

D-branes on \mathbb{C}_6^3 Part I: prepotential and GW-invariants

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Abstract

This is the first of a set of papers having the aim to provide a detailed description of brane configurations on a family of noncompact three-dimensional Calabi–Yau manifolds. The starting point is the singular manifold defined by a given quotient $\mathbb{C}^3/\mathbb{Z}_6$, which we called simply \mathbb{C}_6^3 and which admits five distinct crepant resolutions. Here we apply local mirror symmetry to partially determine the prepotential encoding the GW-invariants of the resolved varieties. It results that such prepotential provides all numbers but the ones corresponding to curves having null intersection with the compact divisor. This is realized by means of a conjecture, due to S. Hosono, so that our results provide a check confirming at least in part the conjecture.

1 Introduction

We use (local) mirror symmetry to compute the Gromov–Witten invariants¹ for a family of noncompact Calabi–Yau manifolds obtained resolving an orbifold $X = \mathbb{C}^3/\mathbb{Z}_6$. The main interest for this example is that it admit five distinct crepant resolutions all birational to X , differing by flops. Going from a resolution to another passing through the singular orbifold realizes a geometrical transition. Geometrically, the transition is obtained by moving the Kähler moduli t through an orbifold point, where the manifold becomes singular with a curve which shrinks down and reemerges as a flopped curve. As it is well known, in string theory such transition can correspond to smooth physical processes. This can be understood for example by means of a (physically equivalent) dual description using mirror symmetry. Because the orbifold was obtained quotienting by an abelian group, the resulting smooth manifold are indeed toric varieties, so that the powerful toric methods can be employed to work out all details. Mirror symmetry for noncompact CY varieties was developed quite recently in [26]. For toric varieties the mirror manifold result to be defined as the zero locus

$$Y_x = \{(\vec{u}, \vec{v}) \in \mathbb{C}^2 \times \mathbb{C}_*^2 \mid F_x(\vec{u}, \vec{v}) = 0\},$$

where x determines a point in the complex structures moduli space of the mirror, corresponding to the point t specifying the Kähler moduli of the starting manifold X_t , and $F_x(\vec{u}, \vec{v}) = u_1^2 + u_2^2 + f_x(\vec{v})$ is a certain polynomial fully determined by the toric data describing the starting orbifold. Thus varying the moduli t corresponds to varying the moduli x of the mirror manifold. However, whereas X_t undergoes a flop transition, Y_t simply changes smoothly its complex moduli.

From the mathematical point of view, the noncompactness of the variety and the particular structure of its cohomology ring introduce some ambiguities in defining the GW invariants and in their interpretation, thus requiring a deeper understanding of the geometrical structures living on a noncompact manifold. From this point of view, a consistent step forward was made by Chiang *et al.* [9], who gave an interpretation to the GW-invariants from an enumerative point of view.

From the physical point of view, mirror symmetry looks like a generalization of T-duality equivalence between different perturbative limits of the, supposed to exist, unique M-theory. However, some nonperturbative enhancements are provided by adding D-brane configurations. At a semi-classical level, D-branes are described by closed cycles (with bundles) which

¹More precisely, the lowest genus Gopakumar–Vafa invariants.

the branes are supposed to wrap on. One may wonder if any possible cycle is a good candidate as a wrapping locus. Indeed, this is actually a still open question, even if many overcomes have been made in the last decade. A consistence check must be stability, at first place perturbative stability. A first step in favour of perturbative stability is supersymmetry. This gives some strong constraint which can depend on the kind of strings to work with. For IIB strings on a CY, supersymmetric configurations are represented by holomorphic (then even-dimensional) cycles, whereas for type IIA branes one finds Lagrangian submanifolds (with respect to the Kähler form) as brane representatives, that are half-dimensional subvarieties. In the case in our interest the lasts are three-dimensional surfaces. Thus, mirror symmetry must be extended in order to take account of nonperturbative brane configurations. An astounding advance in this direction has been proposed by Kontsevich [39] who introduced the concept of homological mirror symmetry. In this case type B branes are described in terms of bounded derived categories of coherent sheaves whereas A branes are substituted by derived Fukaya categories [19]. Such a description reconciles some apparent asymmetry between A- and B-branes. Indeed, whereas A branes result to be half-dimensional, the dimensions of B-branes are heterogeneous so that to any A cycle it can correspond a B cycle of different dimension; on the other hand, Lagrangian cycles can be linearly combined to compose new Lagrangian cycles (monodromies), which, by mirror symmetry, must match with combinations of holomorphic cycles having different dimensions. In the B side, to monodromies correspond autoequivalences of derived categories. In some sense homological mirror symmetry introduce some democracy since all branes are described in terms of higher dimensional branes in an homogeneous way. From a more topological point of view, brane charges (central charges or masses) are thus described in terms of K -theory groups (even if there are many other indications for this beyond and independently from mirror symmetry, see [25, 45]). From the physical point of view, some new insight in this direction for the case of noncompact CY manifolds was done by de la Ossa *et al.* [15] who were able to select a distinguished K -theory basis for B-branes configurations adapted to support monodromy correspondence, generalizing (at least at a conjectural level) the corresponding results quite well established in the compact case.

On this side, further progress is due to Hosono [30] who found an elegant way to describe local mirror symmetry in terms of cohomology valued hypergeometric series. Mirror symmetry identifies the Kähler moduli of a CY variety with the periods of its mirror, which as functions of the complex structure moduli must satisfy a set of Picard–Fuchs equations, the Gel’fand *et al.* [22] system. It results that a particular cohomology valued hypergeometric series w arises naturally providing a basis of solutions for the GKZ system [32–35]. Hosono was able to recognize such series as a

formula identifying the BPS states of the associated physical theory, and proposed an intriguing conjecture, which we dub “*the Hosono conjecture*,” see Conjecture 6.3 in [30], which, beyond identifying the central charge of a brane configuration $F \in K^c(X_t)$ in terms of w , interprets the monodromy of the periods via a naturally associated symplectic form on $K^c(X)$. Hosono checked very carefully his conjecture for the toric quotients \mathbb{C}^2/G , and for the examples $\mathbb{C}^3/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_5$ in three complex dimensions. Among others, a consequence of the Hosono conjecture is to provide a closed formulation of a prepotential for noncompact quotients also. Indeed, at cohomological level, mirror symmetry provides a map

$$\text{mir} : K^c(X_t) \xrightarrow{\sim} H_3(Y_x, \mathbb{Z}),$$

transferring the symplectic form on $K^c(X)$ to a symplectic structure on $H_3(Y, \mathbb{Z})$. This is the noncompact analogue of the symplectic structure that, combined with Griffiths transversality, ensures the existence of a prepotential in the compact cases. However, due to noncompactness, the symplectic structure is generically degenerate. On the X side, it defines a correspondence between $H^2(X, \mathbb{Q})$ and $H^4(X, \mathbb{Q})$ which permits a complete determination of the prepotential (and correspondingly of all GW-invariants) only when it arises as a vector space isomorphism.

At homological level, mirror symmetry is conjectured to define a map

$$\text{Mir} : D^b\text{Coh}(X) \longrightarrow D \text{Fuk}^o(Y, \omega),$$

where the symplectic form ω is the Kähler form corresponding to the fixed complex structure in X . In this way, monodromies of Lagrangian on Y correspond to autoequivalences of derived categories on X described by opportune Mukay transforms which are expected to realize a (quiver) representation of the quotient group by the Mckay correspondence. This has been analysed for example by Karp [38] and Canonaco [8]. The Hosono conjecture indeed works to this higher level too, and gives some hints to get information on the mirror map Mir.

In this paper we will work at the lower level, that is at K -theoretical level, postponing the study of the higher (categorical) level mirror map to a future paper. We apply the Hosono conjecture to compute the GW-invariants for a family of noncompact toric CY varieties obtained as crepant resolutions of an orbifold quotient $\mathbb{C}^3/\mathbb{Z}_6$. We chosen this model because it has quite general properties which make it very interesting to test the conjecture. To begin with, the second and fourth Betti numbers are $b_2 = 4$ and $b_4 = 1$, so that the symplectic structure result to be highly degenerate. Thus it defines a quite poor correspondence between $H^2(X, \mathbb{Q})$ and $H^4(X, \mathbb{Q})$. Nevertheless, we

will see that the Hosono procedure permits to define a partial prepotential containing a lot of information about local geometry. Indeed, from it we are able to read out almost all GW-invariants, leaving out only a three dimensional subcone of the four-dimensional Mori cone.² Indeed, it was proposed by Forbes and Jinzenji [17, 18], a possible way to extend the GKZ system obtaining a complete determination of all GW-invariants. To such extension we will devote a future paper. Here, we will only discuss the possible origin for the ambiguity in defining the lacking GW-invariants. A second interesting peculiarity of our model, yet anticipated at the beginning, is that it admits five distinct crepant resolutions, which differ by flops. Thus one expects monodromy to relate different resolution by means of different Fourier Mukay transforms. This is indeed one of the main targets of this starting studies, but as announced we will not tackle it here. We will limit ourselves to compute the prepotentials and the computable GW-invariants for all resolutions, comparing with themselves.

Thus in some sense this first paper can be thought as a preparatory one. In this spirit we will try to be as much explicit as possible. In Section 2 we include a short overview of the main steps which lead to the introduction of local mirror symmetry to arrive to the Hosono conjecture.

In Section 3 we present a detailed analysis of the first resolution, that is the G -Hilbert resolution. We will use the Hosono's conjecture to construct the cohomological hypergeometric series generating the periods of the mirror manifold. Due to noncompactness, the structure of the cohomology ring does not consent a full definition of the GW-invariants. However, as we will see, the procedure proposed by Hosono permits to equally define a prepotential which generates all GW-invariants associated to the curves in the Mori cone, excluding a codimension one subcone.

In Section 4 we repeat the previous analysis for all the other resolutions, deriving a partial determination of the GW-invariants for all of them.

The results will be commented in Section 5.

2 Local mirror symmetry and the Hosono conjecture

Here we will recall some main step leading to the conjectures we are testing in this and following papers. The literature on the subject is quite huge, so that we will mainly refer to [27] and references therein.

²We are grateful to Professor S. Hosono to give us explanations on this point.

2.1 Dualities and mirror symmetry

Let us consider a string theory having a toric Calabi–Yau variety X as target space. Thus, there is a nice interpretation of mirror symmetry as a T-duality transformation. Indeed, string theory on X can be described in terms of a two-dimensional $U(1)^m$ supersymmetric gauge theory, the so-called “gauged linear sigma model” (see [27, Sections 7.3, 7.4]). It contains a certain number $n > m$ of complex scalar fields $Z = \{Z_\alpha\}_{\alpha=1}^n$ having charges $Q_{\alpha,r}$ $r = 1, 2, \dots, m$ with respect to the gauge group $U(1)^m$, and with potential energy

$$U(Z) = \frac{1}{2} \sum_{r=1}^m g_r^2 \left(\sum_{\alpha=1}^n Q_{\alpha,r} Z_\alpha \bar{Z}_\alpha - r_r \right)^2.$$

Here g_r and r_r are the gauge couplings and the Fayet–Iliopoulos terms, respectively. Supersymmetric ground states require the vanishing of the potential energy:

$$\sum_{\alpha=1}^n Q_{\alpha,r} Z_\alpha \bar{Z}_\alpha = r_r.$$

For a fixed choice of the $F - I$ parameters, these equations define a toric variety X associated to a fan, in an $(n - m)$ -dimensional lattice N , generated by an opportune set $\Sigma(1) = v_1, \dots, v_n$ of vectors in N . From this it is possible to conclude that the supersymmetric vacua are identified with the points of a toric variety X . Each vector v_α determines an invariant divisor,³ D_{v_α} . It is not hard to show (see [27, Section 7.4]) that one can chose a basis $\{C_r\}_{r=1}^m$ of irreducible curves of $H_2(X, \mathbb{Z})$ (which indeed result to be m -dimensional) such that the charges are given by the intersection numbers $Q_{\alpha,r} = D_{v_\alpha} \cdot C_r$. Also note that the $F - I$ parameters rescale as $|Z|^2$ so that, if chosen to be positive, they indeed parameterize the points of the Kähler cone. This means that the supersymmetric configurations are completely characterized in geometrical terms.

At this point mirror symmetry can be realized as a T-duality transformation [27, Section 20]. Indeed, recall that roughly speaking T-duality on a circle transforms a type A string theory on a circle of radius R in a type B string theory on a circle of radius α'/R . If Z_α are taking value on a complex variety (indeed the toric variety in the vacuum configuration) then we can T-dualize their phases which define circles in the target manifold. The

³T.i. invariant under the toric action.

result [27, Section 13] is a Landau–Ginzburg theory with superpotential

$$W(Y, t) = \sum_{\alpha=1}^n e^{-Y_\alpha},$$

for a set of chiral superfields related by the set of constraints

$$\sum_{\alpha=1}^n Q_{\alpha,r} Y_\alpha = t_r,$$

where t_r are the complexified Kähler parameters ($\text{Re}(t_r) = r_r$). In this way, the mirror transformation applied to the two-dimensional sigma model gives rise to a Landau–Ginzburg model with superpotential $W(Y, t)$. To take contact with the Batyrev’s geometric construction of mirror manifolds for toric varieties, let us proceed as follows (see [26]) for the cases when the starting linear sigma model describes strings on a crepant resolution of some abelian quotient \mathbb{C}_3/G . Being crepant, it will be described by a set of vectors v_1, \dots, v_n in a three-dimensional lattice such that for some isomorphism $\phi : N \rightarrow \mathbb{Z}^3$ one has $\phi(v_\alpha) = (n_{\alpha,1}, n_{\alpha,2}, 1)$. The solutions of the constraints can thus be written in terms of three independent fields y_0, y_1, y_2 as $Y_\alpha = y_0 + n_{\alpha,1}y_1 + n_{\alpha,2}y_2 + c_\alpha$, where c_α are some constant satisfying $\sum_{\alpha=1}^n Q_{\alpha,r} Y_\alpha = t_r$. These linear redefinitions do not affect the functional measure, and setting $w_a = \exp(-y_a)$, $a = 0, 1, 2$ and $a_\alpha = \exp(-c_\alpha)$ we get for the superpotential

$$W(w, a) = w_0 \sum_{\alpha=1}^n a_\alpha w_1^{n_{\alpha,1}} w_2^{n_{\alpha,2}}, \quad w_a \in \mathbb{C}_*.$$

As discussed in [26], we can note that, for what concerns the BPS configurations, this LG model is equivalent to another one, where $w_0 \in \mathbb{C}$ and with two extra chiral fields $U, V \in \mathbb{C}$, whose superpotential is

$$\tilde{W}(U, V; w; a) = W(w, a) - w_0 UV.$$

Integrating the field w_0 thus gives a delta function $\delta(\sum_{\alpha=1}^n a_\alpha w_1^{n_{\alpha,1}} w_2^{n_{\alpha,2}} - UV)$ so that the mirror LG model is equivalent to a geometrical theory on a Calabi–Yau manifold

$$Y_a = \{(\vec{u}, \vec{w}) \in \mathbb{C}^2 \times \mathbb{C}_*^2 \mid F_a(\vec{u}, \vec{w}) = 0\},$$

where

$$F_a(\vec{u}, \vec{w}) = u_1^2 + u_2^2 + f_a(\vec{u}, \vec{w}) = u_1^2 + u_2^2 + \sum_{\alpha=1}^n a_\alpha w_1^{n_{\alpha,1}} w_2^{n_{\alpha,2}}.$$

The Kähler parameters t now parameterize the complex moduli of Y . This is indeed local mirror symmetry as discovered for the first time at physical level in [41, 42].

2.2 Branes and homological mirror symmetry

The intuitive picture described above does not take into account the presence of brane configurations. Because we are looking for supersymmetric vacua, we need to know what kind of brane configurations are admitted on a Calabi–Yau manifold X . In other words, one must search for boundary condition compatible with supersymmetry. This is described for example in [26, Section 3]. The answer depends on the type of string theory one considers. For type A strings, supersymmetric branes are represented (at classical level) by halfdimensional subvarieties S , $\iota : S \hookrightarrow X$, where the Kähler form ω of the C–Y manifold vanishes, $\iota^*\omega = 0$, and supporting flat vector bundles. Thus A-branes are Lagrangian submanifolds with respect to the symplectic structure ω . For type B strings one finds that supersymmetric branes must wrap holomorphic cycles of X supporting holomorphic vector bundles. In our models it means that type B-brane configurations will be described classically by compact divisors, curves of the Mori cone and points. Thus mirror symmetry should map BPS states of a model into the BPS states of the mirror model, converting A-branes to B-branes and vice versa. However, there is an odd asymmetry between A and B configurations: indeed all A-branes have the same dimensions, whereas this does not happen for B-branes. Now, the point is that in the LG model description branes configurations can change when moduli vary. In this picture, BPS states will correspond to critical points of the superpotential. Essentially, they determine the points of Y_t around which the supersymmetric three cycles are defined. Varying t , the critical points move on the W -plane; when some of these points moves around a branch point, a monodromy transformation can give rise to a new brane configuration [26]. The boundary states corresponding to the branes are described by the periods of the holomorphic three-form Ω of Y_t (in the geometric picture). The monodromy thus acts on a basis of cycles recasting them in some linear recombination or equivalently on the periods in the same linear recombination. On the mirror X it should correspond to a recombination of the holomorphic cycles, hard to understand in the naïve geometrical picture where they have different dimensions.

