential operator, and, therefore, its contribution can be omitted, more precisely, it can be absorbed in the definition of the path integral measure.

3 Heat kernel method

The effective action is determined by the functional determinants of second-order hyperbolic partial differential operators with Feynman boundary conditions. At this point we can do the analytic continuation to the imaginary time (Wick rotation) and consider instead of hyperbolic operators the elliptic ones. Furthermore, the most important elliptic partial differential operators encountered in quantum field theory are so-called Laplace type operators. That is why we concentrate below on the calculation of the heat kernel for Laplace-type operators (see [9, 10, 13, 15, 19]).

Let (M, g) be a smooth compact Riemannian manifold of dimension n without boundary, equipped with a positive-definite Riemannian metric g. We assume that it is complete simply connected orientable and spin. Let Λ be a vector space and $\mathrm{End}\,(\Lambda)$ be the space of endomorphisms of Λ. Let $\mathcal{T}$ be a spin-tensor bundle with fiber Λ realizing a representation of the spin group $\mathrm{Spin}(n)$. It naturally defines a representation $\Sigma : \mathcal{SO}(n) \to \mathrm{End}\,(\Lambda)$ of the orthogonal algebra $\mathcal{SO}(n)$ in Λ with generators Σ_{ab}. The spin connection induces a connection on the bundle $\mathcal{T}$ defining the covariant derivative of spin-tensor fields.

Let G_{YM} be a compact Lie (gauge) group and $\mathcal{G}_{YM}$ be its Lie algebra. It naturally defines the principal fiber bundle over the manifold M with the structure group G_{YM}. Let W be a vector space and $\mathrm{End}\,(W)$ be the space of its endomorphisms. We consider a representation $X : \mathcal{G}_{YM} \to \mathrm{End}\,(W)$ of the Lie algebra $\mathcal{G}_{YM}$ in W and the associated vector bundle $\mathcal{W}$ through this representation with the same structure group G_{YM} whose typical fiber is W. Then for any spin-tensor bundle $\mathcal{T}$ we define the twisted spin-tensor bundle $\mathcal{V}$ via the twisted product of the bundles $\mathcal{W}$ and $\mathcal{T}$ with the fiber $V = \Lambda \otimes W$.

We assume that the vector bundle $\mathcal{V}$ is equipped with a Hermitian metric. This naturally identifies the dual vector bundle $\mathcal{V}^*$ with $\mathcal{V}$. We assume that the connection ∇ is compatible with the Hermitian metric on the vector bundle $\mathcal{V}$. The connection is given its unique natural extension to bundles in the tensor algebra over $\mathcal{V}$ and $\mathcal{V}^*$. In fact, using the spin connection together with the connection on the bundle $\mathcal{V}$, we naturally obtain connections on all bundles in the tensor algebra over $\mathcal{V}$, $\mathcal{V}^*$, TM and T^*M; the resulting connection will usually be denoted just by ∇. It is usually clear which bundle's connection is being referred to, from the nature of the section being acted upon.

Let $\mathcal{A}$ be a connection one form on the bundle $\mathcal{W}$ (called Yang–Mills or gauge connection) taking values in the Lie algebra $\mathcal{G}_{YM}$. Then for any section of the bundle $\mathcal{V}$ we have

$$[\nabla_\mu, \nabla_\nu]\varphi = \mathcal{R}_{\mu\nu}\varphi, \tag{3.1}$$

where

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}R^{ab}{}_{\mu\nu}\Sigma_{ab} + X(\mathcal{F}_{\mu\nu}) \tag{3.2}$$

is the curvature of the total connection on the bundle $\mathcal{V}$, and

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\mu] \tag{3.3}$$

is the curvature of the Yang–Mills connection. We use Greek indices to denote tensor components in the coordinate basis. We also use Latin indices from the beginning of the alphabet to denote the indices of an orthornomal frame. Both group of indices range over $1, \dots, n$.

The fiber inner product on the bundle $\mathcal{V}$ defines a natural L^2 inner product on $C^\infty(\mathcal{V})$. The completion of $C^\infty(\mathcal{V})$ in this norm defines the Hilbert space $L^2(\mathcal{V})$. Let ∇^* be the formal adjoint to ∇ and Q be a smooth endomorphism of the bundle $\mathcal{V}$. A Laplace-type operator $L : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V})$ is a partial differential operator of the form

$$L = \nabla^*\nabla + Q = -\Delta + Q, \tag{3.4}$$

where $\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant Laplacian. It is easy to show that the Laplacian, Δ, and, therefore, the operator L, is a self-adjoint elliptic partial differential operator [19].

For $t > 0$ the operators $U(t) = \exp(-tL)$ form a semi-group of bounded operators on $L^2(\mathcal{V})$, the heat semi-group. Moreover, the heat semi-group

$U(t)$ is a trace-class operator with a well-defined L^2-trace, the heat trace [19]:

$$\Theta(t) = \mathrm{Tr}_{L^2} \exp(-tL). \tag{3.5}$$

The heat trace is well defined for real positive t. In fact, it can be analytically continued to an analytic function of t in the right half-plane (for $\mathrm{Re}\, t > 0$).

The heat trace determines the zeta-function,

$$\zeta(s, \lambda) = \mu^{2s} \mathrm{Tr}_{L^2} (L - \lambda)^{-s} = \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt\; t^{s-1}\, \mathrm{e}^{t\lambda} \Theta(t), \tag{3.6}$$

where μ is a renormalization parameter introduced to preserve dimensions, λ is a sufficiently large negative constant such that the operator $(L - \lambda)$ is positive and s is a complex parameter with $\mathrm{Re}\, s > n/2$. The zeta-function is a meromorphic function of s analytic at $s = 0$ [19], and, therefore, it enables one to define, in particular, the zeta-regularized determinant of the operator $(L - \lambda)$, via [1, 9, 10, 13]

$$\zeta'(0, \lambda) \equiv \left. \frac{\partial}{\partial s} \zeta(s, \lambda) \right|_{s=0} = -\log \mathrm{Det}\, (L - \lambda), \tag{3.7}$$

which determines the one-loop effective action in quantum field theory. The parameter λ serves here as an infrared regularization parameter. One should take the limit $\lambda \to 0$ at the end of the calculation.

4 Low-energy approximation

Of course, it is impossible to compute the heat kernel in the generic case. That is why, one considers various approximations. To study the structure of the ground state in quantum field theory one needs to evaluate the heat kernel in the low-energy approximation. In this case the curvatures are strong but slowly varying, i.e., the powers of the curvatures are more important than their derivatives. The main terms in this approximation are the terms without any covariant derivatives of the curvatures. We will consider the zeroth order of this approximation which corresponds simply to covariantly constant background

$$\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \quad \nabla_\mu \mathcal{R}_{\alpha\beta} = 0, \quad \nabla_\mu Q = 0. \tag{4.1}$$

Riemannian manifolds with parallel curvature are called symmetric spaces. Vector bundles with parallel curvature are called homogeneous bundles.