To solve this point a first aid comes from a K -theoretical description, where lower dimensional branes can be described in terms of the top dimensional branes and a tachyon field [45]. K -theory mainly captures topological

aspects of the problem, carrying important information on the admissible brane configurations, but it is quite poor from the geometrical point of view. In [16] it was argued that a deeper geometrical understanding of (stable) brane configuration in (topological) type B superstring can be understood in terms of triangulated categories, in particular the derived category of coherent sheaves on the manifold (see also [1], or [4] for a more mathematical point of view). This provided a deep contact between physics and the “homological mirror symmetry” conjectured by Kontsevich [39] who proposed that the usual geometrical mirror symmetry should enhance to homological level as an equivalence between triangulated categories: the derived category of coherent sheaves on a CY manifold X with a fixed complex structure on one side⁴ and the derived \mathcal{A}^∞ Fukaya’s category over the mirror manifold Y on the other side, essentially generated by the Lagrangian submanifolds of $\{Y, \omega\}$, where the symplectic structure ω is given by the fixed Kähler form on Y , dual to the complex form on X :

$$\text{Mir} : D^b\text{Coh}(X) \xrightarrow{\cong} D \text{Fuk}^o(Y, \omega).$$

2.3 The Hosono conjecture

As we said, BPS states in the mirror type A string model are described by periods that are integrals of the holomorphic three form Ω on Y over the Lagrangian cycles. For the noncompact quotients we are describing, the holomorphic three form on the mirror Y is

$$\Omega = \frac{1}{4\pi^3} \text{Res}_{F=0} \left[\frac{du^1 \wedge du^2 \wedge dw^1 \wedge dw^2}{w^1 w^2 F(\vec{u}; \vec{w}; a)} \right].$$

Here we have fixed the Kähler form, however Ω depends explicitly on the complex moduli of Y (as shown by the explicit dependence on a of the polynomial F) so that the periods

$$\Pi_{C_i}(a) = \int_{C_i} \Omega,$$

of any set of Lagrangian cycles C_i , will be locally holomorphic functions of the moduli. Indeed, they are forced to satisfy a set of hypergeometric differential equations known as the GKZ hypergeometric system, largely studied in [22].

⁴For clarity we confine ourselves to the case of Calabi–Yau varieties.

For compact varieties the knowledge of a complete set of solutions for the GKZ system correspond to an exhaustive description of the set of BPS brane configurations on the A side. Furthermore, the special Kähler geometry of the complex structure moduli space of a C–Y manifold can be described in terms of periods [44]. If x parameterizes the structure complex moduli of Y then the Kähler potential of the moduli space can be written as

$$K(x, \bar{x}) = -\log \left[i \sum_{I=0}^{h^{2,1}(Y)} \left(X^I \frac{\partial \bar{G}}{\partial \bar{X}^I} - \bar{X}^I \frac{\partial G}{\partial X^I} \right) \right],$$

where

$$X^I(x) = \int_A^I \Omega(x)$$

are the periods with respect to a canonical symplectic basis $\{A^I, B_I\}$ of $H_3(Y, \mathbb{Z})$. Finally $G(x)$ is the prepotential

$$G(x) = \frac{1}{2} \sum_{I=0}^{h^{1,2}(Y)} \int_{A^I} \Omega \int_{B_I} \Omega.$$

Mirror symmetry gives a correspondence between Kähler moduli t_i of X and complex moduli of Y so that

$$t_i = \frac{X^i}{X^0}, \quad i = 1, \dots, h^{1,2}(Y) = h^{1,1}(X).$$

On the other side, also the Kähler moduli space of X is a special Kähler manifold which can thus be described in terms of a prepotential function $F(t)$. At classical level such geometry is described by the prepotential

$$F^c(t) = \frac{1}{6} d_{ijk} t^i t^j t^k,$$

where $d_{ijk} = J_i \cdot J_j \cdot J_k$ are the intersection numbers of the Kähler cone generators. Physically, they determine the Yukawa couplings of the chiral fields [7]. However these couplings receive quantum corrections which come from worldsheet instantons. At lowest order they corresponds to wrapping of the worldsheet on rational curves in X . The energy of such a wrapping is given by the volume of the wrapped cycle as measured by the Kähler metric. Any given (class of) rational curve of degree \vec{d} results to contribute to the

prepotential with a term

$$n_{\vec{d}} Li_3(e^{2\pi i \vec{d} \cdot t}),$$

$n_{\vec{d}}$ being the number of classes of curves with the given degree, so that it can be shown that the quantum corrected prepotential takes the form [9]

$$F(t) = \frac{1}{6} d_{ijk} t^i t^j t^k - \frac{1}{24} c_2(X) \cdot J_i t^i - i \frac{\zeta(3)}{16\pi^3} c_3(X) + \sum_{d \in \mathbb{Z}_{>}^{h^{1,1}}} n_{\vec{d}} Li_3(e^{2\pi i \vec{d} \cdot t}).$$

More precisely $n_{\vec{d}}$ are the Gromov–Witten invariant (in the Gopakumar–Vafa [23, 24] interpretation). See [9] for a mathematical enumerative interpretation. By means of the identification (making use of the Griffith transversality, [9, 44])

$$\left\{ \int_{A^I} \Omega; \int_{B_I} \Omega \right\}_{I=0}^{h^{2,1}(Y)} = \left\{ 1, t^i; \partial_{t^i} F, 2F - \sum_{j=1}^{h^{1,1}(X)} t^j \partial_{t^i} F \right\}_{i=1}^{h^{1,1}(X)}$$

mirror symmetry thus gives a simple way to compute the *GW*-invariants of X .

In a series of papers (see for example [32–35]) it was provided an efficient strategy to characterize a complete set of the GKZ system for a C–Y hypersurface, which is summarized in [29]. In particular, there was introduced a cohomological valued power series whose expansion in the Chow ring

$$A^*(X) \otimes \mathbb{C}[[x]][\log x]$$

gives a basis for the period integrals of the mirror manifold Y in the large complex structure limit (LCSL) (see [29, Claim 5.11]). Thus the cohomological series encodes many geometrical information on both the manifolds X and Y so summarizing several fundamental aspects of mirror symmetry.

In [30, 31] Hosono extended this picture to local mirror symmetry for noncompact C–Y manifolds, in particular for resolutions of abelian quotients \mathbb{C}^k/G , with $k = 2, 3$. For convenience we will state the conjecture in Section 3.6. In [30], Hosono verified his conjecture carefully for the case $k = 2$ and reported the analysis for the cases $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$, where it was shown the existence of a prepotential for the noncompact cases also.

Here we use Hosono conjecture to analyze the geometry of a quotient $X = \mathbb{C}^3/\mathbb{Z}_6$, which we call for simplicity \mathbb{C}_6^3 . As stated in the introduction, this singular orbifold admits five distinct crepant resolutions. All these resolutions are related by flop transformations. To noncompactness of the

manifold X it corresponds the ambiguity in defining the GW-invariants. In our model this reflects in the fact that the symplectic structure on the half dimensional homology of the mirror Y is degenerate. On the mirror, such structure should determine a pairing between two-dimensional and four-dimensional cohomology, permitting the reconstruction of the prepotential, but which now becomes degenerate. We determine the LCSL cohomological series for all the resolutions. From each of them, using Hosono's prescriptions we will be able to partially determine a prepotential which codifies all the GW-invariants of the (four-dimensional) Mori cone excluding a three-dimensional subcone.

3 The tri-dimensional orbifold \mathbb{C}_6^3 and the G -Hilb resolution

3.1 Definition of \mathbb{C}_6^3

We briefly review the homogeneous coordinates construction of toric varieties [10]. The data of a d -dimensional toric variety $X(\Delta)$ can always be specified in terms of a fan Δ in a lattice N isomorphic to \mathbb{Z}^d . Let ρ_1, \dots, ρ_r be the one-dimensional cones of Δ and let $v_i \in \mathbb{Z}^n$ denote the primitive element of ρ_i , i.e., the generator of $\rho_i \cap \mathbb{Z}^n$. Then introduce variables x_i for $i = 1, \dots, r$ in the affine complex space \mathbb{C}^r . The homogeneous coordinates construction represents $X(\Delta)$ as the quotient

$$X(\Delta) = (\mathbb{C}^r \setminus Z)/G$$

for a certain variety Z and some abelian group $G \subset (C^*)^r$.

Z is determined as follows. We say that a set of edge generators $I = \{v_{i_1}, \dots, v_{i_s}\}$ is primitive if they do not lie in any cone of Δ but every proper subset does. Then

$$Z = \bigcup_{I \text{ primitive}} \{x_{i_1} = 0, \dots, x_{i_s} = 0\}.$$

If $\{e_1, \dots, e_d\}$ is the standard basis of the dual lattice M and $\langle, \rangle: M \times N \rightarrow \mathbb{Z}$ is the natural pairing, the group G is defined as the kernel of the following homomorphism:

$$\Phi: (\mathbb{C}^*)^r \rightarrow (\mathbb{C}^*)^d, \quad (\lambda_1, \dots, \lambda_r) \mapsto \left(\prod_{i=1}^r \lambda_i^{\langle e_1, v_i \rangle}, \dots, \prod_{i=1}^r \lambda_i^{\langle e_n, v_i \rangle} \right)$$

and its actions on $\mathbb{C}^r \setminus Z$ is by multiplication

$$(\lambda_1, \dots, \lambda_r) \cdot (x_1, \dots, x_r) := (\lambda_1 x_1, \dots, \lambda_r x_r).$$

In this paper, we study the three-dimensional orbifold \mathbb{C}_6^3 defined as the toric variety associated to the fan generated by the vectors:

$$v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad (3.1)$$

in $N \simeq \mathbb{Z}^3$. In this case $Z = \emptyset$ and the associated homomorphism is

$$\Phi : (\mathbb{C}^*)^3 \rightarrow (\mathbb{C}^*)^3, \quad (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1^{-1} \lambda_2^2 \lambda_3^{-1}, \lambda_1^{-1} \lambda_2^{-1} \lambda_3, \lambda_1 \lambda_2 \lambda_3), \quad (3.2)$$

which has kernel

$$G := \ker \Phi = \langle (\epsilon, \epsilon^2, \epsilon^3) \rangle \subset (\mathbb{C}^*)^3, \quad \text{with } \epsilon = e^{\frac{2\pi i}{6}}. \quad (3.3)$$

Thus $G \simeq \mathbb{Z}_6$ and $\mathbb{C}_6^3 = \mathbb{C}^3 / \mathbb{Z}_6$ where the action on the coordinates is

$$\epsilon \cdot (x_1, x_2, x_3) = (\epsilon x_1, \epsilon^2 x_2, \epsilon^3 x_3). \quad (3.4)$$

\mathbb{C}_6^3 is a noncompact Calabi–Yau ($K_{\mathbb{C}_6^3}$ is trivial) three-fold with non-isolated singularities, because all vectors v_i lie in the plane $z = 1$ (if (x, y, z) are the coordinates on the lattice).⁵ In this way, all relevant information is included in the two-dimensional intersection of the fan Δ with the plane $z = 1$. In figure 1 we have drawn this section for the fan of \mathbb{C}_6^3 .

3.2 Crepant resolutions of \mathbb{C}_6^3

A crepant resolution of a variety X is a smooth variety Y together with a proper birational morphism $\tau : Y \rightarrow X$ such that $K_Y = \tau^* K_X$. If X is a Calabi–Yau variety this means that K_Y has to be trivial. Any crepant resolution of a toric Calabi–Yau orbifold $X(\Delta) = \mathbb{C}^3 / G$ can be obtained in two simple steps (see [21, 40]). First, add to Δ all possible edges ρ_i that are generated by the integer vectors $v_i \in N$ intersecting the fan and lying on the plane determined by v_1, v_2, v_3 . Next, let one completely triangulate Δ , to obtain the regular fan Δ' of the toric resolution $X(\Delta')$. If there exist

⁵We refer to Section 3.3.1 for an explanation about this CY condition.

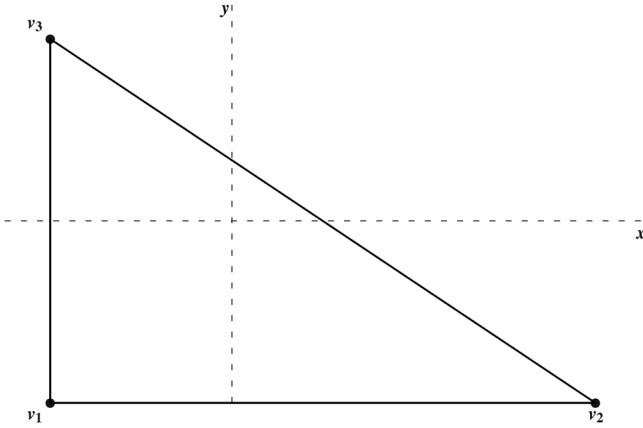


Figure 1: \mathbb{C}_6^3 fan.

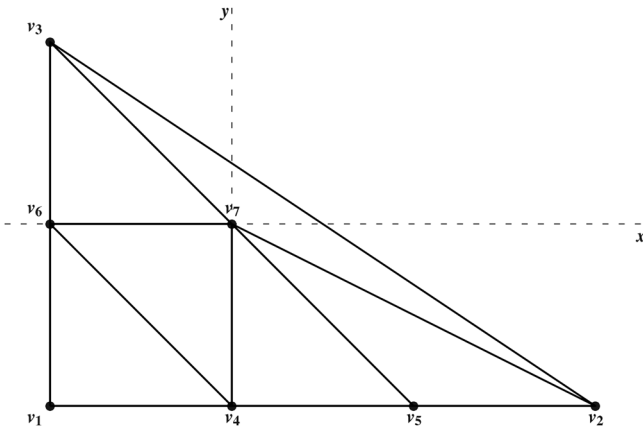


Figure 2: Fan of the G -Hilbert resolution of \mathbb{C}_6^3 .

several complete triangulations this means that the orbifold admits multiple crepant resolutions, all related by flops of curves.

Therefore, to obtain the resolutions of the \mathbb{C}_6^3 singular variety we add to Δ the four vectors

$$v_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_6 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad v_7 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.5)$$

It is easy to show that we have five admissible complete triangulations.

3.2.1 Toric G -Hilbert resolution

We start considering the G -Hilbert resolution (figure 2), which we call $G\text{-}\mathbb{C}_6^3$. Its general toric construction is given in [13] and we refer to it for a detailed explanation. We can think to G -Hilb fan as the “more symmetric” triangulation. We try to illustrate this concept in our case. First, we add to Δ the two-dimensional cones generated by (v_2, v_7) and (v_3, v_7) , that are necessary to obtain any complete triangulation. Then we extend the line (v_3, v_7) to v_5 so obtaining a subdivision of the fan into regular triangles, three of them with edges of length one and the bigger one with edges of length two. Finally, we complete the triangulation subdividing this last triangle with a regular tessellation, obtained by drawing all possible internal lines parallel to its edges.

3.2.2 G -Hilbert resolution as the moduli space of G -clusters of \mathbb{C}^3

Given an algebraic variety M and a finite group G with an action on M , the $G\text{-Hilb}(M)$ is defined as the moduli space of G -clusters $Z \subset M$. A G -cluster is a G -invariant zero-dimensional subscheme Z , with defining ideal $\mathcal{I}_Z \subset \mathcal{O}_M$ and structure sheaf $\mathcal{O}_Z = \mathcal{O}_M/\mathcal{I}_Z$ isomorphic to the regular representation of G , i.e., $H^0(Z, \mathcal{O}_Z) \simeq R(G)$ with $\dim H^0(Z, \mathcal{O}_Z) = |G|$. The simplest example of G -cluster is a general orbit of G consisting of N distinct point.