Thus, the most general covariantly constant background is described by homogeneous vector bundles over symmetric spaces.

4.1 Holonomy group

A generic symmetric space has the structure $M = M_0 \times M_s$, where $M_0 = \mathbb{R}^{n_0}$, $M_s = M_+ \times M_-$, and M_+ and M_- are compact and non-compact symmetric spaces, respectively [20, 23, 26]. The components of the curvature tensor can be presented in the form [3, 6, 23]

$$R_{abcd} = \beta_{ik} E^i{}_{ab} E^k{}_{cd}, \tag{4.2}$$

where $E^i{}_{ab}$ is a collection of p anti-symmetric matrices and β_{ik} is a symmetric non-degenerate $p \times p$ matrix. The number p is determined by the curvature tensor. In the following the Latin indices from the middle of the alphabet range over $1, \dots, p$ and will be raised and lowered with the matrix β_{ik} and its inverse β^{ik}. They should not be confused with the Latin indices from the beginning of the alphabet.

Next, we define the traceless $n \times n$ matrices $D_i = (D^a{}_{ib})$, by

$$D^a{}_{ib} = -\beta_{ik} E^k{}_{cb} \delta^{ca}. \tag{4.3}$$

The matrices D_i are known to be the generators of the holonomy algebra, $\mathcal{H}$, i.e., the Lie algebra of the restricted holonomy group, H, [6, 23, 26]

$$[D_i, D_j] = F^k{}_{ij} D_k, \tag{4.4}$$

where $F^j{}_{ik}$ are the structure constants.

The holonomy algebra is a subalgebra of the orthogonal algebra $\mathcal{SO}(n)$ [14, 24, 26]. The embedding of the holonomy algebra $\mathcal{H}$ in the orthogonal algebra $\mathcal{SO}(n)$ is described as follows [12]. Let Y_{ab} be the generators of the orthogonal algebra $\mathcal{SO}(n)$ in the representation $Y : \mathcal{SO}(n) \to \mathrm{End}\,(W)$ of the orthogonal algebra $\mathcal{SO}(n)$ in a vector space W and let T_i be the matrices defined by

$$T_i = -\frac{1}{2} D^a{}_{ib} Y^b{}_a. \tag{4.5}$$

Then T_i form a representation of the holonomy algebra $\mathcal{H}$ in W, that is, they satisfy the commutation relations

$$[T_i, T_j] = F^k{}_{ij} T_k. \tag{4.6}$$

Vice versa, for every representation $T : \mathcal{H} \to \mathrm{End}\,(W)$ of the holonomy algebra $\mathcal{H}$ in a vector space W there is a representation $Y : \mathcal{SO}(n) \to \mathrm{End}\,(W)$ of the orthogonal algebra $\mathcal{SO}(n)$ in W such that the generators T_i of the representation T are given by (4.5).

The structure constants $F^j{}_{ik}$ of the holonomy group define the $p \times p$ matrices F_i, by $(F_i)^j{}_k = F^j{}_{ik}$, which generate the adjoint representation of the holonomy algebra. The scalar curvature of the holonomy group is given by the invariant [20, 26]

$$R_H = -\frac{1}{4}\beta^{ij} F^k{}_{il} F^l{}_{jk}. \tag{4.7}$$

4.2 Homogeneous vector bundles

Let $h^a{}_b$ be the projection to the subspace $T_x M_s$ of the tangent space $T_x M$ and

$$q^a{}_b = \delta^a{}_b - h^a{}_b \tag{4.8}$$

be the projection tensor to the flat subspace $\mathbb{R}^{n_0}$. Since the curvature exists only in the semi-simple submanifold M_s, the components of the curvature tensor R_{abcd}, as well as the tensors $E^i{}_{ab}$, are non-zero only in the semi-simple subspace M_s. Moreover, the condition (4.1) imposes strong constraints on the curvature of the homogeneous bundle $\mathcal{W}$. We decompose the Yang–Mills curvature according to

$$\mathcal{F}_{ab} = \mathcal{B}_{ab} + \mathcal{E}_{ab}, \tag{4.9}$$

where

$$\mathcal{B}_{ab} = \mathcal{F}_{cd} q^c{}_a q^d{}_b, \quad \mathcal{E}_{ab} = \mathcal{F}_{cd} h^c{}_a h^d{}_b. \tag{4.10}$$

Then, one can show [12] that if $\mathcal{B}_{ab}$ is non-zero then it takes values in an Abelian ideal of the gauge algebra $\mathcal{G}_{YM}$ (that is, commutes with everyhting else) and if $\mathcal{E}_{ab}$ is non-zero then it takes values in a representation of the holonomy algebra. More precisely, the existence of a non-zero component $\mathcal{E}_{ab}$ is possible only if the holonomy algebra $\mathcal{H}$ is an ideal of the gauge algebra $\mathcal{G}_{YM}$. That is, the gauge algebra $\mathcal{G}_{YM}$ must be big enough to have a subalgebra $\mathcal{C} \oplus \mathcal{H}$, where $\mathcal{C}$ is an Abelian ideal. Below we will assume that this is the case.

Since the curvature $\mathcal{E}_{ab}$ takes values in the holonomy algebra, it has the form [12] $X(\mathcal{E}_{ab}) = -E^i_{ab}T_i$, where T_i are the generators of the holonomy algebra in some representation T of the holonomy algebra in the vector space W. Since the holonomy algebra is a subalgebra of the orthogonal algebra $\mathcal{SO}(n)$ it can be embedded in the orthogonal algebra $\mathcal{SO}(n)$ via a representation $Y : \mathcal{SO}(n) \to \mathrm{End}\,(W)$ of the orthogonal algebra $\mathcal{SO}(n)$ in W, equation (4.5). Thus, the curvature of the homogeneous bundle $\mathcal{W}$ is given by

$$X(\mathcal{F}_{ab}) = -E^i{}_{ab}T_i + X(\mathcal{B}_{ab}) = \frac{1}{2}R^{cd}{}_{ab}Y_{cd} + X(\mathcal{B}_{ab}), \tag{4.11}$$

where $X(\mathcal{B}_{ab})$ satisfies the commutation relations $[X(\mathcal{B}_{ab}), X(\mathcal{B}_{cd})] = [X(\mathcal{B}_{ab}), Y_{cd}] = 0$ and Y_{ab} are the generators of the orthogonal algebra $\mathcal{SO}(n)$ in the representation Y.