We will study the simple example of $\mathbb{Z}_2\text{-Hilb}(\mathbb{C}^2)$. Let us consider $\mathbb{C}^2 = \text{Spec } \mathbb{C}[X, Y]$ and the action of \mathbb{Z}_2 , with generator $\epsilon = -1$, defined on the coordinates as

$$\epsilon \cdot (X, Y) = (\epsilon X, \epsilon Y). \tag{3.6}$$

The orbits of \mathbb{Z}_2 are the sets of couple of points

$$\{(p_1, p_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid X(p_1) = -X(p_2), Y(p_1) = -Y(p_2)\}. \tag{3.7}$$

If p_1 has coordinates (a, b) , on the open set $X \neq 0$ the \mathbb{Z}_2 -cluster Z with support over (p_1, p_2) is defined by the equations

$$X^2 = a, \quad Y = \frac{b}{a}X \implies \mathcal{O}_Z = \frac{\mathbb{C}[X, Y]}{(X^2 - a, Y - \frac{b}{a}X)} \simeq \mathbb{C} \oplus \mathbb{C} \cdot X, \tag{3.8}$$

and on the open set $Y \neq 0$ by

$$Y^2 = b, \quad X = \frac{a}{b}Y \implies \mathcal{O}_Z = \frac{\mathbb{C}[X, Y]}{(Y^2 - b, X - \frac{a}{b}Y)} \simeq \mathbb{C} \oplus \mathbb{C} \cdot Y. \tag{3.9}$$

It is easy to verify all the properties of the \mathbb{Z}_2 -clusters. Thus we have a bijective relation between generic orbits and \mathbb{Z}_2 -cluster having support on

them. On the set $\{X = 0, Y = 0\}$ we have \mathbb{Z}_2 -clusters Z of type

$$X^2 = 0, \quad X = \frac{\beta}{\alpha} Y \implies \mathcal{O}_Z = \frac{\mathbb{C}[X, Y]}{(X^2, X - \frac{\beta}{\alpha} Y)} \simeq \mathbb{C} \oplus \mathbb{C} \cdot X, \quad (3.10)$$

for any (α, β) with $\alpha \neq 0$, or, in alternative, of type

$$Y^2 = 0, \quad Y = \frac{\alpha}{\beta} X \implies \mathcal{O}_Z = \frac{\mathbb{C}[X, Y]}{(Y^2, Y - \frac{\alpha}{\beta} X)} \simeq \mathbb{C} \oplus \mathbb{C} \cdot Y, \quad (3.11)$$

for any (α, β) with $\beta \neq 0$. It is evident that the \mathbb{Z}_2 -Hilb(\mathbb{C}^2) has the structure of the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin and, with the map

$$\tau : \mathbb{Z}_2\text{-Hilb}(\mathbb{C}^2) \longrightarrow \mathbb{C}^2/G \quad Z(p_1, p_2) \longmapsto (p_1, p_2), \quad (3.12)$$

it becomes the (crepant) resolution of the orbifold. Let us prove this fact explicitly using toric geometry.

We will follow the construction of toric orbifold given in [21]. Let $L = \mathbb{Z}^2 + \frac{1}{2}(1, 1)$ be the lattice over \mathbb{Z}^2 ; in L the fan of $\mathbb{C}^2/\mathbb{Z}_2$ is the junior simplex Δ_{junior} generated by the standard base (e_1, e_2) of \mathbb{Z}_2 .

Using the “old construction” of toric variety [21], we have

$$X_{\Delta_{\text{junior}}} = \text{Spec } \mathbb{C}[X^2, XY, Y^2] = \text{Spec } \frac{\mathbb{C}[U, V, W]}{(UW - V^2)}. \quad (3.13)$$

The toric resolution of $X_{\Delta_{\text{junior}}}$ is obtained adding to Δ_{junior} the edge generated by $\frac{1}{2}(e_1 + e_2)$. In the right side of figure 3 we have drawn the toric fan of the resolution marked with the coordinates related to the toric curves, expressed as \mathbb{Z}_2 -invariant ratios of monomials in the orbifold coordinates.

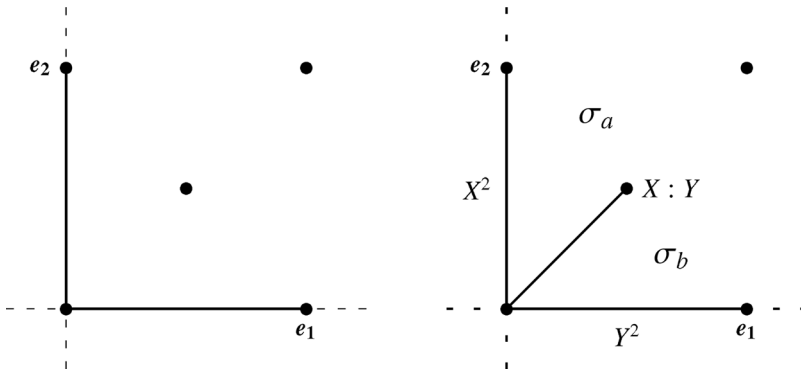


Figure 3: Fan for $\mathbb{C}^2/\mathbb{Z}_2$ and \mathbb{Z}_2 -Hilb(\mathbb{C}^2) in $L = \mathbb{Z}^2 + \frac{1}{2}(1, 1)$.

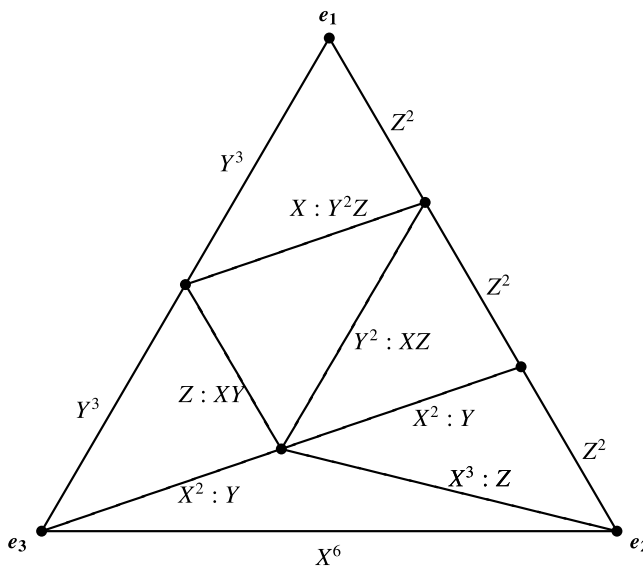


Figure 4: Fan for $\mathbb{Z}_6\text{-Hilb}(\mathbb{C}^3)$ in lattice $L = \mathbb{Z}^3 + \frac{1}{6}(1, 2, 3)$.

Geometrically, this is the blow up of $X_{\Delta_{\text{junior}}} = \mathbb{C}^2/\mathbb{Z}_2$ in the origin. The two affine open sets are

$$U_{\sigma_a} = \text{Spec } \mathbb{C}[X^2, Y/X], \quad U_{\sigma_b} = \text{Spec } \mathbb{C}[Y^2, X/Y]. \tag{3.14}$$

Thus U_{σ_a} , for example, parameterizes equations of the form

$$X^2 = \xi_a, \quad Y = \eta_a X, \tag{3.15}$$

which define the \mathbb{Z}_2 -clusters (3.8). Similar U_{σ_b} parameterizes clusters (3.9) and their intersection $U_{\sigma_a \cap \sigma_b}$ the clusters (3.10, 3.11). Therefore, the crepant toric resolution of $\mathbb{C}^2/\mathbb{Z}_2$ is exactly $\mathbb{Z}_2\text{-Hilb}(\mathbb{C}^2)$.

In a similar way, it has been proved in [13,36] that the toric resolutions of \mathbb{C}^3/G defined in the previous section (for $G \subset SL(3, \mathbb{C})$ abelian) are exactly the $G\text{-Hilb}(\mathbb{C}^3)$. In figure 4 we report the $\mathbb{Z}_6\text{-Hilb}(\mathbb{C}^3)$ fan marked with the \mathbb{Z}_6 -invariant ratios associated to the curves of the resolution.

3.3 Intersection theory for $G\text{-}\mathbb{C}_6^3$

We are interested in finding the Chow ring $A^*(G\text{-}\mathbb{C}_6^3)$, the module $A_*^c(G\text{-}\mathbb{C}_6^3)$ and the intersection pairing $A^*(G\text{-}\mathbb{C}_6^3) \otimes A_*^c(G\text{-}\mathbb{C}_6^3) \rightarrow A_*^c(G\text{-}\mathbb{C}_6^3)$ (for this section we refer to [20, 21, 40]).

3.3.1 Chow ring $A^*(G-\mathbb{C}_6^3)$

On any variety X the Chow group $A_k(X)$ is defined to be the free abelian group on the k -dimensional irreducible closed subvarieties of X , modulo the subgroup generated by the cycles of the form (f) , where f is a nonzero rational function on a $(k + 1)$ -dimensional subvariety of X .⁶ For a toric variety $X = X(\Delta)$, the Chow group $A_k(X)$ is generated by the classes of the closures $V(\sigma) = \overline{O(\sigma)}$ of orbits of the $(n - k)$ -dimensional cones $\sigma \in \Delta$ under the action of the torus \mathbb{C}_*^n . If τ is a cone of Δ , we define $N_\tau := \mathbb{Z} \cdot \tau$ and $N(\tau) := N/N_\tau$. The relations in $A_k(X)$ are generated by the cycles of the form $(\chi^u) := \sum_i \langle u, v_i \rangle V(\rho_i)$, where u is an element in the dual lattice $M(\tau) = N(\tau)^*$, ρ_i are the one-dimensional subcones of the projection of τ in $N(\tau)$ with primitive vectors v_i , $\langle \cdot, \cdot \rangle$ is the natural pairing between $M(\tau)$ and $N(\tau)$, for any cone $\tau \in \Delta$ of dimension $n - k - 1$.

We will study explicitly this construction for $X = G-\mathbb{C}_6^3$.

Let us first decorate the fan in figure 5 with labels for the toric invariant subvarieties related to the cones of Δ :

$A_3(X)$ has only one generator, corresponding to the unique zero-dimensional cone of Δ , and obviously without relations.

$$A_3(X) = \mathbb{Z} \cdot X, \quad X = V(0) \tag{3.16}$$

$A_2(X)$ has seven generators, related to the seven one-dimensional cones of the fan. The relations are generated by the cycles (χ^u) for u in $M(0) = M$:

$$A_2(X) = \frac{\bigoplus_{i=1}^7 \mathbb{Z} \cdot D_i}{\langle \chi^u \rangle}, \quad D_i = V(\rho_i), \quad \rho_i = \mathbb{R}_{\geq 0} \cdot v_i. \tag{3.17}$$

We choose as u the standard basis of the lattice M , e_1^*, e_2^*, e_3^* , so we obtain these three independent relations

$$-D_1 + 2D_2 - D_3 + D_5 - D_6 = 0, \tag{3.18}$$

$$-D_1 - D_2 + D_3 - D_4 - D_5 = 0, \tag{3.19}$$

$$D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 = 0. \tag{3.20}$$

⁶Recall that (f) is the cycle obtained as the sum of the zeros of f minus the poles of f , each counted with its multiplicity.

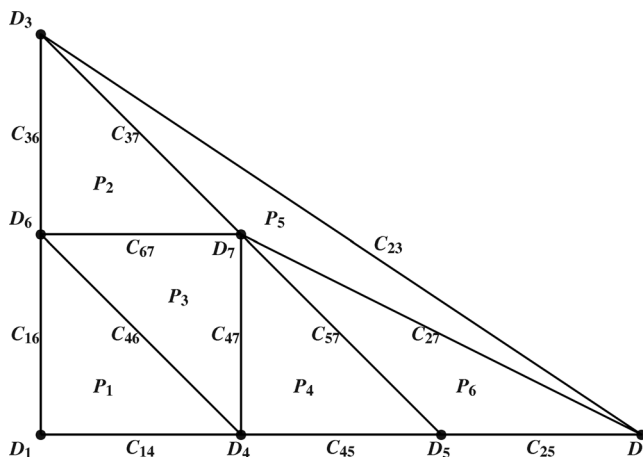


Figure 5: The toric invariant subvarieties of $\mathbb{Z}_6 - \text{Hilb}(\mathbb{C}^3)$.

In $A_2(X)$ the divisors D_5, D_6, D_7 can be expressed in terms of the others

$$D_5 = -D_1 - D_2 + D_3 - D_4, \tag{3.21}$$

$$D_6 = -2D_1 + D_2 - D_4, \tag{3.22}$$

$$D_7 = 2D_1 - D_2 - 2D_3 + D_4. \tag{3.23}$$

It follows that

$$A_2(X) = \bigoplus_{i=1}^4 \mathbb{Z} \cdot D_i \simeq \mathbb{Z}^4. \tag{3.24}$$

Since the variety $G - \mathbb{C}_6^3$ is nonsingular, we have

$$\text{Pic}(G - \mathbb{C}_6^3) \simeq A_2(G - \mathbb{C}_6^3) \simeq \mathbb{Z}^4. \tag{3.25}$$

Let us make a remark about the canonical divisor. It is a standard fact in toric geometry that the canonical divisor of a variety X is given by $K_X = -\sum_i D_i$ where the sum is over all toric invariant divisors. By relation (3.20) it then follows that $K_{G - \mathbb{C}_6^3} = 0$ in $\text{Pic}(G - \mathbb{C}_6^3)$ so that $G - \mathbb{C}_6^3$ is a Calabi–Yau variety. This is true for any toric variety with all integer vectors generating the fan lying on the same (hyper)plane.

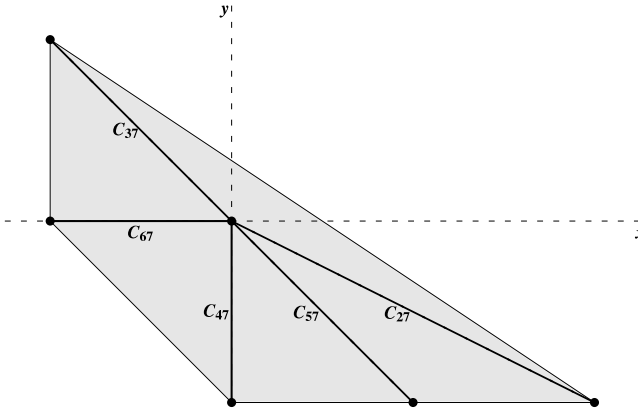


Figure 6: Fan for D_7 ($\text{Star}(\rho_7)$).

$A_1(X)$ has twelve generators, the toric invariant curves. The relations are generated by the cycles (χ^u) for u in $M(\rho_i)$:

$$A_1(X) = \frac{\bigoplus \mathbb{Z} \cdot C_{ij}}{\langle \chi^u \rangle}. \tag{3.26}$$

Therefore, to find the relations we have to study the geometry of any toric invariant divisor. Recall that $D_i = V(\rho_i)$ is a toric variety for any i ; its fan is called $\text{Star}(\rho_i)$ and is obtained by the projection of the cones containing ρ_i into the quotient lattice $N(\rho_i)$. As an example we plot $\text{Star}(\rho_7)$ in figure 6. D_7 has five toric invariant divisors and two relations between them

$$-C_{27} + C_{37} - C_{47} - C_{57} = 0, \tag{3.27}$$

$$2C_{27} - C_{37} + C_{57} - C_{67} = 0. \tag{3.28}$$

Doing the same for any divisor D_i we obtain all relations between curves. At the end we find that any two given curves are equivalent:

$$A_1(X) = \mathbb{Z} \cdot C, \quad C = [C_{46}]. \tag{3.29}$$

Any other invariant curve is related to C by the relations expressed in the decorated fan of figure 7.

$A_0(X)$ is generated by the six toric invariant points of X . Every toric variety contains only two kinds of toric curves: compact curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ and noncompact curves isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. Any compact curve gives a rational relation between two invariant points, and in such way we find that any two given points are rationally equivalent. Finally, linear equivalence on

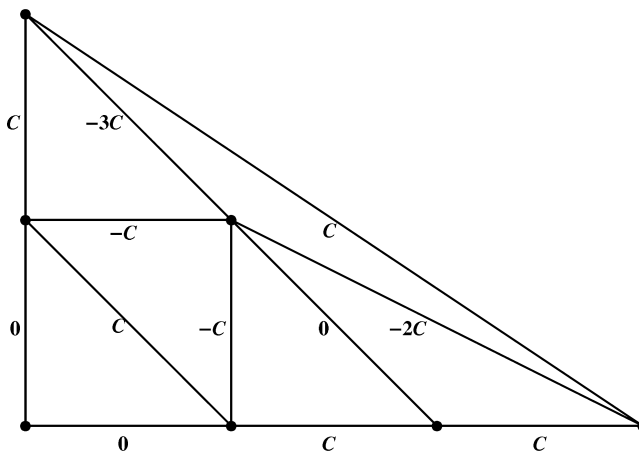


Figure 7: The rational equivalence between curves of $\mathbb{Z}_6 - Hilb(\mathbb{C}^3)$.

affine curves says us that points are rational equivalent to zero. Therefore $A_0(X)$ is the trivial group (this is true for any noncompact toric variety).

On a nonsingular n -dimensional variety X , one sets $A^p(X) := A_{n-p}(X)$. There is an intersection product $A^p(X) \times A^q(X) \rightarrow A^{p+q}(X)$, making $A^*(X) := \bigoplus A^p(X)$ into a commutative graded ring. For a general toric variety $X(\Delta)$, if σ and τ are cones in Δ , then

$$V(\sigma) \cap V(\tau) = \begin{cases} V(\gamma) & \text{if } \sigma \text{ and } \tau \text{ span the cone } \gamma, \\ \emptyset & \text{if } \sigma \text{ and } \tau \text{ do not span a cone in } \Delta. \end{cases}$$

If $X(\Delta)$ is nonsingular and the intersection is proper, i.e., each component of the intersection has codimension equal to the sum of the codimension of the two subvarieties, or empty, then $V(\sigma)$ and $V(\tau)$ meet transversally in $V(\gamma)$ (or \emptyset). In this case we define $[V(\sigma)] \cdot [V(\tau)] = [V(\gamma)]$ (or 0). Otherwise if $V(\sigma)$ and $V(\tau)$ do not meet properly, we can always use rational equivalence to replace in $A^*(X)$ a subvariety (i.e., $V(\sigma)$) with another one in the same class and such that it meets $V(\tau)$ in a proper way.