Now, we consider the representation $\Sigma : \mathcal{SO}(n) \to \mathrm{End}\,(\Lambda)$ of the orthogonal algebra $\mathcal{SO}(n)$ in the vector space Λ (defining the spin-tensor bundle $\mathcal{T}$) and define the generators

$$G_{ab} = \Sigma_{ab} \otimes \mathbb{I}_Y + \mathbb{I}_\Sigma \otimes Y_{ab} \tag{4.12}$$

of the orthogonal algebra $\mathcal{SO}(n)$ in the product representation $G = \Sigma \otimes Y : \mathcal{SO}(n) \to \mathrm{End}\,(V)$ in the vector space $V = \Lambda \otimes W$.

Then the matrices

$$\mathcal{R}_i = -\frac{1}{2}D^a{}_{ib}G^b{}_a \tag{4.13}$$

form a representation $\mathcal{R} : \mathcal{H} \to \mathrm{End}\,(V)$ of the holonomy algebra in V and the total curvature of the twisted spin-tensor bundle $\mathcal{V}$ is

$$\mathcal{R}_{ab} = -E^i{}_{ab}\mathcal{R}_i + X(\mathcal{B}_{ab}) = \frac{1}{2}R^{cd}{}_{ab}G_{cd} + X(\mathcal{B}_{ab}). \tag{4.14}$$

The Casimir operator of the holonomy group in this representation is [12]

$$\mathcal{R}^2 = \beta^{ij}\mathcal{R}_i\mathcal{R}_j = \frac{1}{4}R^{abcd}G_{ab}G_{cd}. \tag{4.15}$$

4.3 Heat trace

The heat trace of the operator L was computed in [12]. It has the form

$$\begin{aligned}\Theta(t) = &\int_M dx\, g^{1/2}\, (4\pi t)^{-n/2} \exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H\right) t\right\}\\ &\times \int_{\mathbb{R}^n_{\rm reg}} \frac{d\omega}{(4\pi t)^{p/2}}\, \beta^{1/2} \exp\left\{-\frac{1}{4t}\,\langle\omega, \beta\omega\rangle\right\} \Psi(t,\omega)\\ &\times \left[\det{}_{\mathcal{H}}\left(\frac{\sinh\left[\,F(\omega)/2\right]}{F(\omega)/2}\right)\right]^{1/2}\\ &\times \left[\det{}_{TM}\left(\frac{\sinh\left[\,D(\omega)/2\right]}{D(\omega)/2}\right)\right]^{-1/2}, \end{aligned} \tag{4.16}$$

where $\beta = \det\beta_{ij}$, $\langle\omega, \beta\omega\rangle = \beta_{ij}\omega^i\omega^j$ and

$$\begin{aligned}\Psi(t,\omega) = \operatorname{tr}{}_W &\left[\det{}_{TM}\left(\frac{\sinh(tX(\mathcal{B}))}{tX(\mathcal{B})}\right)\right]^{-1/2}\\ &\times \operatorname{tr}{}_\Lambda \exp\left[-t\left(\mathcal{R}^2 + Q\right)\right] \exp\left[\mathcal{R}(\omega)\right]. \end{aligned} \tag{4.17}$$

Here $D(\omega)$, $F(\omega)$, $\mathcal{R}(\omega)$ and $\mathcal{B}$ are matrices defined by $D(\omega) = \omega^i D_i$, $F(\omega) = \omega^i F_i$, $\mathcal{R}(\omega) = \omega^i \mathcal{R}_i$ and $\mathcal{B} = (\mathcal{B}^a{}_b)$, where the matrices D_i, F_i, $\mathcal{R}_i$ and $\mathcal{B}_{ab}$ were defined above in Sections 3.1 and 3.2. Notice that the whole structure of this expression is the same for all vector bundles (all representations), the only difference is in the function $\Psi(t,\omega)$.

We need to explain the meanning of the integral over ω^i in (4.16). In the derivation of this formula in [12] we used a certain regularization procedure. The point is that the integrals over the holonomy group in canonical coordinates ω^i have singularities that need to be avoided (or regularized) by deforming the contour of integration. This procedure with the non-standard contour of integration is necessary for the convergence of the integrals since we are treating both the compact and the non-compact symmetric spaces simultaneously. We complexify the holonomy group by extending the canonical coordinates ω^i to be complex, more precisely, to take values in the p-dimensional subspace $\mathbb{R}^p_{\rm reg}$ of $\mathbb{C}^p$ obtained by rotating $\mathbb{R}^p$ counterclockwise by $\pi/4$ in $\mathbb{C}^p$, that is, we replace each variable ω^j by $e^{i\pi/4}\omega^j$. We also make an analytic continuation in the complex plane of t with a cut along the negative imaginary axis so that $-\pi/2 < \arg t < 3\pi/2$ and consider t to be real negative, $t < 0$. Remember, that, in general, the non-degenerate diagonal matrix β_{ij} is not positive definite. The space $\mathbb{R}^p_{\rm reg}$ is chosen in such a way to make the Gaussian exponent purely imaginary. Then the indefiniteness of the matrix β does not cause any problems. Moreover, the integrand does not have any singularities on these contours. The convergence of the integral is guaranteed by the exponential growth of the sine for imaginary argument. These integrals can be computed then in the following way. The coordinates ω^j corresponding to the compact directions are rotated further by another $\pi/4$ to imaginary axis and the coordinates ω^j corresponding to the non-compact directions are rotated back to the real axis. Then, for $t<0$ all the integrals are well defined and convergent and define an analytic function of t in a complex plane with a cut along the negative imaginary axis.

5 General relativity

Einstein's theory of general relativity is a gauge theory with the gauge group $\mathcal{G}$ being the group of diffeomorphisms of the spacetime manifold M. The gravitational field can be parametrized by the metric tensor of the spacetime $g_{\mu\nu}$. The Hilbert–Einstein action of general relativity has the form

$$S_{\rm GR} = \frac{1}{k^2}\int_M dx\; g^{1/2}\left(R-2\Lambda\right). \tag{5.1}$$

The tangent bundle to the bundle of Riemannian metrics is the bundle $\mathcal{T}_{(2)} = T^*M \vee T^*M$ of symmetric covariant 2-tensors. (Here $\vee = \mathrm{Sym}\otimes$ is the symmetric tensor product.) An invariant fiber metric on the vector bundle $\mathcal{T}_{(2)}$ is defined by

$$E^{\mu\nu\alpha\beta} = g^{\mu(\alpha}g^{\beta)\nu} - \varkappa g^{\mu\nu}g^{\alpha\beta}, \tag{5.2}$$

where $\varkappa \neq 1/n$ is a real parameter. The inverse metric is then

$$E^{-1}_{\mu\nu\alpha\beta} = g_{\mu(\alpha}g_{\beta)\nu} - \frac{\varkappa}{n\varkappa-1}g_{\mu\nu}g_{\alpha\beta}. \tag{5.3}$$

The tangent bundle to the group of diffeomorphisms is the tangent bundle TM itself. We define an invariant metric on the gauge algebra by

$$\gamma_{\mu\nu} = \frac{k^2}{\alpha}g_{\mu\nu}, \tag{5.4}$$

where $\alpha \neq 0$ is a real parameter.

The invariant flows of the action are the infinitesimal diffeomorphisms, which define the first-order differential operators $N : C^\infty(TM) \to C^\infty(\mathcal{T}_{(2)})$

and $\bar N : C^\infty(\mathcal{T}_{(2)}) \to C^\infty(TM)$ by

$$(N\xi)_{\mu\nu} = 2g_{\lambda(\nu}\nabla_{\mu)}\xi^\lambda, \tag{5.5}$$

$$(\bar N h)^\alpha = -2\frac{\alpha}{k^2}\left(g^{\alpha(\nu}\nabla^{\mu)} - \varkappa g^{\mu\nu}\nabla^\alpha\right)h_{\mu\nu}. \tag{5.6}$$

Therefore, the ghost operator $\tilde F = \bar N N : C^\infty(TM) \to C^\infty(TM)$ is a second-order differential operator defined by

$$\tilde F = 2\frac{\alpha}{k^2}F, \tag{5.7}$$

where

$$F^\mu_\nu = -\delta^\mu_\nu\Delta + (2\varkappa - 1)\nabla^\mu\nabla_\nu - R^\mu_\nu. \tag{5.8}$$

The second variation of the action defines a second-order partial differential operator $P : C^\infty(\mathcal{T}_{(2)}) \to C^\infty(\mathcal{T}_{(2)})$ by

$$\frac{d^2}{d\varepsilon^2}S_{\rm GR}(g+\varepsilon h)\Big|_{\varepsilon=0} = (h, Ph)_{\mathcal{T}_{(2)}}, \tag{5.9}$$

where

$$\begin{aligned}
P_{\mu\nu}{}^{\alpha\beta} = -\frac{1}{2k^2}\Bigg\{ &-\left(\delta^{(\alpha}{}_{(\mu}\delta^{\beta)}{}_{\nu)} + \frac{1-2\varkappa}{n\varkappa-1}g_{\mu\nu}g^{\alpha\beta}\right)\Delta + \frac{1-2\varkappa}{n\varkappa-1}g_{\mu\nu}\nabla^{(\alpha}\nabla^{\beta)} \\
&- g^{\alpha\beta}\nabla_{(\mu}\nabla_{\nu)} + 2\nabla_{(\mu}\delta^{(\alpha}{}_{\nu)}\nabla^{\beta)} - 2R^\alpha{}_{(\mu}{}^\beta{}_{\nu)} - 2\delta^{(\alpha}{}_{(\mu}R^{\beta)}{}_{\nu)} + R_{\mu\nu}g^{\alpha\beta} \\
&+ \frac{4\varkappa-1}{n\varkappa-1}g_{\mu\nu}R^{\alpha\beta} + (R-2\Lambda)\delta^\alpha{}_{(\mu}\delta^\beta{}_{\nu)} \\
&+ \frac{1}{2(n\varkappa-1)}\Big[(1-4\varkappa)R + 2(2\varkappa-1)\Lambda\Big]g_{\mu\nu}g^{\alpha\beta}\Bigg\}.
\end{aligned} \tag{5.10}$$

The operator $N\bar N : C^\infty(\mathcal{T}_{(2)}) \to C^\infty(\mathcal{T}_{(2)})$ is a second-order operator of the form

$$(N\bar N)^{\alpha\beta}_{\mu\nu} = -4\frac{\alpha}{k^2}\left\{\nabla_{(\mu}\delta^{(\alpha}{}_{\nu)}\nabla^{\beta)} - \varkappa g^{\alpha\beta}\nabla_{(\mu}\nabla_{\nu)}\right\}. \tag{5.11}$$

The graviton operator $\tilde L = -P - N\bar N : C^\infty(\mathcal{T}_{(2)}) \to C^\infty(\mathcal{T}_{(2)})$ now reads

$$\tilde L = \frac{1}{2k^2}L, \tag{5.12}$$

where

$$
\begin{aligned}
L_{\mu\nu}{}^{\alpha\beta} = &-\left(\delta^{(\alpha}{}_{(\mu}\delta^{\beta)}{}_{\nu)} + \frac{1-2\varkappa}{n\varkappa-1}g_{\mu\nu}g^{\alpha\beta}\right)\Delta + \frac{1-2\varkappa}{n\varkappa-1}g_{\mu\nu}\nabla^{(\alpha}\nabla^{\beta)}\\
&-(1+8\alpha\varkappa)g^{\alpha\beta}\nabla_{(\mu}\nabla_{\nu)} + 2(1+4\alpha)\nabla_{(\mu}\delta^{(\alpha}{}_{\nu)}\nabla^{\beta)} - 2R^{\alpha}{}_{(\mu}{}^{\beta}{}_{\nu)}\\
&-2\delta^{(\alpha}{}_{(\mu}R^{\beta)}{}_{\nu)} + R_{\mu\nu}g^{\alpha\beta} + \frac{4\varkappa-1}{n\varkappa-1}g_{\mu\nu}R^{\alpha\beta} + (R-2\Lambda)\delta^{\alpha}{}_{(\mu}\delta^{\beta}{}_{\nu)}\\
&+\frac{1}{2(n\varkappa-1)}\Big[(1-4\varkappa)R + 2(2\varkappa-1)\Lambda\Big]g_{\mu\nu}g^{\alpha\beta}.
\end{aligned}
\qquad (5.13)
$$

The most convenient choice of the parameters of the fiber metrics (gauge parameters) is

$$
\varkappa = \frac{1}{2}, \quad \alpha = -\frac{1}{4}. \qquad (5.14)
$$

In this gauge the non-diagonal derivatives in both the operators F and L vanish so that they both are of Laplace type

$$
\begin{aligned}
F^{\mu}{}_{\nu} &= -\delta^{\mu}{}_{\nu}\Delta - R^{\mu}{}_{\nu},\\
L &= -\Delta + Q,
\end{aligned}
\qquad (5.15)
$$

where

$$
\begin{aligned}
Q_{\mu\nu}{}^{\alpha\beta} = &-2R_{\mu}{}^{(\alpha}{}_{\nu}{}^{\beta)} - 2\delta^{(\alpha}{}_{(\mu}R^{\beta)}{}_{\nu)} + R_{\mu\nu}g^{\alpha\beta} + \frac{2}{n-2}g_{\mu\nu}R^{\alpha\beta}\\
&-\frac{1}{(n-2)}g_{\mu\nu}g^{\alpha\beta}R + (R-2\Lambda)\delta^{\alpha}{}_{(\mu}\delta^{\beta}{}_{\nu)}.
\end{aligned}
\qquad (5.16)
$$

In the Euclidean formulation the zeta-regularized effective action has the form

$$
\Gamma^{\mathrm{GR}}_{(1)} = -\frac{1}{2}\zeta'_{\mathrm{GR}}(0), \qquad (5.17)
$$

where

$$
\zeta_{\mathrm{GR}}(s) = \zeta_L(s) - 2\zeta_F(s), \qquad (5.18)
$$

where $\zeta_L(s)$ and $\zeta_F(s)$ are the zeta functions of the graviton operator L and the ghost operator F. Next, by using the definition of the zeta function we obtain

$$
\zeta_{\mathrm{GR}}(s) = \frac{\mu^{2s}}{\Gamma(s)}\int_0^\infty dt\; t^{s-1}e^{t\lambda}\Theta_{\mathrm{GR}}(t), \qquad (5.19)
$$

where

$$\Theta_{\rm GR}(t) = \Theta_L(t) - 2\Theta_F(t), \tag{5.20}$$

and $\Theta_L(t)$ and $\Theta_F(t)$ are the heat traces of the operators L and F.