Again, let us apply these considerations to our example $X = G - \mathbb{C}_6^3$.

First, note that the intersection between X and any subvarieties $V(\sigma)$ is obviously equal to $V(\sigma)$. Therefore X is the multiplicative identity in $A^*(X)$.

Any divisor D_i meets each other properly (or not at all), and so their products give the curves $D_i \cdot D_j = [C_{ij}]$ (or 0). We have to use the linear equivalences (3.18) only to find the autointersections $D_i \cdot D_i$.

Table 1: Intersection product in $A^*(X)$

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	0	C	0	0
D_3	D_3	0	C	C	0	0
D_4	D_4	0	0	0	$-C$	0
C	C	0	0	0	0	0

Finally, when we intersect divisors and curves we obtain a point (or \emptyset), but, as we have seen, they are rational equivalent to zero. Any other intersection is always equivalent to the empty set.

The intersection products in $A^*(X)$ are summarized in table 1.

We can see that the product is symmetric and respects the grading. If we call R the set of relations given by the intersection product we find

$$A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R. \tag{3.30}$$

3.3.2 Group $A_*^c(G-\mathbb{C}_6^3)$ of compactly supported subvarieties

On a noncompact variety X the group $A_*^c(X)$ is defined to be the direct limit of the groups $A^*(Z)$, where Z are the closed and compact subvarieties of X ordered by inclusion. This means $A_*^c(X) = \bigoplus A^*(Z)/R$, where the direct sum is over all compact subvarieties of X and the relations R say that two elements $[Z_1]$ and $[Z_2]$ of $\bigoplus A^*(Z)$ must be identified if exists a compact subvariety Z_3 that contains them and such that in $A^*(Z_3)$ they represent the same cycle class. As usual in toric geometry we can restrict our analysis to compact toric invariant subvarieties $V(\sigma)$; recall that $X(\Delta)$ is compact in the classical topology if and only if its support $|\Delta|$ is the whole space $N_{\mathbb{R}}$.

Our example has one compact invariant divisor D_7 , six compact curves $(C_{27}, C_{37}, C_{47}, C_{57}, C_{67}, C_{46})$ and six points P_i . It is easy to see that (as a group)

$$A^*(P_i) = \mathbb{Z} \cdot P_i^c, \quad A^*(C_{ij}) = \mathbb{Z} \cdot C_{ij}^c \oplus \mathbb{Z} \cdot P_{ij}^c, \tag{3.31}$$

where P_{ij}^c represents the point class in the curve C_{ij}^c .

The divisor D_7 is the toric variety associated to the fan of figure 6, therefore

$$A^*(D_7) = \mathbb{Z} \cdot D_7^c \oplus \mathbb{Z} \cdot C_{47}^c \oplus \mathbb{Z} \cdot C_{57}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot P_7^c \tag{3.32}$$

and the relations with other curves are

$$C_{27}^c = C_{47}^c + C_{67}^c, \tag{3.33}$$

$$C_{37}^c = 2C_{47}^c + C_{57}^c + C_{67}^c. \tag{3.34}$$

Now we have to sum all these groups and find relations between different generators. It results that all point classes have to be identified, exactly as the classes of the same curve. The group of compact subvarieties of $G-\mathbb{C}_6^3$ is then isomorphic to \mathbb{Z}^6 :

$$A_*^c(X) = \mathbb{Z} \cdot D_7^c \oplus \mathbb{Z} \cdot C_{46}^c \oplus \mathbb{Z} \cdot C_{57}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot C_{47}^c \oplus \mathbb{Z} \cdot P^c. \tag{3.35}$$

3.3.3 Intersection pairing

There is a well-defined intersection pairing $A^*(X) \otimes A_*^c(X) \rightarrow A_*^c(X)$. For any two generators $[Z_1] \in A^*(X)$ and $[Z_2] \in A_*^c(X)$ it is possible to find two representatives which meet properly. Their intersection is a compact subvariety and defines the above pairing $[Z_1].[Z_2] := [Z_1 \cap Z_2] \in A_*^c(X)$, which is extendable by linearity to all elements in $A^*(X) \otimes A_*^c(X)$. This product gives the group $A_*^c(X)$ the structure of an $A^*(X)$ -module.

For $X = G-\mathbb{C}_6^3$ we obtain the intersection pairing of table 2:⁷

3.4 Homology, cohomology, Mori and Kähler cones

For any compact smooth variety X we have two natural homomorphisms

$$cl_X : A_*(X) \rightarrow H_*(X, \mathbb{Z}), \quad cl^X : A^*(X) \rightarrow H^*(X, \mathbb{Z}).$$

The map cl_X sends the representant V of an algebraic cycle to the homological cycle $[V]$; it is well-defined because algebraic equivalence implies homological equivalence. The map cl^X is defined by composition of cl_X with

⁷We can quickly obtain the product between divisors and curves which do not intersect properly in this way: suppose v_1, v_2 are the minimal lattice points on the edges of σ_C and let v', v'' be the minimal lattice points of the three-dimensional cones containing σ_C , then $v' + v'' = a_1 v_1 + a_2 v_2$ and $D_k \cdot C = -a_k P^c$.

Table 2: Intersection pairing $A^*(X) \otimes A_*^c(X) \rightarrow A_*^c(X)$

	D_7^c	C_{46}^c	C_{57}^c	C_{67}^c	C_{47}^c	P^c
X	D_7^c	C_{46}^c	C_{57}^c	C_{67}^c	C_{47}^c	P^c
D_1	0	P^c	0	0	0	0
D_2	C_{27}^c	0	P^c	0	0	0
D_3	C_{37}^c	0	0	P^c	0	0
D_4	C_{47}^c	$-P^c$	P^c	P^c	$-P^c$	0
C	P^c	0	0	0	0	0

Poincaré duality, which associate to an homological k -cycle V the $(n - k)$ -form η_V such that

$$\int_V \theta = \int_X \theta \wedge \eta_V.$$

In the case of crepant resolutions of toric orbifolds, it is possible to prove [11] that there exists the following module isomorphism:

$$A_*^c(X) \simeq H_*^c(X, \mathbb{Z}), \quad A^*(X) \simeq H^*(X, \mathbb{Z})$$

which respects the intersection product⁸

$$H^*(X, \mathbb{Z}) \otimes H_*^c(X, \mathbb{Z}) \rightarrow H_*^c(X, \mathbb{Z}).$$

We are interested in determining the Kähler cone of X , which is the set of all forms J in $H^2(X, \mathbb{Q})$ such that

$$\int_C J \geq 0$$

for all effective cycles in $H_2^c(X, \mathbb{Q})$. We describe the Kähler cone using the module isomorphism of the previous paragraph. We begin defining the Mori cone, i.e., the polyhedral cone in $A_2^c(X) \otimes \mathbb{Q}$ generated by effective toric invariant compact curves of X , which are the compact algebraic cycles $\sum_{i=1}^l a_{ij} C_{ij}^c$ where all the a_{ij} are nonnegative. Now we can think at the Kähler cone of X as the dual polyhedral cone in $A^2(X) \otimes \mathbb{Q}$ of the Mori cone with respect to the intersection pairing.

⁸The restriction to compact homology is necessary because of the problem in defining integration over noncompact cycles.

For $X = G-\mathbb{C}_6^3$, in view of relations (3.33) and (3.34), the Mori cone is generated by

$$C_1 := C_{46}^c, \quad C_2 := C_{57}^c, \quad C_3 := C_{67}^c, \quad C_4 := C_{47}^c. \tag{3.36}$$

Then the Kähler cone has the following dual generators, that satisfied $T_a \cdot C_b = \delta_{ab} P^c$:

$$T_1 := D_1, \quad T_2 := D_2, \quad T_3 := D_3, \quad T_4 := -D_1 + D_2 + D_3 - D_4. \tag{3.37}$$

For completeness we report in table 3 the products between the T_i in the Chow ring $A^*(X)$:

If we call J_i the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the T_i , then we find the cohomology ring

$$H^*(X, \mathbb{Q}) = \frac{\mathbb{Q}[J_1, J_2, J_3, J_4]}{(J_1^2, J_1 J_2, J_1 J_3, J_1 J_4, J_2^2, J_2 J_4 - J_2 J_3, J_3^2 - J_2 J_3, J_3 J_4 - 2J_2 J_3, J_4^2 - 2J_2 J_3)}. \tag{3.38}$$

3.5 K-theory

3.5.1 Preliminaries

The K -theory ring of a variety X is related to the cohomology via the Chern character map. More precisely, this is an injective homomorphism of rings from $K(X)$ to the Chow ring with rational coefficients $A^*(X)_{\mathbb{Q}}$. Then composition with cl^X gives the homomorphism with $H^*(X, \mathbb{Q})$:

$$\text{ch} : K(X) \rightarrow A^*(X) \otimes \mathbb{Q} \simeq H^*(X, \mathbb{Q}).$$

Here we summarize some general properties of Chern map that will be useful in the next sections. The Chern class c_i is a map from $K(X)$ to $A^i(X) \otimes \mathbb{Q}$;

Table 3: Products between the Kähler generators in $A^*(X)$

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	C	C
T_3	0	C	C	$2C$
T_4	0	C	$2C$	$2C$

the total Chern class is defined as the sum of all Chern class

$$c(\mathcal{F}) := c_0(\mathcal{F}) + c_1(\mathcal{F}) + \cdots + c_n(\mathcal{F}),$$

where n is the dimension of X . A divisorial $\mathcal{O}_X(D)$ sheaf has a very simple total Chern class

$$c(\mathcal{O}_X(D)) = X + D$$

and, using multiplicative properties of the Chern classes, this implies

$$\mathcal{F} = \bigoplus_{i=1}^r \mathcal{O}_X(D_i) \implies c(\mathcal{F}) = \prod_{i=1}^r (X + D_i).$$

The Chern character is defined for such sheaf as

$$\text{ch}(\mathcal{F}) := \sum_{i=1}^r e^{D_i} = rX + c_1(\mathcal{F}) + \frac{1}{2}(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})),$$

where the expansion is stopped to second order in view of the cohomology ring structure of our non compact threefold varieties. In particular for a divisorial sheaf we have

$$\text{ch}(\mathcal{O}_X(D)) = X + D + \frac{1}{2}D^2.$$

We recall also the definition of the Todd class:

$$\text{td}(\mathcal{F}) := \prod_{i=1}^r \frac{D_i}{1 - e^{-D_i}} = X + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}(c_1(\mathcal{F})^2 + c_2(\mathcal{F})).$$

In particular, we need the Todd class of the tangent bundle T_X ; for a toric variety its total Chern class is

$$c(T_X) = \sum_{\sigma \in \Delta} [V(\sigma)]$$

and so if X is a Calabi–Yau noncompact toric three-fold we have

$$c(T_X) = X + \sum [C_{ij}] \implies \text{td}(X) = \text{td}(T_X) = X + \frac{1}{12} \sum [C_{ij}], \quad (3.39)$$

where the sum is over all (compact and non compact) toric curves in X .

In the context of noncompact varieties we also have to work with the compactly supported K -theory group $K^c(X)$. This group is related to the

compactly supported Chow group with rational coefficients $A_*^c(X)_{\mathbb{Q}}$, and therefore to $H_*^c(X, \mathbb{Q})$, via the local Chern character map [37]:

$$\text{ch}^c : K^c(X) \rightarrow A_*^c(X) \otimes \mathbb{Q} \simeq H_*^c(X, \mathbb{Q}).$$

Let us briefly review its definition and properties. Any element S of $K^c(X)$ can be represented by coherent sheaves S_V on a compact subvariety V of X . If $i : V \hookrightarrow X$ is the embedding of V in X , we can define the local Chern character of S by

$$\text{ch}^c(S) = \text{ch}(i_*S_V).$$

Actually, we can compute the local Chern characters with the help of the Grothendieck–Riemann–Roch theorem:

$$i_*(\text{ch}(S_V)\text{td}(V)) = \text{ch}(i_*S_V)\text{td}(X)$$

for any compact subvarieties V of X , which implies

$$\text{ch}^c(S) = \text{td}(X)^{-1} i_*(\text{ch}(S_V)\text{td}(V)).$$

The local Chern classes of the divisorial sheaves over the compact subvarieties in a Calabi–Yau noncompact toric threefold X are:

$$\begin{aligned} \text{ch}^c(\mathcal{O}_{p^c}) &= p^c, & \text{ch}^c(\mathcal{O}_{C^c}(n)) &= C^c + (n + 1)p^c, \\ \text{ch}^c(\mathcal{O}_{D^c}(C)) &= i_* \left(D^c + \left(C + \frac{1}{2}c_1(D^c) \right) \right. \\ &\quad \left. + \frac{1}{2} \left(C^2 + c_1(D^c)C + \frac{1}{6}(c_1(D^c)^2 + c_2(D^c)) \right) \right) \\ &\quad - \frac{1}{12}c_2(X)D^c. \end{aligned} \tag{3.40}$$

In the last character, C is a divisor in D^c and the $c_i(D^c)$ are the Chern classes $c_i(T_{D^c})$, which naturally live in $A^*(D^c)$ and that can be calculated using formula (3.39). Moreover all the products excepted the last are in $A^*(D^c)$.

3.5.2 K-theory generators

Let G be an abelian subgroup of $SL(3, \mathbb{C})$ which acts on the affine space \mathbb{C}^3 . We write $\pi : \mathbb{C}^3 \rightarrow Y = \mathbb{C}^3/G$ for the quotient, $X = G\text{-Hilb}(\mathbb{C}^3)$ for the Hilbert scheme with crepant resolution $\tau : X \rightarrow Y$ and the universal scheme

$\mathcal{Z} = \{(Z(x), x) \in X \times \mathbb{C}^3\}$ where $Z(x)$ is the G -cluster over x (see Section 3.2.2). Thus we have the commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & \mathbb{C}^3 \\ \downarrow p & & \downarrow \pi \\ X & \xrightarrow{\tau} & Y \end{array}$$

Let us consider the sheaf $\mathcal{R} := p_*\mathcal{O}_{\mathcal{Z}}$ on the resolution X . Over any point $x \in X$ the fibre of \mathcal{R} is $H^0(Z(x), \mathcal{O}_{Z(x)})$ which supports the regular representation of G . In particular, the rank of \mathcal{R} is equal to the order of the group G . The decomposition of the regular representation into irreducible submodules induces the decomposition

$$\mathcal{R} = \bigoplus_k \mathcal{R}_k \otimes \rho_k \quad \text{for } \mathcal{R}_k = \text{Hom}_G(\rho_k, \mathcal{R})$$

into locally free sheaves of rank $\mathcal{R}_i = \dim \rho_i = 1$. We called \mathcal{R}_k the tautological line bundle on X associated to the irreducible representation ρ_k of G . At the level of K -theory the McKay correspondence states the equivalence of the G -equivariant K -theory of \mathbb{C}^n and the K -theory of the crepant resolutions. In [36] it has been determined the ring isomorphism

$$\varphi : K^G(\mathbb{C}^3) \xrightarrow{\sim} K(X)$$

showing that $\varphi(\rho_i \otimes \mathcal{O}_{\mathbb{C}^3}) = \mathcal{R}_i$ and therefore, that the tautological line bundles form a \mathbb{Z} -basis of $K(X)$.

In Section 3.2.2 we studied the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ and its crepant resolution $\mathbb{Z}^2\text{-Hilb}(\mathbb{C}^2)$. We have given a description of \mathcal{R} and its decomposition into line bundles on the two open sets U_{σ_a} and U_{σ_b} . In figure 8 we report the monomial generators of \mathcal{R}_i on the affine pieces.

With the same procedure we can give the generators of \mathcal{R}_i on $G\text{-Hilb}(\mathbb{C}^3)$ for any abelian $G \subset SL(3, \mathbb{C})$. We report in figure 9 the monomial generators for $\mathbb{Z}_6\text{-Hilb}(\mathbb{C}^3)$.

The action of \mathbb{Z}_6 on the coordinate ring of \mathbb{C}^3 is

$$\epsilon \cdot (X, Y, Z) = (\epsilon X, \epsilon^2 Y, \epsilon^3 Z), \quad \epsilon = e^{\frac{2\pi i}{6}}. \tag{3.41}$$

Therefore, it is simple to verify that each \mathcal{R}_i supports the irreducible representation ρ_i of \mathbb{Z}_6 .

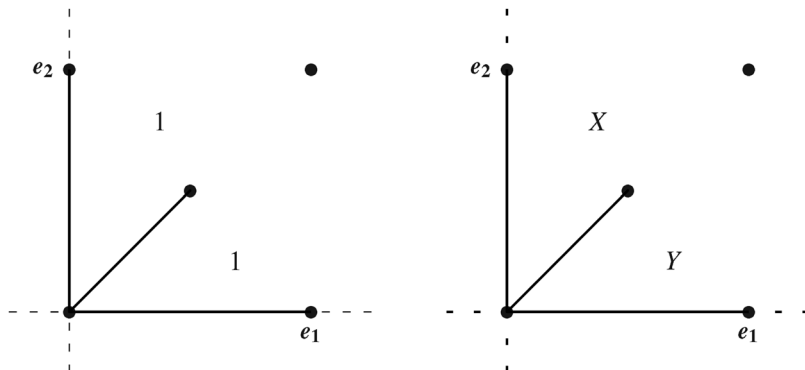


Figure 8: Monomial generators of \mathcal{R}_0 and \mathcal{R}_1 for $\mathbb{Z}_2\text{-Hilb}(\mathbb{C}^2)$.