By using the results for the heat traces described above we obtain the total heat trace

$$\Theta_{\rm GR}(t) = (4\pi t)^{-n/2} \int_M d\text{vol} \;\; \exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H\right)t\right\} \tag{5.21}$$

$$\times \int_{\mathbb{R}^n_{\rm reg}} \frac{d\omega}{(4\pi t)^{p/2}}\, \beta^{1/2} \exp\left\{-\frac{1}{4t}\langle \omega, \beta\omega\rangle\right\} \Psi_{\rm GR}(t,\omega)$$

$$\times \left[\det{}_{\mathcal{H}}\left(\frac{\sinh\left[\,F(\omega)/2\right]}{F(\omega)/2}\right)\right]^{1/2} \left[\det{}_{TM}\left(\frac{\sinh\left[\,D(\omega)/2\right]}{D(\omega)/2}\right)\right]^{-1/2}, \tag{5.22}$$

where

$$\Psi_{\rm GR}(t,\omega) = \Psi_L(t,\omega) - 2\Psi_F(t,\omega). \tag{5.23}$$

Thus, all we need to compute is the functions $\Psi_L(t,\omega)$ and $\Psi_F(t,\omega)$ for the operators L and F.

Notice that both operators L and F act on pure (untwisted) tensor bundles. Therefore, there is no Yang–Mills group, that is, $\mathcal{F}_{ab} = \mathcal{E}_{ab} = \mathcal{B}_{ab} = 0$. The generators of the orthogonal algebra $\mathcal{SO}(n)$ in the vector and the symmetric 2-tensor representation are

$$\left(\Sigma_{(1)ab}\right)^c{}_d = 2\delta^c_{[a} g_{b]d}, \tag{5.24}$$

$$\left(\Sigma_{(2)ab}\right)^{ef}{}_{cd} = -4\delta^{(e}_{[a} g_{b](d}\delta^{f)}_{c)}. \tag{5.25}$$

Therefore, the generators of the holonomy group are

$$\mathcal{R}_{(1)i} = D_i, \tag{5.26}$$

$$\mathcal{R}_{(2)i} = -2D_i \vee I_{(1)}, \tag{5.27}$$

which, in component language, reads

$$\left(\mathcal{R}_{(1)i}\right)^a{}_b = D^a{}_{ib}, \tag{5.28}$$

$$\left(\mathcal{R}_{(2)i}\right)^{ab}{}_{cd} = -2D^{(a}{}_{i(d}\delta^{b)}{}_{c)}. \tag{5.29}$$

Now, it is easy to compute the Casimir operators

$$\left(\mathcal{R}^2_{(1)}\right)^a_{\ b} = -R^a_b, \tag{5.30}$$

$$\left(\mathcal{R}^2_{(2)}\right)^{ab}_{\ \ cd} = 2R^{(a}{}_d{}^{b)}{}_c - 2\delta^{(a}{}_{(c}R^{b)}{}_{d)}. \tag{5.31}$$

The potentials for both operators are obviously read off from their definition

$$(Q_F)^a_b = -R^a{}_b, \tag{5.32}$$

$$\begin{aligned}(Q_L)^{ab}_{cd} = &-2R^{(a}{}_c{}^{b)}{}_d - 2\delta^{(a}{}_{(c}R^{b)}{}_{d)} + R_{cd}g^{ab} + \frac{2}{n-2}g_{cd}R^{ab} \\ &- \frac{1}{(n-2)}g_{cd}g^{ab}R + \delta^a{}_{(c}\delta^b{}_{d)}(R-2\Lambda).\end{aligned} \tag{5.33}$$

By substituting these expressions in the general formula (4.17) we obtain

$$\begin{aligned}\Psi_L(t,\omega) &= \exp\left[-t(R-2\Lambda)\right] \operatorname{tr}_{\mathcal{T}_{(2)}} \exp\left(t\mathcal{M}_L\right) \exp\left[2D(\omega)\vee I_{(1)}\right], \\ \Psi_F(t,\omega) &= \operatorname{tr}_{TM} \exp\left(t\mathcal{M}_F\right) \exp\left[\,D(\omega)\right],\end{aligned} \tag{5.34}$$

where the endomorphisms $\mathcal{M}_L$ and $\mathcal{M}_F$ are defined by

$$(\mathcal{M}_F)^a_b = 2R^a{}_b, \tag{5.35}$$

$$(\mathcal{M}_L)^{ab}_{cd} = 4\delta^{(a}{}_{(c}R^{b)}{}_{d)} - R_{cd}g^{ab} - \frac{2}{n-2}g_{cd}R^{ab} + \frac{1}{(n-2)}g_{cd}g^{ab}R. \tag{5.36}$$

6 Yang–Mills theory in curved space

Let G_{YM} be a compact simple Lie group. Yang–Mills theory is a gauge theory with the gauge group being the group of transformations of sections of the principal bundle over the spacetime manifold M with structure group G_{YM} and the configuration space being the space of all connections on this principal bundle valued in the Lie algebra $\mathcal{G}_{YM}$ of the group G_{YM}. Let $Ad : \mathcal{G}_{YM} \to \operatorname{End}(W_{Ad})$ be the adjoint representation of the gauge algebra $\mathcal{G}_{YM}$ in the vector space W_{Ad} and $\mathcal{W}_{Ad}$ be the associated vector bundle over M with structure group G_{YM} and the fiber $\operatorname{End}(W_{Ad})$ realizing the adjoint representation of the gauge group. The Yang–Mills gauge field can be parametrized by the local components of the connection $\mathcal{A}_\mu$ taking values in End (W_{Ad}). Then the Yang–Mills action has the form [18]

$$S_{YM} = \frac{1}{8e^2} \int_M dx\, g^{1/2} \mathrm{tr}\,_{W_{Ad}} g^{\mu\alpha} g^{\nu\beta} \mathcal{F}_{\mu\nu} \mathcal{F}_{\alpha\beta}. \tag{6.1}$$