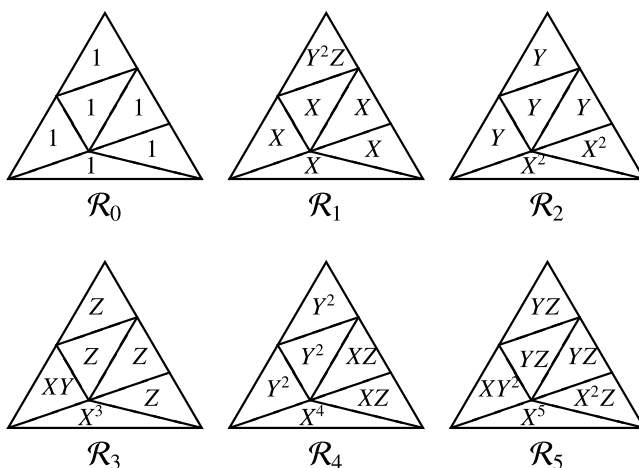


Figure 9: Monomial generators of \mathcal{R}_i for $\mathbb{Z}_6\text{-Hilb}(\mathbb{C}^3)$.

Any line bundle on a smooth algebraic variety is a divisorial bundle. We briefly sketch the standard procedure to find the divisor related to the \mathcal{R}_i defined by Reid and proved by Craw, and refer to [11] for a detailed explanation.

The first step consists in decorating the G -Hilb fan with the characters of the group. Any curve has to be marked with the character of the monomials in its associated ratio. For any internal vertex v there exists a recipe to associate one or two characters of G , depending primarily on the valency of v (i.e., the number of lines meeting at v). For a G -Hilb fan this is always 3, 4, 5 or 6. There are the following cases:

- A vertex v of valency 3 defines an exceptional \mathbb{P}^2 . A single character χ_k marks all three lines meeting at v . Mark the vertex v with the character $\chi_m := \chi_k \otimes \chi_k$.

- A vertex v of valency 4 defines an exceptional Hirzebruch surface \mathbb{F}_r . There are distinct characters χ_k and χ_l each one marking a pair of lines meeting at v . Mark the vertex v with the character $\chi_m := \chi_k \otimes \chi_l$.
- A vertex v of valency 5 or 6 (excluding three straight lines meeting at a point) defines an Hirzebruch surface \mathbb{F}_r blown-up in one or two points. There are uniquely determined characters χ_k and χ_l each one marking a pair of lines meeting at v . Mark the vertex v with $\chi_m := \chi_k \otimes \chi_l$.
- A vertex v at the intersection of three straight lines defines an exceptional Del Pezzo surface of degree six, denoted dP_6 . The monomials defining the pair of morphisms $dP_6 \rightarrow \mathbb{P}^2$ lie in uniquely determined character spaces χ_l and χ_m satisfying

$$\chi_l \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_k,$$

where χ_i, χ_j and χ_k mark the straight lines through the vertex v . Mark the vertex v with both χ_l and χ_m .

Each character of G appears once on the fan Δ .

By analysing of the monomial generators of the tautological line bundles \mathcal{R}_i , in [11] the author proved that:

- If χ_k marks the line defining the compact curve $C_k \in H_2^c(X, \mathbb{Z})$ on the resolution X , the first Chern class $c_1(\mathcal{R}_k)$ is the dual to C_k in $H^2(X, \mathbb{Z})$:

$$\int_{C_l} c_1(\mathcal{R}_k) = \delta_{lk}.$$

This means that $R_k = \mathcal{O}_X(T_k)$, where T_k is the generator of the Kähler cone dual to C_k .

- In $\text{Pic}(X)$ all relations between tautological line bundles are of the following forms:
 - $\mathcal{R}_m = \mathcal{R}_k \otimes \mathcal{R}_k$ when $\chi_m = \chi_k \otimes \chi_k$ marks a vertex v of valency 3;
 - $\mathcal{R}_m = \mathcal{R}_k \otimes \mathcal{R}_l$ when $\chi_m = \chi_k \otimes \chi_l$ marks a vertex v of valency 4;
 - $\mathcal{R}_m = \mathcal{R}_k \otimes \mathcal{R}_l$ when $\chi_m = \chi_k \otimes \chi_l$ marks a vertex v of valency 5 or 6 (excluding three straight lines meeting at a point);
 - $\mathcal{R}_l \otimes \mathcal{R}_m = \mathcal{R}_i \otimes \mathcal{R}_j \otimes \mathcal{R}_k$ when the pair of characters χ_l and χ_m satisfying $\chi_l \otimes \chi_m = \chi_i \otimes \chi_j \otimes \chi_k$ marks the intersection point v of three straight lines.

As usual we apply these considerations to our case $X = \mathbb{Z}_6\text{-Hilb}(\mathbb{C}_3^6)$ and we summarize them in the decorated fan of figure 10.

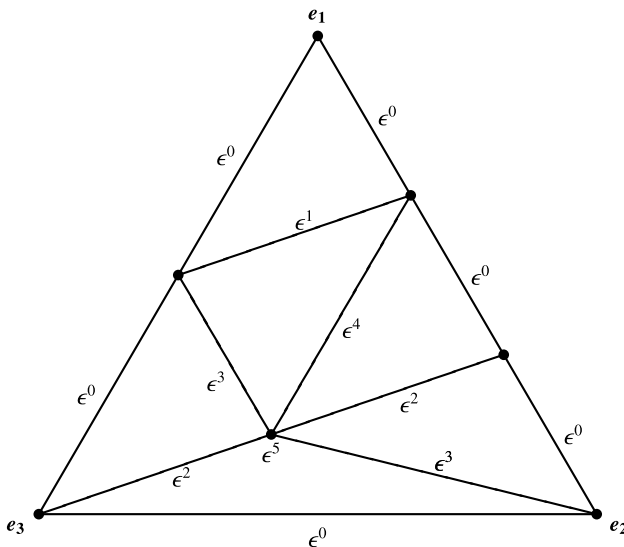


Figure 10: Fan for $\mathbb{Z}_6\text{-Hilb}(C^3)$ decorated with Reid's recipe.

The resulting tautological line bundles, that give a \mathbb{Z} -basis of $K(X)$, are:

$$\begin{aligned}
 \mathcal{R}_0 &= \mathcal{O}_X, & \mathcal{R}_1 &= \mathcal{O}_X(D_1), & \mathcal{R}_2 &= \mathcal{O}_X(D_2), & \mathcal{R}_3 &= \mathcal{O}_X(D_3), \\
 \mathcal{R}_4 &= \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), & \mathcal{R}_5 &= \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3).
 \end{aligned}
 \tag{3.42}$$

3.5.3 $K(X)$ and $K^c(X)$

Chosen a base of generators for $K(X)$ we can find the dual basis for the compact K -theory $K^c(X)$ as in [36] using the perfect pairing

$$(\mid) : K(X) \times K^c(X) \longrightarrow \mathbb{Z}, \quad (\mathcal{R}, \mathcal{S}) \longmapsto (\mathcal{R} \mid \mathcal{S}) = \int_X \text{ch}(\mathcal{R}) \text{ch}^c(\mathcal{S}) \text{td}(X),$$

so that

$$(\mathcal{R}_i \mid \mathcal{S}_j) = \delta_{ij}.
 \tag{3.43}$$

As usual, the integral is by definition the coefficient of the point class. Using this fact, the standard computations of Chern and Todd characters and the intersection product table 2, from condition (3.43) we find

$$\begin{aligned}
 \text{ch}^c(\mathcal{S}_0) &= D_7^c - (C_{46}^c + C_{57}^c + \frac{3}{2}C_{67}^c + 2C_{47}^c) + \frac{7}{6}P^c, \\
 \text{ch}^c(\mathcal{S}_1) &= C_{46}^c,
 \end{aligned}$$

$$\begin{aligned}
 \text{ch}^c(\mathcal{S}_2) &= -D_7^c + (C_{57}^c + \frac{1}{2}C_{67}^c + C_{47}^c) - \frac{1}{6}P^c, \\
 \text{ch}^c(\mathcal{S}_3) &= -D_7^c + (\frac{3}{2}C_{67}^c + C_{47}^c) - \frac{1}{6}P^c, \\
 \text{ch}^c(\mathcal{S}_4) &= C_{47}^c, \\
 \text{ch}^c(\mathcal{S}_5) &= D_7^c - (\frac{1}{2}C_{67}^c + C_{47}^c) + \frac{1}{6}P^c.
 \end{aligned}
 \tag{3.44}$$

In the spirit of the paper [30] we now express the elements \mathcal{S}_i in terms of a symplectic D-brane basis of $K^c(X)$. Such basis can be constructed starting from the generators of the compact Chow ring. We choose

$$B_0 := \mathcal{O}_P^c, \quad B_a := \mathcal{O}_{C_a^c}(-T_a), \quad B_5 := \mathcal{O}_{D_7^c}(-T_2 - T_3),
 \tag{3.45}$$

with $a = 1, \dots, 4$ and $\mathcal{O}_{C_a^c}(-T_a) := \mathcal{O}_{C_a^c} \otimes \mathcal{O}_X(-T_a)$, $\mathcal{O}_{D_7^c}(-T_2 - T_3) := \mathcal{O}_{D_7^c} \otimes \mathcal{O}_X(-T_2 - T_3)$. To express the basis \mathcal{S}_i in terms of B_j we can compare their compact Chern characters. Using (3.40) and the multiplicative property of Chern character we find

$$\begin{aligned}
 \text{ch}^c(B_0) = P^c, \quad \text{ch}^c(B_a) = C_a^c, \quad \text{ch}^c(B_5) = D_7^c - \left(\frac{1}{2}C_{67}^c + C_{47}^c \right) \\
 + \frac{1}{6}P^c
 \end{aligned}
 \tag{3.46}$$

and then

$$\begin{aligned}
 \mathcal{S}_0 &= B_0 - B_1 - B_2 - B_3 - B_4 + B_5, \\
 \mathcal{S}_1 &= B_1, \\
 \mathcal{S}_2 &= B_2 - B_5, \\
 \mathcal{S}_3 &= B_3 - B_5, \\
 \mathcal{S}_4 &= B_4, \\
 \mathcal{S}_5 &= B_5,
 \end{aligned}
 \tag{3.47}$$

Finally, we write the B_i basis of $K^c(X)$ in terms of the \mathcal{S}_i and its dual basis Φ_i of $K(X)$ in term of \mathcal{R}_i :

$$\begin{aligned}
 B_0 &= \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_0 &= \mathcal{R}_0, \\
 B_1 &= \mathcal{S}_1, & \Phi_1 &= -\mathcal{R}_0 + \mathcal{R}_1, \\
 B_2 &= \mathcal{S}_2 + \mathcal{S}_5, & \Phi_2 &= -\mathcal{R}_0 + \mathcal{R}_2, \\
 B_3 &= \mathcal{S}_3 + \mathcal{S}_5, & \Phi_3 &= -\mathcal{R}_0 + \mathcal{R}_3, \\
 B_4 &= \mathcal{S}_4, & \Phi_4 &= -\mathcal{R}_0 + \mathcal{R}_4, \\
 B_5 &= \mathcal{S}_5, & \Phi_5 &= \mathcal{R}_0 - \mathcal{R}_2 - \mathcal{R}_3 + \mathcal{R}_5.
 \end{aligned}
 \tag{3.48}$$

3.6 The Hosono conjecture

We will restate shortly here the Hosono conjecture [30], for convenience. The main point is that the periods for the mirror manifold are solutions of a set of Picard–Fuchs equations, and the general solution can be expressed in terms of an hypergeometric function with value in the cohomology of X :

$$w = w \left(x_1, \dots, x_4; \frac{J_1}{2\pi i}, \dots, \frac{J_4}{2\pi i} \right).$$

Then the conjecture (adapted to our case) states as follows.

3.6.1 Hosono conjecture

Define the basis for $H^*(X, \mathbb{Q})$

$$Q_i := \text{ch}(\Phi_i), \quad i = 0, \dots, 5$$

and expand the cohomology-valued hypergeometric series w with respect to this basis:

$$w \left(x_1, \dots, x_4; \frac{J_1}{2\pi i}, \dots, \frac{J_4}{2\pi i} \right) = \sum_{i=0}^5 w_i(x_1, \dots, x_4) Q_i.$$

Thus

- (1) the coefficient hypergeometric series $w_i(x_1, \dots, x_4)$ may be identified with the period integrals over the cycles $\text{mir}(B_i)$,

$$w_i(x_1, \dots, x_4) = \int_{\text{mir}(B_i)} \Omega(Y_x);$$

- (2) the monodromy of the hypergeometric series is integral and symplectic with respect to the symplectic form defined in $K^c(X)$

$$\chi(B_i, B_j) = \int_X \text{ch}(B_i^\vee) \text{ch}(B_j) \text{td}(X);$$

- (3) the central charge of an element $F \in K^c(X)$ is expressed in terms of the cohomology valued hypergeometric w as

$$Z(F) = \int_X \text{ch}(F) w \left(x_1, \dots, x_4; \frac{J_1}{2\pi i}, \dots, \frac{J_4}{2\pi i} \right) \text{td}(X).$$

The symplectic form of point 2 can be easily computed with respect to the basis \mathcal{S}_i^- following the paper of Ito–Nakajima [36]. Let Q be the three-dimensional representation given by the inclusion $G \subset SL(3, \mathbb{C})$ and $\{\rho_i\}_{i=0}^r$ be the irreducible representations. The decomposition

$$Q \otimes \rho_j = \bigoplus_k a_{kj} \rho_k$$

is related to the symplectic form by

$$\chi(\mathcal{S}_i, \mathcal{S}_j) = a_{ji} - a_{ij}.$$

In our example

$$\chi(\mathcal{S}_i, \mathcal{S}_j) = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}, \tag{3.49}$$

and then, for our chosen basis,

$$\chi(B_i, B_j) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{3.50}$$

This matrix gives a symplectic correspondence between the space $H^4(X, \mathbb{Q})$ and a one-dimensional subspace of $H^2(X, \mathbb{Q})$. There is an obvious ambiguity in such a correspondence, but we will turn back to it later.

3.6.2 The cohomological hypergeometric series

The vectors ℓ_a , $a = 1, \dots, 4$ are given by the intersection numbers between the Mori cone generators and the invariant divisors of X , so that we find

$$\begin{aligned} C_1 & : \ell_1 = (1, 0, 0, -1, 0, -1, 1), \\ C_2 & : \ell_2 = (0, 1, 0, 1, -2, 0, 0), \\ C_3 & : \ell_3 = (0, 0, 1, 1, 0, -1, -1), \\ C_4 & : \ell_4 = (0, 0, 0, -1, 1, 1, -1). \end{aligned} \tag{3.51}$$

The hypergeometric series is then

$$w = \sum_{\vec{m} \in \mathbb{Z}_{\geq 0}^4} \frac{x_1^{m_1+\rho_1} x_2^{m_2+\rho_2} x_3^{m_3+\rho_3} x_4^{m_4+\rho_4}}{\prod_{i=1}^7 \Gamma_i(\vec{m} + \vec{\rho})} \Bigg|_{\vec{\rho} = \frac{\vec{J}}{2\pi i}}, \tag{3.52}$$

$$\begin{aligned} \Gamma_1(\vec{m}) &= \Gamma(1 + m_1), \\ \Gamma_2(\vec{m}) &= \Gamma(1 + m_2), \\ \Gamma_3(\vec{m}) &= \Gamma(1 + m_3), \\ \Gamma_4(\vec{m}) &= \Gamma(1 - m_1 + m_2 + m_3 - m_4), \\ \Gamma_5(\vec{m}) &= \Gamma(1 - 2m_2 + m_4), \\ \Gamma_6(\vec{m}) &= \Gamma(1 - m_1 - m_3 + m_4), \\ \Gamma_7(\vec{m}) &= \Gamma(1 + m_1 - m_3 - m_4). \end{aligned} \tag{3.53}$$

We need to expand this function in power series in \vec{J} . Because of the ring relations for $H^*(X, \mathbb{Q})$, we see that the expansion stops at order two. The coefficient functions, with respect to the basis $\{1, \vec{J}, C\}$ of $H^*(X, \mathbb{Q})$, are computed in the appendix.