The (ghost) operator $K : C^\infty(\mathcal{W}_{Ad}) \to C^\infty(\mathcal{W}_{Ad})$ and the (gluon) operator $H : C^\infty(\mathcal{W}_{Ad} \otimes TM) \to C^\infty(\mathcal{W}_{Ad} \otimes TM)$ are second-order partial differential operators acting on scalar and vector fields valued in End (W_{Ad}). In the minimal gauge these operators are [5, 7]

$$H^\mu{}_\nu = -\delta^\mu{}_\nu \Delta + R^\mu{}_\nu - 2\mathcal{F}^\mu{}_\nu, \tag{6.2}$$

$$K = -\Delta. \tag{6.3}$$

Thus, the zeta-regularized one-loop effective action of quantum Yang–Mills theory in the Euclidean formulation is given by

$$\Gamma^{YM}_{(1)} = -\frac{1}{2}\zeta'_{YM}(0), \tag{6.4}$$

where

$$\zeta_{YM}(s) = \zeta_H(s) - 2\zeta_K(s), \tag{6.5}$$

and $\zeta_H(s)$ and $\zeta_K(s)$ are the zeta functions of the gluon operator H and the ghost operator K. Next, by using the definition of the zeta function we obtain

$$\zeta_{YM}(s) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{e}^{t\lambda} \Theta_{YM}(t), \tag{6.6}$$

where

$$\Theta_{YM}(t) = \Theta_H(t) - 2\Theta_K(t), \tag{6.7}$$

and $\Theta_H(t)$ and $\Theta_K(t)$ are the heat traces of the operators H and K.

Since both operators H and K are of Laplace type we can use the results for the heat trace described above. We obtain the total heat trace

$$\Theta_{YM}(t) = (4\pi t)^{-n/2} \int_M dx\, g^{1/2} \exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H\right)t\right\} \tag{6.8}$$

$$\times \int_{\mathbb{R}^n_{\rm reg}} \frac{d\omega}{(4\pi t)^{p/2}}\, \beta^{1/2} \exp\left\{-\frac{1}{4t}\langle\omega, \beta\omega\rangle\right\} \Psi_{YM}(t,\omega)$$

$$\times \left[\det{}_{\mathcal{H}} \left(\frac{\sinh\left[F(\omega)/2\right]}{F(\omega)/2}\right)\right]^{1/2} \left[\det{}_{TM} \left(\frac{\sinh\left[D(\omega)/2\right]}{D(\omega)/2}\right)\right]^{-1/2}, \tag{6.9}$$

where

$$\Psi_{YM}(t,\omega) = \Psi_H(t,\omega) - 2\Psi_K(t,\omega). \tag{6.10}$$

Thus all we have to do now is to compute the functions $\Psi_H(t,\omega)$ and $\Psi_K(t,\omega)$ for the operators H and K.

We assume that the gauge algebra is big enough to include the holonomy algebra as a subalgebra (as discussed above). Further, we assume that $\mathcal{B}_{ab}$ takes values in the (Abelian) Cartan subalgebra of the gauge algebra. The other part $\mathcal{E}_{ab}$ of the Yang–Mills curvature is described by a representation $Y_{Ad} : \mathcal{SO}(n) \to \mathrm{End}\,(W_{Ad})$ of the orthogonal algebra $\mathcal{SO}(n)$ in W_{Ad} with generators Y^{Ad}_{ab} so that the total Yang–Mills curvature is given by (6.11)

$$Ad(\mathcal{F}_{ab}) = \frac{1}{2} R^{cd}{}_{ab} Y^{Ad}_{cd} + Ad(\mathcal{B}_{ab}). \tag{6.11}$$

For the ghost operator K we have $Q_K = 0$ and $\Sigma^K_{ab} = 0$. Therefore,

$$\mathcal{R}^K_i = -\frac{1}{2} D^a{}_{ib} {Y_{Ad}}^b{}_a, \tag{6.12}$$

$$\mathcal{R}^2_K = \frac{1}{4} R^{abcd} Y^{Ad}_{ab} Y^{Ad}_{cd}. \tag{6.13}$$

Thus, we obtain

$$\Psi_K(t,\omega) = \mathrm{tr}_{\,W_{Ad}} \left[\det{}_{TM}\left(\frac{\sinh(tAd(\mathcal{B}))}{tAd(\mathcal{B})}\right)\right]^{-1/2} \times \exp\left(-\mathcal{R}^2_K t\right) \exp\left[\mathcal{R}_K(\omega)\right], \tag{6.14}$$

where $\mathcal{R}_K(\omega) = \mathcal{R}^K_i \omega^i$.

For the gluon operator H we have

$$(Q_H)^a{}_b = R^a{}_b - 2Ad(\mathcal{F}^a{}_b) = R^a{}_b - R^a{}_{bcd} Y^{cd}_{Ad} - 2Ad(\mathcal{B}^a{}_b), \tag{6.15}$$

$$\left(\Sigma^H{}_{ab}\right)^c{}_d = 2\delta^c{}_{[a} g_{b]d}. \tag{6.16}$$

Therefore,

$$\left(\mathcal{R}^H{}_i\right)^a_b = D^a{}_{ib} + \delta^a{}_b \mathcal{R}^K_i, \tag{6.17}$$

and

$$\left(\mathcal{R}^2_H\right)^a_{\ b} = -R^a{}_b + R^a{}_{bcd} Y^{cd}_{Ad} + \delta^a{}_b \mathcal{R}^2_K. \tag{6.18}$$

Thus

$$\begin{aligned}\Psi_H(t,\omega) = \operatorname{tr}_{W_{Ad}} &\left[\det{}_{TM}\left(\frac{\sinh(tAd(\mathcal{B}))}{tAd(\mathcal{B})}\right)\right]^{-1/2} \exp\left(-\mathcal{R}^2_K t\right)\\ &\times \exp\left[\mathcal{R}_K(\omega)\right] \operatorname{tr}_{TM} \exp\left[2Ad(\mathcal{B})t\right] \exp\left[D(\omega)\right]. \end{aligned}\tag{6.19}$$

Finally, we obtain the total function $\Psi(t,\omega)$:

$$\begin{aligned}\Psi_{YM}(t,\omega) = \operatorname{tr}_{W_{Ad}} &\left[\det{}_{TM}\left(\frac{\sinh(tAd(\mathcal{B}))}{tAd(\mathcal{B})}\right)\right]^{-1/2} \exp\left(-\mathcal{R}^2_K t\right) \exp\left[\mathcal{R}_K(\omega)\right]\\ &\times \operatorname{tr}_{TM}\Big\{\exp\left[2Ad(\mathcal{B})t\right] \exp\left[D(\omega)\right] - 2\Big\}. \end{aligned}\tag{6.20}$$

7 Matter fields

Now, we assume that M is a spin manifold. Let $\Lambda_{\rm spin}$ be the spinor vector space and $\operatorname{End}(\Lambda_{\rm spin})$ be the space of endomorphisms of $\Lambda_{\rm spin}$. Let $\mathcal{S}$ be the spinor bundle with fiber $\Lambda_{\rm spin}$ realizing the spinor representation of the spin group $\mathrm{Spin}(n)$. It defines the spinor representation $\gamma : \mathcal{SO}(n) \to \operatorname{End}(\Lambda_{\rm spin})$ of the orthogonal algebra $\mathcal{SO}(n)$ in $\Lambda_{\rm spin}$. The spin connection induces a connection on the bundle $\mathcal{S}$ defining the covariant derivative of spin-tensor fields. Let G_{YM} be a compact simple Lie group and $\mathcal{G}_{YM}$ be its Lie algebra. It naturally defines the principal fiber bundle over the manifold M with the structure group G_{YM}. Let $W_{\rm spin}$ be a vector space and $\operatorname{End}(W_{\rm spin})$ be the space of its endomorphisms. We consider a representation $X_{\rm spin} : \mathcal{G}_{YM} \to \operatorname{End}(W_{\rm spin})$ of the Lie algebra $\mathcal{G}_{YM}$ in $W_{\rm spin}$ and the associated vector bundle $\mathcal{W}_{\rm spin}$ through this representation with the same structure group G_{YM} whose typical fiber is $W_{\rm spin}$. Then we define the twisted spinor bundle $\mathcal{V}_{\rm spin}$ via the twisted product of the bundles $\mathcal{W}_{\rm spin}$ and $\mathcal{S}$ with the fiber $V_{\rm spin} = \Lambda_{\rm spin} \otimes W_{\rm spin}$. The spin connection on the spinor bundle and the Yang–Mills connection on the bundle $\mathcal{W}_{\rm spin}$ define the twisted spin connection on the bundle $\mathcal{V}_{\rm spin}$.

Let $\mathcal{W}_0$ be another associated vector bundle over M with the structure group G_{YM} and typical fiber W_0 realizing a representation $X_0 : \mathcal{G}_{YM} \to \operatorname{End}(W_0)$ of the Lie algebra $\mathcal{G}_{YM}$ in W_0.

The sections of the bundles $\mathcal{W}_0$ and $\mathcal{V}_{\rm spin}$ are multiplets of scalar, φ, and spinor, ψ, fields that we call matter fields. The action of matter fields reads

$$S_{\rm matter} = \int_M dx\, g^{1/2} \Bigg\{ \langle \psi, [\gamma^\mu \nabla_\mu + M(\varphi)]\,\psi \rangle_{V_{\rm spin}} - \frac{1}{2} g^{\mu\nu} \langle \nabla_\mu \varphi, \nabla_\nu \varphi \rangle_{W_0} - V(\varphi) \Bigg\}, \tag{7.1}$$

where $\langle\ ,\ \rangle_{V_{\rm spin}}$ and $\langle\ ,\ \rangle_{W_0}$ are the (Hermitian) inner products in the vector spaces $V_{\rm spin}$ and W_0, $M(\varphi) \in \operatorname{End}(V_{\rm spin})$ is an endomorphism of $V_{\rm spin}$ and $V(\varphi)$ is a scalar function of φ.

The contribution of the matter fields to the one-loop effective action has the form [18]

$$\Gamma^{\rm matter}_{(1)} = -\log \operatorname{Det} D + \frac{1}{2} \log \operatorname{Det} L_0, \tag{7.2}$$

where D is the Dirac-type operator and L_0 is a Laplace-type operator defined by

$$D = \gamma^\mu \nabla_\mu + M(\phi), \tag{7.3}$$

$$L_0 = -\Delta + Q_0(\phi), \tag{7.4}$$

where ϕ is a background scalar field and $Q_0(\phi)$ is the mass matrix of the scalar fields. Here the background scalar fields realize the minimum of the potential $V(\varphi)$, and the matrix Q_0 is defined by

$$V(\varphi) = V(\phi) + \frac{1}{2} \langle (\varphi - \phi), Q_0(\phi)(\varphi - \phi) \rangle_{W_0} + O((\varphi - \phi)^3). \tag{7.5}$$

As we mentioned above it is assumed that the endomorphism Q_0 is covariantly constant.

We also assume that the mass matrix M does not contain the Dirac matrices or contains only even number of them, so that

$$[M, \gamma_\mu] = 0. \tag{7.6}$$

Then one can show that the spinor contribution can be expressed in terms of the squared Dirac operator

$$\log \operatorname{Det} D = \frac{1}{2} \log \operatorname{Det} L_{\rm spin}, \tag{7.7}$$

where

$$L_{\text{spin}} = -\Delta + \frac{1}{4}R - \frac{1}{2}\gamma^{ab}X(\mathcal{F}_{ab}) + M^2, \tag{7.8}$$

where $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$.

Thus, the zeta-regularized contribution of the matter fields to the one-loop effective action in the Euclidean formulation is given by

$$\Gamma^{\text{matter}}_{(1)} = -\frac{1}{2}\zeta'_{\text{matter}}(0), \tag{7.9}$$

where

$$\zeta_{\text{matter}}(s) = \zeta_0(s) - \zeta_{\text{spin}}(s), \tag{7.10}$$

where $\zeta_0(s)$ and $\zeta_{\text{spin}}(s)$ are the zeta functions of the operators L_0 and L_{spin}. Next, by using the definition of the zeta function we obtain

$$\zeta_{\text{matter}}(s) = \frac{\mu^{2s}}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{t\lambda}\Theta_{\text{matter}}(t), \tag{7.11}$$

where

$$\Theta_{\text{matter}}(t) = \Theta_0(t) - \Theta_{\text{spin}}(t), \tag{7.12}$$

where $\Theta_0(t)$ and $\Theta_{\text{spin}}(t)$ are the heat traces of the operators L_0 and L_{spin}.

Since both operators L_0 and L_{spin} are of Laplace type we can use the results for the heat trace described above. We obtain the total heat trace

$$\begin{aligned}\Theta_{\text{matter}}(t) &= (4\pi t)^{-n/2}\int_M dx\, g^{1/2}\ \exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H\right)t\right\}\\ &\times \int_{\mathbb{R}^n_{\text{reg}}} \frac{d\omega}{(4\pi t)^{p/2}}\ \beta^{1/2}\exp\left\{-\frac{1}{4t}\langle\omega,\beta\omega\rangle\right\}\Psi_{\text{matter}}(t,\omega)\\ &\times\left[\det{}_{\mathcal{H}}\left(\frac{\sinh[F(\omega)/2]}{F(\omega)/2}\right)\right]^{1/2}\left[\det{}_{TM}\left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2}\right)\right]^{-1/2},\end{aligned} \tag{7.13}$$

where

$$\Psi_{\text{matter}}(t,\omega) = \Psi_0(t,\omega) - \Psi_{\text{spin}}(t,\omega). \tag{7.14}$$

Thus all we have to do now is to compute the functions $\Psi_0(t,\omega)$ and Ψ_{spin} (t,ω) for the operators L_0 and L_{spin}.