However we chosen the basis B_i in $K^c(X)$ so that we need to rewrite the expansion in terms of the dual basis $Q_i = \text{ch}(\Phi_i)$:

$$Q_0 = 1, \quad Q_1 = J_1, \quad Q_2 = J_2, \quad Q_3 = J_3 + \frac{1}{2}C, \quad Q_4 = J_4 + C, \quad Q_5 = C. \tag{3.54}$$

If we make this change of basis and use the mirror symmetry identification

$$w \left(\vec{x}, \frac{\vec{J}}{2\pi i} \right) = Q_0 1 + \sum_{a=1}^4 Q_a t_a + Q_5 g(t_1, \dots, t_4), \tag{3.55}$$

then we find

$$\begin{aligned} 2\pi i t_1 &= \log x_1 + \Psi(x_1 x_3) + \Phi(x_2, x_1 x_4) + \aleph(\vec{x}), \\ 2\pi i t_2 &= \log x_2 - \Phi(x_2, x_1 x_4) + 2\Phi(x_1 x_4, x_2), \\ 2\pi i t_3 &= \log x_3 - \Phi(x_2, x_1 x_4) + \Psi(x_1 x_3) - \aleph(\vec{x}), \\ 2\pi i t_4 &= \log x_4 - \Phi(x_1 x_4, x_2) + \Phi(x_2, x_1 x_4) - \Psi(x_1 x_3) - \aleph(\vec{x}), \end{aligned} \tag{3.56}$$

and

$$\begin{aligned}
 (2\pi i)^2 g(\vec{t}) = & -\frac{\pi^2}{3} - \pi i(\log x_3 - \Phi(x_2, x_1 x_4) + \Psi(x_1 x_3) - \aleph(\vec{x})) \\
 & - 2\pi i(\log x_4 - \Phi(x_1 x_4, x_2) + \Phi(x_2, x_1 x_4) - \Psi(x_1 x_3) - \aleph(\vec{x})) \\
 & + 7\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(4)}(\vec{x}) - \aleph^{(5)}(\vec{x}) \\
 & - \Psi_1(x_2, x_1 x_4) + \Psi_2(x_2, x_1 x_4) - \Psi_1(x_1 x_4, x_2) + \Psi_2(x_1 x_4, x_2) \\
 & - \Psi_1(x_1 x_4, x_2) + \Psi_3(x_1 x_4, x_2) - \Psi_4(x_1 x_3) + \Psi_5(x_1 x_3) + \Psi_6(x_2, x_4) \\
 & + \Lambda_1(\vec{x}) - \Lambda_2(\vec{x}) - \Lambda_3(\vec{x}) \\
 & + \frac{1}{2}(\log x_3)^2 + \log x_3[\Psi(x_1 x_3) - \Phi(x_2, x_1 x_4) - \aleph(\vec{x})] \\
 & + (\log x_4)^2 + 2 \log x_4[\Phi(x_2, x_1 x_4) - \Phi(x_1 x_4, x_2) - \Psi(x_1 x_3) - \aleph(\vec{x})] \\
 & + \log x_2 \log x_3 + \log x_2[\Psi(x_1 x_3) - \Phi(x_2, x_1 x_4) - \aleph(\vec{x})] \\
 & + \log x_3[2\Phi(x_1 x_4, x_2) - \Phi(x_2, x_1 x_4)] + \log x_2 \log x_4 \\
 & + \log x_2[\Phi(x_2, x_1 x_4) - \Phi(x_1 x_4, x_2) - \Psi(x_1 x_3) - \aleph(\vec{x})] \\
 & + \log x_4[2\Phi(x_1 x_4, x_2) - \Phi(x_2, x_1 x_4)] + 2 \log x_3 \log x_4 \\
 & + 2 \log x_3[\Phi(x_2, x_1 x_4) - \Phi(x_1 x_4, x_2) - \Psi(x_1 x_3) - \aleph(\vec{x})] \\
 & + 2 \log x_4[\Psi(x_1 x_3) - \Phi(x_2, x_1 x_4) - \aleph(\vec{x})]. \tag{3.57}
 \end{aligned}$$

Using the above expressions we find

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2} \phi(\vec{t}), \tag{3.58}$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{12} - \frac{1}{2}t_3 - t_4 + \frac{1}{2}t_3^2 + t_4^2 + t_2 t_3 + t_2 t_4 + 2t_3 t_4, \tag{3.59}$$

and

$$\begin{aligned}
 \phi(\vec{t}) = & 7\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(4)}(\vec{x}) - \aleph^{(5)}(\vec{x}) \\
 & - \Psi_1(x_2, x_1 x_4) + \Psi_2(x_2, x_1 x_4) - \Psi_1(x_1 x_4, x_2) + \Psi_2(x_1 x_4, x_2) \\
 & - \Psi_1(x_1 x_4, x_2) + \Psi_3(x_1 x_4, x_2) - \Psi_4(x_1 x_3) + \Psi_5(x_1 x_3) + \Psi_6(x_2, x_4) \\
 & + \Lambda_1(\vec{x}) - \Lambda_2(\vec{x}) - \Lambda_3(\vec{x}) \\
 & + \frac{1}{2}\Psi^2(x_1 x_3) + \frac{1}{2}\Phi^2(x_2, x_1 x_4) + \Phi^2(x_1 x_4, x_2) - \frac{7}{2}\aleph^2(\vec{x}) \\
 & - \Psi(x_1 x_3)\aleph(\vec{x}) - \Phi(x_2, x_1 x_4)\aleph(\vec{x}) \\
 & - \Phi(x_2, x_1 x_4)\Psi(x_1 x_3) - \Phi(x_2, x_1 x_4)\Phi(x_1 x_4, x_2), \tag{3.60}
 \end{aligned}$$

with \vec{x} expressed as a function of \vec{t} by inverting system (3.56), is the part corresponding to instantonic contributions. Following Hosono and using

(3.50) we find

$$(\partial_{t_1} - \partial_{t_3} - \partial_{t_4})F(\vec{t}) = g(\vec{t}), \tag{3.61}$$

where F is the prepotential. To integrate this equation we must expect for the prepotential to be as usual the sum of a classical term, a cubic polynomial in \vec{t} and a quantum instantonic contribution. Setting

$$q_k := e^{2\pi i t_k}, \tag{3.62}$$

we then find

$$F(\vec{t}) = -\frac{t_4}{12} + \frac{t_4^2}{4} + \frac{t_3 t_4}{2} - \frac{t_4^3}{6} - \frac{t_3 t_4}{2}(t_3 + t_4 + 2t_2) + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_2, t_1 + t_3, t_1 + t_4) + Q_{\text{inst}}(q_2, q_1 q_3, q_1 q_4), \tag{3.63}$$

where F_{inst} are the instantonic corrections, obtained integrating ϕ , P_{class} is an arbitrary cubic polynomial of three variables and Q_{inst} an arbitrary function of three variables which we assume analytic in $(0, 0, 0)$. P_{class} and Q_{inst} represent the contributions which are undetermined by the equation (3.61).

As an example, let us compute the Gromov–Witten (GW) invariants up to degree six. Here we use the Gopakumar–Vafa (G–V) reinterpretation, so that by GW-invariants we mean the G–V integral invariants for rational curves, in place of the original fractional GW-invariants.⁹ We will use $[d_1, d_2, d_3, d_4]$ to indicate the degree of the curves in the Mori cone, corresponding to the generators J_1, \dots, J_4 . Thus we consider the curves with degree $d_1 + d_2 + d_3 + d_4 \leq 6$. The curves with degree in the integer cone generated by $[0, 1, 0, 0]$, $[1, 0, 1, 0]$, $[1, 0, 0, 1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$\begin{aligned} GW_{[0,0,0,1]} &= GW_{[0,0,1,0]} = GW_{[1,0,0,0]} \\ &= GW_{[1,0,1,1]} = GW_{[0,1,0,1]} = GW_{[1,1,1,1]} = 1; \\ GW_{[0,0,1,1]} &= GW_{[0,1,1,1]} = GW_{[1,1,1,2]} = -2; \\ GW_{[0,1,1,2]} &= GW_{[1,1,2,2]} = 3; \\ GW_{[0,1,2,2]} &= -4; \quad GW_{[0,1,2,3]} = 5. \end{aligned} \tag{3.64}$$

⁹See [3] for a nice introduction to GW invariants and their interpretation in physics and in mathematics.

4 The flopped resolutions

In this section, we study the remaining four crepant resolutions X of the orbifold \mathbb{C}_6^3 .

4.1 Intersection theory

For any resolution we have

$$\begin{aligned} A^0(X) &= \mathbb{Z} \cdot X, \\ A^1(X) &= \bigoplus_{i=1}^4 \mathbb{Z} \cdot D_i \simeq \mathbb{Z}^4, \\ A^2(X) &= \mathbb{Z} \cdot C, \\ A^3(X) &= 0. \end{aligned} \tag{4.1}$$

In $A_2(X)$ the divisors D_5, D_6, D_7 can be expressed in terms of the others

$$D_5 = -D_1 - D_2 + D_3 - D_4, \tag{4.2}$$

$$D_6 = -2D_1 + D_2 - D_4, \tag{4.3}$$

$$D_7 = 2D_1 - D_2 - 2D_3 + D_4. \tag{4.4}$$

The curve C depends on the resolution, as well as the intersection product in $A^*(X)$. If we call R the set of relations given by the intersection product, we have

$$A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R. \tag{4.5}$$

$A_*^c(X)$ is an $A^*(X)$ -module, it is generated as group by the compact divisor D_7^c , the four compact curves depending on the resolution and the point class P^c . Finally, the intersection pairing $A^*(X) \otimes A_*^c(X) \rightarrow A_*^c(X)$ depends on the resolution.

Resolution $X = R_2 - \mathbb{C}_6^3$ (figure 11): This resolution differs from the G -Hilb by the flop

$$C_{46} \longrightarrow C_{17}. \tag{4.6}$$

We define $C = [C_{14}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 12.

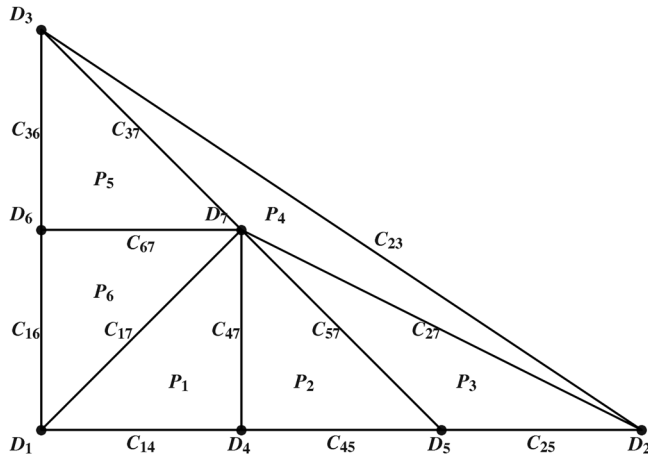


Figure 11: The toric invariant subvarieties of $R_2-(\mathbb{C}_6^3)$.

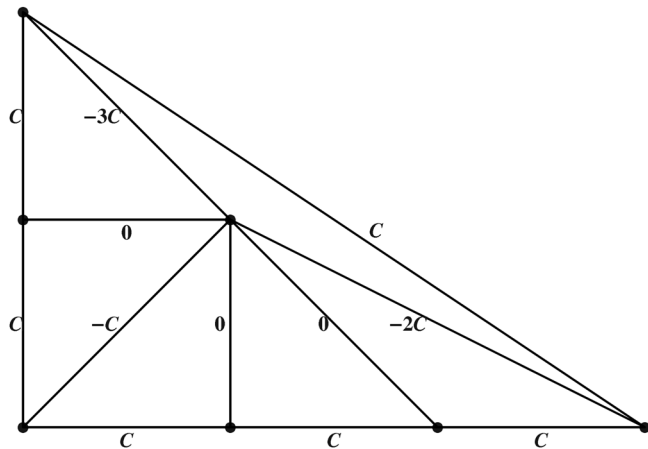


Figure 12: The rational equivalence between curves of $R_2-(\mathbb{C}_6^3)$.

In table 4 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

The group of compact subvarieties of X is

$$A_*^c(X) = \mathbb{Z} \cdot D_7^c \oplus \mathbb{Z} \cdot C_{17}^c \oplus \mathbb{Z} \cdot C_{57}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot C_{47}^c \oplus \mathbb{Z} \cdot P^c, \quad (4.7)$$

with relations to other compact curves

$$C_{27}^c = 2C_{17}^c + C_{67}^c + C_{47}^c, \quad (4.8)$$

$$C_{37}^c = 3C_{17}^c + C_{57}^c + C_{67}^c + 2C_{47}^c. \quad (4.9)$$

Table 4: Intersection product in $A^*(X)$

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	C	0	0	$-C$	0
D_2	D_2	0	0	$-C$	0	0
D_3	D_3	0	$-C$	$-C$	0	0
D_4	D_4	$-C$	0	0	$2C$	0
C	C	0	0	0	0	0

Table 5: Intersection pairing $A^*(X) \otimes A_*^c(X) \rightarrow A_*^c(X)$

	D_7^c	C_{17}^c	C_{57}^c	C_{67}^c	C_{47}^c	P^c
X	D_7^c	C_{17}^c	C_{57}^c	C_{67}^c	C_{47}^c	P^c
D_1	C_{17}^c	$-P^c$	0	P^c	P^c	0
D_2	C_{27}^c	0	P^c	0	0	0
D_3	C_{37}^c	0	0	P^c	0	0
D_4	C_{47}^c	P^c	P^c	0	$-2P^c$	0
C	P^c	0	0	0	0	0

In table 5 we summarize the intersection pairing.

The Mori cone generators are C_a , $a = 1, \dots, 4$, with

$$C_1 = C_{17}, \quad C_2 = C_{57}, \quad C_3 = C_{67}, \quad C_4 = C_{47}. \tag{4.10}$$

The Kähler cone is generated by the dual elements T_a , $a = 1, \dots, 4$ with

$$\begin{aligned} T_1 &= -2D_1 + D_2 + 2D_3 - D_4, & T_2 &= D_2, \\ T_3 &= D_3, & T_4 &= -D_1 + D_2 + D_3 - D_4. \end{aligned} \tag{4.11}$$

If we call J_a the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \dots, J_4] / \sim, \tag{4.12}$$

with \sim given by table 6.

Table 6: Intersection between Kähler generators

	T_1	T_2	T_3	T_4
T_1	$6C$	$2C$	$3C$	$4C$
T_2	$2C$	0	C	C
T_3	$3C$	C	C	$2C$
T_4	$4C$	C	$2C$	$2C$

Resolution $X = \mathbb{R}_3 - \mathbb{C}_6^3$ (figure 13): This resolution differs from the G -Hilb by the flop

$$C_{67} \longrightarrow C_{34}. \tag{4.13}$$

We set $C = [C_{34}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 14.

In table 7 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

The group of compact subvarieties of X is

$$A_*^c(X) = \mathbb{Z} \cdot D_7^c \oplus \mathbb{Z} \cdot C_{46}^c \oplus \mathbb{Z} \cdot C_{57}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot C_{47}^c \oplus \mathbb{Z} \cdot P^c, \tag{4.14}$$

with relations to other compact curves

$$C_{27}^c = C_{47}^c, \tag{4.15}$$

$$C_{37}^c = C_{57}^c + 2C_{47}^c. \tag{4.16}$$

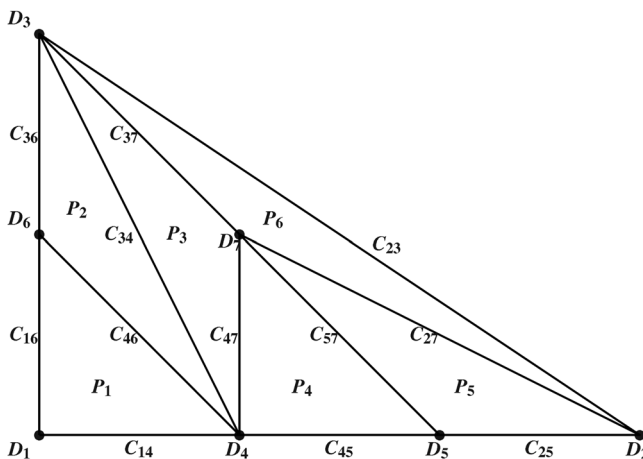


Figure 13: The toric invariant subvarieties of $\mathbb{R}_3 - (\mathbb{C}_6^3)$.

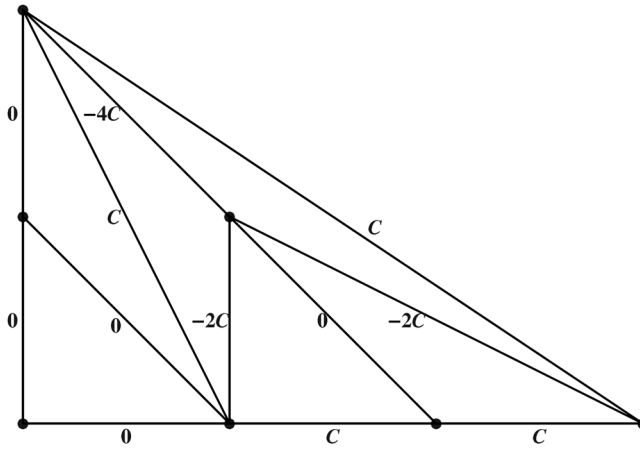


Figure 14: The rational equivalence between curves of $R_3-(\mathbb{C}_6^3)$.

Table 7: Intersection product in $A^*(X)$

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	0	C	0	0
D_3	D_3	0	C	$2C$	C	0
D_4	D_4	0	0	C	0	0
C	C	0	0	0	0	0

Table 8: Intersection pairing $A^*(X) \otimes A_*^c(X) \rightarrow A_*^c(X)$

	D_7^c	C_{46}^c	C_{57}^c	C_{34}^c	C_{47}^c	P^c
X	D_7^c	C_{46}^c	C_{57}^c	C_{34}^c	C_{47}^c	P^c
D_1	0	P^c	0	0	0	0
D_2	C_{27}^c	0	P^c	0	0	0
D_3	C_{37}^c	P^c	0	$-P^c$	P^c	0
D_4	C_{47}^c	0	P^c	$-P^c$	0	0
C	P^c	0	0	0	0	0

In table 8 we summarize the intersection pairing.