We assume that the gauge algebra is big enough to include the holonomy algebra as a subalgebra (as discussed above). Further, we assume that $\mathcal{B}_{ab}$ takes values in the (Abelian) Cartan subalgebra of the gauge algebra and $\mathcal{E}_{ab}$ takes values in the corresponding repesentation of the holonomy algebra. More precisely, we define two representations of the orthogonal algebra $Y_0 : \mathcal{SO}(n) \to \mathrm{End}\,(W_0)$ and $Y_{\rm spin} : \mathcal{SO}(n) \to \mathrm{End}\,(W_{\rm spin})$ with generators Y^0_{ab} and $Y^{\rm spin}_{ab}$ so that the total Yang–Mills curvature in the representations X_0 and $X_{\rm spin}$ is given by (6.11)

$$X_0(\mathcal{F}_{ab}) = \frac{1}{2} R^{cd}{}_{ab} Y^0_{cd} + X_0(\mathcal{B}_{ab}), \tag{7.15}$$

$$X_{\rm spin}(\mathcal{F}_{ab}) = \frac{1}{2} R^{cd}{}_{ab} Y^{\rm spin}_{cd} + X_{\rm spin}(\mathcal{B}_{ab}). \tag{7.16}$$

Now, for the scalar operator L_0 we have $\Sigma^0_{ab} = 0$, and, therefore,

$$\mathcal{R}^0_i = -\frac{1}{2} D^a{}_{ib} Y_0{}^b{}_a, \tag{7.17}$$

$$\mathcal{R}^2_0 = \frac{1}{4} R^{abcd} Y^0_{ab} Y^0_{cd}. \tag{7.18}$$

Thus, we obtain

$$\begin{aligned}\Psi_0(t,\omega) = \mathrm{tr}\,_{W_0} &\left[\det{}_{TM} \left(\frac{\sinh(tX_0(\mathcal{B}))}{tX_0(\mathcal{B})} \right) \right]^{-1/2} \\ &\times \exp\left[-\left(\frac{1}{4} R^{abcd} Y^0_{ab} Y^0_{cd} + Q_0 \right) t \right] \exp\left[-\frac{1}{2} D^a{}_{ib} Y_0{}^b{}_a \omega^i \right].\end{aligned} \tag{7.19}$$

For the spinor operator $L_{\rm spin}$ we have

$$\begin{aligned} Q_{\rm spin} &= \frac{1}{4} R - \frac{1}{2} \gamma^{ab} X_{\rm spin}(\mathcal{F}_{ab}) + M^2 \\ &= \frac{1}{4} R - \frac{1}{4} R^{abcd} \gamma_{ab} Y^{\rm spin}_{cd} - \frac{1}{2} \gamma^{ab} X_{\rm spin}(\mathcal{B}_{ab}) + M^2. \end{aligned} \tag{7.20}$$

The generators of the orthogonal algebra in the spinor representation are

$$\Sigma^{\rm spin}_{ab} = \frac{1}{2} \gamma_{ab}. \tag{7.21}$$

Therefore,

$$\mathcal{R}^{\rm spin}_i = -\frac{1}{2} D^a{}_{ib} \left(\gamma^b{}_a + Y_{\rm spin}{}^b{}_a \right) \tag{7.22}$$

and

$$\mathcal{R}^2_{\text{spin}} = -\frac{1}{8}R + \frac{1}{4}R^{abcd}\gamma_{ab}Y^{\text{spin}}_{cd} + \frac{1}{4}R^{abcd}Y^{\text{spin}}_{ab}Y^{\text{spin}}_{cd}. \tag{7.23}$$

Thus the endomorphism $\mathcal{R}^2_{\text{spin}} + Q_{\text{spin}}$ has the form

$$\mathcal{R}^2_{\text{spin}} + Q_{\text{spin}} = \frac{1}{8}R + \frac{1}{4}R^{abcd}Y^{\text{spin}}_{ab}Y^{\text{spin}}_{cd} - \frac{1}{2}\gamma^{ab}X_{\text{spin}}(\mathcal{B}_{ab}) + M^2. \tag{7.24}$$

Finally, we obtain

$$\begin{aligned}\Psi_{\text{spin}}(t,\omega) = \exp\left(-\frac{1}{8}Rt\right) \operatorname{tr}_{W_{\text{spin}}}\left[\det{}_{TM}\left(\frac{\sinh(tX_{\text{spin}}(\mathcal{B}))}{tX_{\text{spin}}(\mathcal{B})}\right)\right]^{-1/2} \\ \times \exp\left[-\left(\frac{1}{4}R^{abcd}Y^{\text{spin}}_{ab}Y^{\text{spin}}_{cd} + M^2\right)t\right] \exp\left[-\frac{1}{2}D^a{}_{ib}\omega^i Y_{\text{spin}}{}^b{}_a\right] \\ \times \operatorname{tr}_{\Lambda_{\text{spin}}} \exp\left[-\frac{1}{2}\left(X_{\text{spin}}(\mathcal{B}^a{}_b)t + D^a{}_{ib}\omega^i\right)\gamma^b{}_a\right],\end{aligned} \tag{7.25}$$

where $\operatorname{tr}_{\Lambda_{\text{spin}}}$ indicates the trace over the spinor indices. It is interesting to note that the scalar curvature term $\exp(-\frac{1}{8}R)$ in the function $\Psi_{\text{spin}}(t,\omega)$ precisely cancels the prefactor $\exp(\frac{1}{8}R)$ in the heat trace (7.13).

8 Conclusion

In the present paper we used the results for the heat kernel on homogeneous bundles over symmetric spaces obtained in our recent paper [12] by using sophisticated algebraic methods to evaluate the low-energy effective action in quantum gravity and gauge (Yang–Mills) theory. There always exists a minimal gauge such that both the gauge field operator and the ghost operator are of laplace type, and, therefore, the evaluation of the zeta-regularized effective action reduces to the calculation of the corresponding heat traces. Of course, one could try to go further and compute the functions $\Psi(t;\omega)$ for the relevant operators by finding the eigenvalues of the corresponding endomorphisms, etc. However, we will not do this in this paper and leave the result in the general form it was presented above.

We would like to stress two more points here. First of all, quantum general relativity is a non-renormalizable theory. Therefeore, even if one gets a final result via the zeta-regularization one should not take it too seriously. Secondly, our results for the heat kernel and, hence, for the effective action are essentially non-perturbative. They contain an infinite series of Feynmann diagrams with low momenta and cannot be obtained in any perturbation theory. One could try now to use this result for the analysis of the ground state in quantum gravity. But this is a rather ambitious program for the future.

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