The Mori cone generators are C_a , $a = 1, \dots, 4$, with

$$C_1 = C_{46}, \quad C_2 = C_{57}, \quad C_3 = C_{34}, \quad C_4 = C_{47}. \tag{4.17}$$

Table 9: Intersection between Kähler generators

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	0	C
T_3	0	0	0	0
T_4	0	C	0	$2C$

The Kähler cone is generated by the dual elements T_a , $a = 1, \dots, 4$ with

$$\begin{aligned} T_1 &= D_1, & T_2 &= D_2, \\ T_3 &= D_2 - D_4, & T_4 &= -D_1 + D_2 + D_3 - D_4. \end{aligned} \tag{4.18}$$

If J_a are the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \dots, J_4] / \sim, \tag{4.19}$$

with \sim given by table 9.

Resolution $X = \mathbb{R}_4 - \mathbb{C}_6^3$ (figure 15): This resolution differs from the G -Hilb by the flop

$$C_{47} \longrightarrow C_{56}. \tag{4.20}$$

We set $C = [C_{56}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 16.

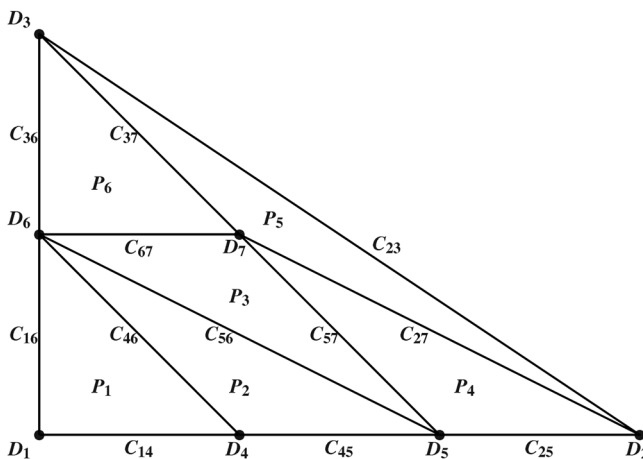


Figure 15: The toric invariant subvarieties of $\mathbb{R}_4 - (\mathbb{C}_6^3)$.

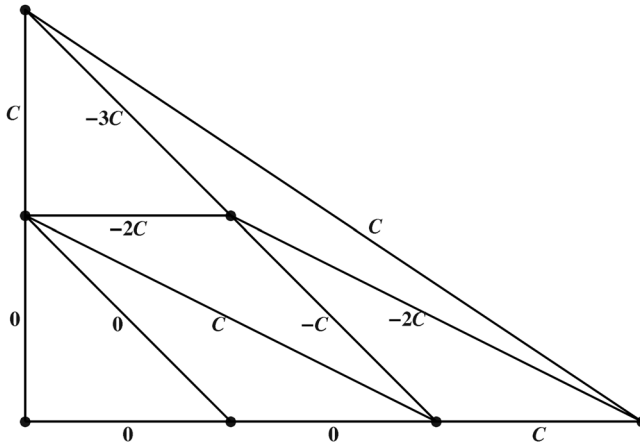


Figure 16: The rational equivalence between curves of $R_4-(\mathbb{C}_6^3)$.

Table 10: Intersection product in $A^*(X)$

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	0	C	0	0
D_3	D_3	0	C	C	0	0
D_4	D_4	0	0	0	0	0
C	C	0	0	0	0	0

In table 10 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

The group of compact subvarieties of X is

$$A_*^c(X) = \mathbb{Z} \cdot D_7^c \oplus \mathbb{Z} \cdot C_{46}^c \oplus \mathbb{Z} \cdot C_{57}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot C_{56}^c \oplus \mathbb{Z} \cdot P^c, \quad (4.21)$$

with relations to other compact curves

$$C_{27}^c = C_{67}^c, \quad (4.22)$$

$$C_{37}^c = C_{57}^c + C_{67}^c. \quad (4.23)$$

In table 11 we summarize the intersection pairing.

Table 11: Intersection pairing $A^*(X) \otimes A_*^c(X) \rightarrow A_*^c(X)$

	D_7^c	C_{46}^c	C_{57}^c	C_{67}^c	C_{56}^c	P^c
X	D_7^c	C_{46}^c	C_{57}^c	C_{67}^c	C_{56}^c	P^c
D_1	0	P^c	0	0	0	0
D_2	C_{27}^c	0	P^c	0	0	0
D_3	C_{37}^c	0	0	P^c	0	0
D_4	0	$-2P^c$	0	0	P^c	0
C	P^c	0	0	0	0	0

The Mori cone generators are C_a , $a = 1, \dots, 4$, with

$$C_1 = C_{46}, \quad C_2 = C_{57}, \quad C_3 = C_{67}, \quad C_4 = C_{56}. \tag{4.24}$$

The Kähler cone is generated by the dual elements T_a , $a = 1, \dots, 4$ with

$$T_1 = D_1, \quad T_2 = D_2, \quad T_3 = D_3, \quad T_4 = 2D_1 + D_4. \tag{4.25}$$

If J_a are the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \dots, J_4] / \sim, \tag{4.26}$$

with \sim given by table 12.

Resolution $X = \mathbb{R}_5 - \mathbb{C}_6^3$ (figure 17): This resolution differs from the G -Hilb by the flops

$$C_{47} \longrightarrow C_{56}, \quad C_{57} \longrightarrow C_{26}. \tag{4.27}$$

We set $C = [C_{56}]$ and we report the relations between any other toric curve and C in the decorated fan of figure 18.

Table 12: Intersection between Kähler generators

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	C	0
T_3	0	C	C	0
T_4	0	0	0	0

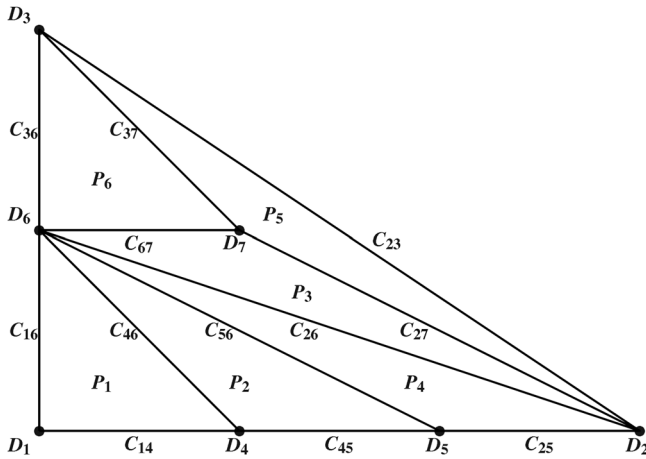


Figure 17: The toric invariant subvarieties of $R_5-(\mathbb{C}_6^3)$.

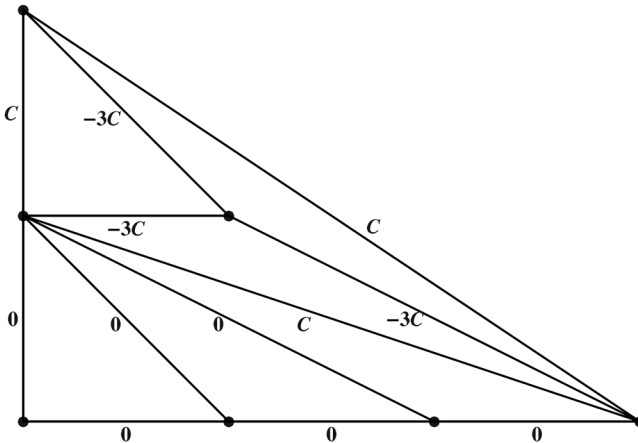


Figure 18: The rational equivalence between curves of $R_5-(\mathbb{C}_6^3)$.

In table 13 we summarize the intersection products, which give the relations R in the Chow ring $A^*(X) = \mathbb{Z}[X, D_1, D_2, D_3, D_4, C]/R$.

The group of compact subvarieties of X is

$$A_*^c(X) = \mathbb{Z} \cdot D_7^c \oplus \mathbb{Z} \cdot C_{46}^c \oplus \mathbb{Z} \cdot C_{26}^c \oplus \mathbb{Z} \cdot C_{67}^c \oplus \mathbb{Z} \cdot C_{56}^c \oplus \mathbb{Z} \cdot P^c, \tag{4.28}$$

with relations to other compact curves

$$C_{27}^c = C_{67}^c, \tag{4.29}$$

$$C_{37}^c = C_{67}^c. \tag{4.30}$$

Table 13: Intersection product in $A^*(X)$

	X	D_1	D_2	D_3	D_4	C
X	X	D_1	D_2	D_3	D_4	C
D_1	D_1	0	0	0	0	0
D_2	D_2	0	C	C	0	0
D_3	D_3	0	C	C	0	0
D_4	D_4	0	0	0	0	0
C	C	0	0	0	0	0

Table 14: Intersection pairing $A^*(X) \otimes A_*^c(X) \rightarrow A_*^c(X)$

	D_7^c	C_{46}^c	C_{26}^c	C_{67}^c	C_{56}^c	P^c
X	D_7^c	C_{46}^c	C_{26}^c	C_{67}^c	C_{56}^c	P^c
D_1	0	P^c	0	0	0	0
D_2	C_{27}^c	0	$-P^c$	P^c	P^c	0
D_3	C_{37}^c	0	0	P^c	0	0
D_4	0	$-2P^c$	0	0	P^c	0
C	P^c	0	0	0	0	0

In table 14 we summarize the intersection pairing.

The Mori cone generators are $C_a, a = 1, \dots, 4$, with

$$C_1 = C_{46}, \quad C_2 = C_{26}, \quad C_3 = C_{67}, \quad C_4 = C_{56}. \tag{4.31}$$

The Kähler cone is generated by the dual elements $T_a, a = 1, \dots, 4$ with

$$\begin{aligned} T_1 &= D_1, & T_2 &= 2D_1 - D_2 + D_3 + D_4, \\ T_3 &= D_3, & T_4 &= 2D_1 + D_4. \end{aligned} \tag{4.32}$$

If we call J_a the Kähler generators in $H^2(X, \mathbb{Q})$ corresponding to the T_a then the cohomology ring is

$$H^*(X, \mathbb{Q}) = \mathbb{Q}[J_1, \dots, J_4]/\sim, \tag{4.33}$$

with \sim given by table 15.

4.2 K-theory generators

We have seen that G -Hilb is the moduli space of G -cluster in \mathbb{C}^3 . The natural generalization of G -cluster is G -constellation. For a finite group

Table 15: Intersection between Kähler generators

	T_1	T_2	T_3	T_4
T_1	0	0	0	0
T_2	0	0	0	0
T_3	0	0	C	0
T_4	0	0	0	0

$G \subset GL(n, \mathbb{C})$, a G -constellation is a G -equivariant coherent sheaf F on \mathbb{C}^n with global sections $H^0(F)$ isomorphic as a $C[G]$ -module to the regular representation R of G . Set

$$\Theta := \{\theta \in \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(R) = 0\},$$

where $R(G)$ is the representation ring of G . This is an hyperplane in \mathbb{Q}^r , where r is the order of G . For $\theta \in \Theta$, a G -constellation F is said to be θ -stable (or θ -semistable) if every proper G -equivariant coherent subsheaf $0 \subset E \subset F$ satisfies $\theta(E) > 0$ (or $\theta(E) \geq 0$). The moduli space \mathcal{M}_θ of θ -stable constellations is constructed using GIT (cf. [43]). The space Θ is subdivided into polyhedral convex cones C called GIT chamber. Given θ and θ' in the same chamber C the moduli spaces \mathcal{M}_θ and $\mathcal{M}_{\theta'}$ are isomorphic, so we write \mathcal{M}_C in place of \mathcal{M}_θ for any $\theta \in C$. Ito and Nakajima [36] observed that $G\text{-Hilb} = \mathcal{M}_{C_0}$ for some chamber $C_0 \subset \Theta$ and more generally the method of [5] shows that for any chamber $C \subset \Theta$ there is a crepant resolution $\tau : \mathcal{M}_C \rightarrow \mathbb{C}^3/G$ and an equivalence of $\Phi_C : D(\mathcal{M}_C) \rightarrow D^G(\mathbb{C}^3)$ between derived categories of coherent sheaves on \mathcal{M}_C and derived categories of G -equivariant sheaves on \mathbb{C}^3 . Craw and Ishii [12] proved that in the Abelian case every crepant resolution may be realized as a moduli space \mathcal{M}_C for some chamber. Moreover, they uncovered explicit equivalence between the derived categories of moduli \mathcal{M}_θ for parameters lying in adjacent GIT chambers. Therefore starting from $G\text{-Hilb}$ and by analysing the chamber structure of Θ , we can define the tautological bundles \mathcal{R}_ρ that generate $K(X)$ on every flopped resolution X and that are the Fourier–Mukai transforms of the original tautological bundles on $G\text{-Hilb}$.

Here we summarize how calculate the chamber structure of Θ and the transformation induced by crossing the walls W of the chambers C . We refer to [12, 14] for detailed explanations. The derived equivalence Φ_C induces a \mathbb{Z} -linear isomorphism

$$\varphi_C : K^c(\mathcal{M}_C) \rightarrow R(G), \quad \sum a_i \mathcal{S}_i \longmapsto \bigoplus a_i \rho_i, \tag{4.34}$$

where as usual \mathcal{S}_i is the element of the basis of $K^c(\mathcal{M}_C)$ dual of the tautological bundle \mathcal{R}_i and ρ_i is the irreducible representation of character i . Let $C \subset \Theta$ be a chamber. Then $\theta \in C$ if and only if

- for every exceptional curve ℓ , we have $\theta(\varphi_C(\mathcal{O}_\ell)) = \sum_i \theta(\rho_i) \deg(\mathcal{R}_{\rho_i}|_\ell) > 0$;¹⁰
- for every compact reduced divisor D^{11} and irreducible representation ρ , we have

$$\theta(\varphi_C(\mathcal{R}_\rho^{-1} \otimes \omega_D)) < 0 \quad \text{and} \quad \theta(\varphi_C(\mathcal{R}_\rho^{-1}|_D)) > 0,$$

where ω_D is the canonical bundle of D .¹²

These inequalities determine the walls of the chamber C , but we have to pay attention that some of them may be redundant.

Let $\theta \in \Theta$ be a generic parameter, C the chamber containing it and θ_0 a parameter on its wall W . The wall is said to be of type 0, I, II or III as follows:

- type 0 if \mathcal{M}_{θ_0} is isomorphic to \mathcal{M}_θ ,
- type I if \mathcal{M}_{θ_0} is obtained from \mathcal{M}_θ by the contraction of a curve to a point,
- type II if \mathcal{M}_{θ_0} is obtained from \mathcal{M}_θ by the contraction of a divisor to a point,
- type III if \mathcal{M}_{θ_0} is obtained from \mathcal{M}_θ by the contraction of a divisor to a curve.

The inequalities coming from curves determine walls of type I or III, while the others determine walls of type 0. There are no walls of type II.

If C' is the chamber behind the wall W , the relation between $\mathcal{M}_{C'}$ and \mathcal{M}_C and their tautological bundles depends on the type of the wall.

- W of type 0: $\mathcal{M}_{\theta'}$ is isomorphic to \mathcal{M}_θ ; the wall $W \subset \Theta$ is the zero locus of an equation of the form $R(\theta_0(\rho_1), \dots, \theta_0(\rho_r)) = a_1\theta_0(\rho_1) + \dots + a_r\theta_0(\rho_r) = 0$ and, if D is the divisor defining the wall, the tautological bundles \mathcal{R}_i and \mathcal{R}'_i are related

¹⁰Recall that if $\mathcal{R}_\rho = \mathcal{O}_X(D')$ then $\deg(\mathcal{R}_\rho|_\ell) = D' \cdot \ell$

¹¹I.e., $D = \sum a_i D_i$ where D_i are compact invariant divisors and the coefficient $a_i \in \{0, 1, -1\}$.

¹²If $\mathcal{R}_\rho = \mathcal{O}_X(D')$ and $\omega_D = \mathcal{O}_D(K_D)$ then $\mathcal{R}_\rho^{-1} \otimes \omega_D = \mathcal{O}_D(-D' \cdot D + K_D)$ and $\mathcal{R}_\rho^{-1}|_D = \mathcal{O}_D(-D' \cdot D)$. Then we calculate the inequalities with the help of (3.40) and (4.38).

as follows:

– Case +: if $R(\theta(\rho_1), \dots, \theta(\rho_r)) > 0$ then

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i & \text{if } a_i = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(D) & \text{if } a_i \neq 0; \end{cases}$$

– Case -: if $R(\theta(\rho_1), \dots, \theta(\rho_r)) < 0$ then

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i & \text{if } a_i = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(-D) & \text{if } a_i \neq 0. \end{cases}$$

- W of type I: $\mathcal{M}_{\theta'}$ is the variety obtained from \mathcal{M}_{θ} by the flop of the curve ℓ determining the wall; the tautological bundles \mathcal{R}'_i are the proper transform of \mathcal{R}_i .
- W of type III: $\mathcal{M}_{\theta'}$ is isomorphic to \mathcal{M}_{θ} ; if D is the divisor contracted in \mathcal{M}_{θ_0} , the tautological bundles \mathcal{R}_i and \mathcal{R}'_i are related as follows:
 - Case +: if $\{\deg(\mathcal{R}_i|_{\ell})\} = \{0, 1\}$ then

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i & \text{if } \deg(\mathcal{R}_i|_{\ell}) = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(D) & \text{if } \deg(\mathcal{R}_i|_{\ell}) = 1; \end{cases}$$

– Case -: if $\{\deg(\mathcal{R}_i|_{\ell})\} = \{0, -1\}$ then

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i & \text{if } \deg(\mathcal{R}_i|_{\ell}) = 0, \\ \mathcal{R}_i \otimes \mathcal{O}_{\mathcal{M}_{\theta'}}(-D) & \text{if } \deg(\mathcal{R}_i|_{\ell}) = -1. \end{cases}$$

Thus, crossing walls of type I induces flops, while walls of type 0 and III induce self-equivalence of the derived category of the resolved variety. One can start from the chamber of the G -Hilb resolution, follow the change of the tautological bundles crossing the walls and reconstruct the chamber structure of Θ .

In our example the tautological bundles for the \mathbb{Z}_6 -Hilb are

$$\begin{aligned} \mathcal{R}_0 &= \mathcal{O}_X, & \mathcal{R}_1 &= \mathcal{O}_X(D_1), & \mathcal{R}_2 &= \mathcal{O}_X(D_2), & \mathcal{R}_3 &= \mathcal{O}_X(D_3), \\ \mathcal{R}_4 &= \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), \\ \mathcal{R}_5 &= \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3). \end{aligned} \tag{4.35}$$

We write parameters θ as $(\theta_0, \dots, \theta_5)$, where $\theta_i := (\theta(\rho_i))$. The inequalities defining the \mathbb{Z}_6 -Hilb chamber are

$$\theta_1 > 0 \quad \text{wall of type I related to the flop of the curve } C_1; \tag{4.36}$$

$$\theta_2 + \theta_5 > 0 \quad \text{wall of type III+ related to the contraction of the divisor } D_5; \tag{4.37}$$

$$\theta_3 + \theta_5 > 0 \quad \text{wall of type I related to the flop of the curve } C_3; \tag{4.38}$$

$$\theta_4 > 0 \quad \text{wall of type I related to the flop of the curve } C_4; \tag{4.39}$$

$$\theta_5 > 0 \quad \text{wall of type 0+ defined by } \theta(\varphi_C(\mathcal{R}_5^{-1}|_{D_7})) > 0; \tag{4.40}$$

$$\theta_2 + \theta_3 + \theta_4 + \theta_5 > 0 \quad \text{wall of type 0+ defined by } \theta(\varphi_C(\mathcal{R}_5^{-1} \otimes \omega_{D_7})) < 0. \tag{4.41}$$

Any other inequality is redundant. As it is proven in Section 9 of [12], the flop of any single curve in the G -Hilb is achieved by crossing a wall of the chamber (generally if we are in a chamber different from the G -Hilb's it may be necessary first cross a type 0 wall to realize a flop).

Resolution $R_2 - \mathbb{C}_6^3$: Starting from the G -Hilb chamber we obtain this resolution by crossing the wall (4.36). The tautological bundles are again

$$\begin{aligned} \mathcal{R}_0 &= \mathcal{O}_X, & \mathcal{R}_1 &= \mathcal{O}_X(D_1), & \mathcal{R}_2 &= \mathcal{O}_X(D_2), & \mathcal{R}_3 &= \mathcal{O}_X(D_3), \\ \mathcal{R}_4 &= \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), \\ \mathcal{R}_5 &= \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3). \end{aligned} \tag{4.42}$$

while the pure D-brane basis is

$$B_0 := \mathcal{O}_p; \quad B_a := \mathcal{O}_{C_a}(-T_a); \quad B_5 := \mathcal{O}_{D_7}(-T_2 - T_3), \tag{4.43}$$

with $a = 1, \dots, 4$. In terms of the \mathcal{R}_i and their duals \mathcal{S}_i , the B_i -basis of $K(X)$ and its dual Φ -basis of $K^c(X)$ are thus:

$$\begin{aligned}
 B_0 &= \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_0 &= \mathcal{R}_0, \\
 B_1 &= -\mathcal{S}_1, & \Phi_1 &= -\mathcal{R}_0 - \mathcal{R}_1 + \mathcal{R}_3 + \mathcal{R}_4, \\
 B_2 &= \mathcal{S}_2 + \mathcal{S}_5, & \Phi_2 &= -\mathcal{R}_0 + \mathcal{R}_2, \\
 B_3 &= \mathcal{S}_1 + \mathcal{S}_3 + \mathcal{S}_5, & \Phi_3 &= -\mathcal{R}_0 + \mathcal{R}_3, \\
 B_4 &= \mathcal{S}_1 + \mathcal{S}_4, & \Phi_4 &= -\mathcal{R}_0 + \mathcal{R}_4, \\
 B_5 &= \mathcal{S}_5. & \Phi_5 &= \mathcal{R}_0 - \mathcal{R}_2 - \mathcal{R}_3 + \mathcal{R}_5.
 \end{aligned} \tag{4.44}$$

The symplectic form in the selected basis is

$$\chi(B_i, B_j) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.45}$$

Resolution $R_3\text{-}\mathbb{C}_6^3$: Starting from the G -Hilb chamber we obtain this resolution by crossing the wall (4.38). The tautological bundles are again

$$\begin{aligned}
 \mathcal{R}_0 &= \mathcal{O}_X, & \mathcal{R}_1 &= \mathcal{O}_X(D_1), & \mathcal{R}_2 &= \mathcal{O}_X(D_2), & \mathcal{R}_3 &= \mathcal{O}_X(D_3), \\
 \mathcal{R}_4 &= \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), & \mathcal{R}_5 &= \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3).
 \end{aligned} \tag{4.46}$$

while the pure D-brane basis is

$$B_0 := \mathcal{O}_p; \quad B_a := \mathcal{O}_{C_a}(-T_a); \quad B_5 := \mathcal{O}_{D_7}(-T_1 - T_2 + T_3 - T_4), \tag{4.47}$$

with $a = 1, \dots, 4$. In terms of the \mathcal{R}_i and their duals \mathcal{S}_i , the B_i -basis of $K(X)$ and its dual Φ -basis of $K^c(X)$ are thus:

$$\begin{aligned}
 B_0 &= \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_0 &= \mathcal{R}_0, \\
 B_1 &= \mathcal{S}_1 + \mathcal{S}_3 + \mathcal{S}_5, & \Phi_1 &= -\mathcal{R}_0 + \mathcal{R}_1, \\
 B_2 &= \mathcal{S}_2 + \mathcal{S}_5, & \Phi_2 &= -\mathcal{R}_0 + \mathcal{R}_2, \\
 B_3 &= -\mathcal{S}_3 - \mathcal{S}_5, & \Phi_3 &= -\mathcal{R}_0 + \mathcal{R}_1 - \mathcal{R}_3 + \mathcal{R}_4, \\
 B_4 &= \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_4 &= -\mathcal{R}_0 + \mathcal{R}_4, \\
 B_5 &= \mathcal{S}_5. & \Phi_5 &= \mathcal{R}_0 - \mathcal{R}_2 - \mathcal{R}_3 + \mathcal{R}_5.
 \end{aligned} \tag{4.48}$$

The symplectic form in the selected basis is

$$\chi(B_i, B_j) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix}. \tag{4.49}$$

Resolution $R_4-\mathbb{C}_6^3$: Starting from the G -Hilb chamber we obtain this resolution by crossing the wall (4.39). The tautological bundles are again

$$\begin{aligned} \mathcal{R}_0 &= \mathcal{O}_X, & \mathcal{R}_1 &= \mathcal{O}_X(D_1), & \mathcal{R}_2 &= \mathcal{O}_X(D_2), & \mathcal{R}_3 &= \mathcal{O}_X(D_3), \\ \mathcal{R}_4 &= \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), & \mathcal{R}_5 &= \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3). \end{aligned} \tag{4.50}$$

while the pure D-brane basis is

$$B_0 := \mathcal{O}_p; \quad B_a := \mathcal{O}_{C_a}(-T_a); \quad B_5 := \mathcal{O}_{D_7}(-T_2 - T_3), \tag{4.51}$$

with $a = 1, \dots, 4$. In terms of the \mathcal{R}_i and their duals \mathcal{S}_i , the B_i -basis of $K(X)$ and its dual Φ -basis of $K^c(X)$ are thus:

$$\begin{aligned} B_0 &= \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_0 &= \mathcal{R}_0, \\ B_1 &= \mathcal{S}_1 + \mathcal{S}_4, & \Phi_1 &= -\mathcal{R}_0 + \mathcal{R}_1, \\ B_2 &= \mathcal{S}_2 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_2 &= -\mathcal{R}_0 + \mathcal{R}_2, \\ B_3 &= \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_3 &= -\mathcal{R}_0 + \mathcal{R}_3, \\ B_4 &= -\mathcal{S}_4, & \Phi_4 &= -\mathcal{R}_0 + \mathcal{R}_1 - \mathcal{R}_4 + \mathcal{R}_5, \\ B_5 &= \mathcal{S}_4 + \mathcal{S}_5. & \Phi_5 &= \mathcal{R}_0 - \mathcal{R}_2 - \mathcal{R}_3 + \mathcal{R}_5. \end{aligned} \tag{4.52}$$

The symplectic form in the selected basis is

$$\chi(B_i, B_j) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{pmatrix}. \tag{4.53}$$

Resolution $R_5-\mathbb{C}_6^3$: Starting from the above chamber of the resolution $R_4-\mathbb{C}_6^3$ we obtain this resolution by crossing a single wall of type I. The

tautological bundles are again

$$\begin{aligned} \mathcal{R}_0 &= \mathcal{O}_X, & \mathcal{R}_1 &= \mathcal{O}_X(D_1), & \mathcal{R}_2 &= \mathcal{O}_X(D_2), & \mathcal{R}_3 &= \mathcal{O}_X(D_3), \\ \mathcal{R}_4 &= \mathcal{O}_X(-D_1 + D_2 + D_3 - D_4), & \mathcal{R}_5 &= \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{O}_X(D_2 + D_3). \end{aligned} \tag{4.54}$$

while the pure D-brane basis is

$$B_0 := \mathcal{O}_p; \quad B_a := \mathcal{O}_{C_a}(-T_a); \quad B_5 := \mathcal{O}_{D_7}(T_2 - 2T_3 - T_4), \tag{4.55}$$

with $a = 1, \dots, 4$. In terms of the \mathcal{R}_i and their duals \mathcal{S}_i , the B_i -basis of $K(X)$ and its dual Φ -basis of $K^c(X)$ are thus:

$$\begin{aligned} B_0 &= \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5, & \Phi_0 &= \mathcal{R}_0, \\ B_1 &= \mathcal{S}_1 + \mathcal{S}_4, & \Phi_1 &= -\mathcal{R}_0 + \mathcal{R}_1, \\ B_2 &= -\mathcal{S}_2 - \mathcal{S}_4 - \mathcal{S}_5, & \Phi_2 &= -\mathcal{R}_0 + \mathcal{R}_1 - \mathcal{R}_2 + \mathcal{R}_3 - \mathcal{R}_4 + \mathcal{R}_5, \\ B_3 &= \mathcal{S}_2 + \mathcal{S}_3 + 2\mathcal{S}_4 + 2\mathcal{S}_5, & \Phi_3 &= -\mathcal{R}_0 + \mathcal{R}_3, \\ B_4 &= \mathcal{S}_2 + \mathcal{S}_5, & \Phi_4 &= -\mathcal{R}_0 + \mathcal{R}_1 - \mathcal{R}_4 + \mathcal{R}_5, \\ B_5 &= \mathcal{S}_4 + \mathcal{S}_5. & \Phi_5 &= \mathcal{R}_0 - \mathcal{R}_2 - \mathcal{R}_3 + \mathcal{R}_5. \end{aligned} \tag{4.56}$$

The symplectic form in the selected basis is

$$\chi(B_i, B_j) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 & 0 \end{pmatrix}. \tag{4.57}$$

4.3 The cohomological hypergeometric series and GW-invariants

The hypergeometric series are specified by the ℓ vectors corresponding to large Kähler parameters and the hypergeometric coefficients are determined expanding them with respect to the basis $Q_i = \text{ch}(\Phi_i)$ of $H^*(X, \mathbb{Q})$.

Invariants for $R_2 - \mathbb{C}_6^3$: The vectors ℓ_a , $a = 1, \dots, 4$ are

$$\begin{aligned} C_1 &: & \ell_1 &= (-1, 0, 0, 1, 0, 1, -1), \\ C_2 &: & \ell_2 &= (0, 1, 0, 1, -2, 0, 0), \\ C_3 &: & \ell_3 &= (1, 0, 1, 0, 0, -2, 0), \\ C_4 &: & \ell_4 &= (1, 0, 0, -2, 1, 0, 0). \end{aligned} \tag{4.58}$$

The selected basis of the cohomology is

$$\begin{aligned} Q_0 &= 1, & Q_1 &= J_1 - 2C, & Q_2 &= J_2, \\ Q_3 &= J_3 - \frac{1}{2}C, & Q_4 &= J_4 - C, & Q_5 &= -C. \end{aligned} \tag{4.59}$$

If we make this change of basis and use the mirror symmetry identification

$$w\left(\vec{x}, \frac{\vec{J}}{2\pi i}\right) = Q_0 1 + \sum_{a=1}^4 Q_a t_a + Q_5 g(t_1, \dots, t_4), \tag{4.60}$$

then we find

$$\begin{aligned} 2\pi i t_1 &= \log x_1 - \Psi(x_3) + \Phi(x_2, x_4) - \aleph(\vec{x}), \\ 2\pi i t_2 &= \log x_2 + \Phi(x_2, x_4) - 2\Phi(x_4, x_2), \\ 2\pi i t_3 &= \log x_3 + 2\Psi(x_3), \\ 2\pi i t_4 &= \log x_4 - 2\Phi(x_2, x_4) + \Phi(x_4, x_2), \end{aligned} \tag{4.61}$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2} \phi(\vec{t}), \tag{4.62}$$

where P_2 is the degree two polynomial part

$$\begin{aligned} P_2(\vec{t}) &= -2t_1 - \frac{1}{2}t_3 - t_4 + 3t_1^2 + \frac{1}{2}t_3^2 + t_4^2 \\ &\quad + 2t_1t_2 + 3t_1t_3 + 4t_1t_4 + t_2t_3 + t_2t_4 + 2t_3t_4, \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} \phi(\vec{t}) &= 6\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(6)}(\vec{x}) - \Lambda_4(\vec{x}) - \Lambda_5(\vec{x}) \\ &\quad + \Psi_6(x_2, x_4) - 2\Psi_4(x_3) + 2\Psi_5(x_3) + 2\Psi_1(x_2, x_4) + 2\Psi_1(x_4, x_2) \\ &\quad - \Psi_2(x_2, x_4) - \Psi_2(x_4, x_2) - \Psi_3(x_2, x_4) - \Psi_3(x_4, x_2) \\ &\quad - 3\aleph^2(\vec{x}) + \Psi^2(x_3) + \Phi^2(x_2, x_4) + \Phi^2(x_4, x_2) \\ &\quad - \Phi(x_2, x_4)\Phi(x_4, x_2), \end{aligned} \tag{4.64}$$

with \vec{x} expressed as a function of \vec{t} by inverting system (4.61), is the part corresponding to instantonic contributions. Following Hosono and using

(4.45) we find

$$(-\partial_{t_1})F(\vec{t}) = g(\vec{t}), \tag{4.65}$$

where F is the prepotential. Setting

$$q_k := e^{2\pi i t_k}, \tag{4.66}$$

we then find

$$\begin{aligned} F(\vec{t}) &= t_1^2 + \frac{1}{2}t_1t_3 - t_1t_4 - t_1^3 - \frac{1}{2}t_1t_3^2 - t_1t_4^2 - t_1^2t_2 - \frac{3}{2}t_1^2t_3 - 2t_1^2t_4 \\ &\quad - t_1t_2t_3 - t_1t_2t_4 - 2t_1t_3t_4 + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_2, t_3, t_4) \\ &\quad + Q_{\text{inst}}(q_2, q_3, q_4). \end{aligned} \tag{4.67}$$

P and Q are the undetermined parts. We list the Gromov–Witten invariants for rational curves up to degree six. The curves with degree in the integer cone generated by $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$\begin{aligned} GW_{[1,0,0,0]} &= GW_{[1,0,1,0]} = GW_{[1,0,0,1]} \\ &= GW_{[1,0,1,1]} = GW_{[1,1,0,1]} = GW_{[1,1,1,1]} = 1; \\ GW_{[2,0,1,1]} &= GW_{[2,1,1,1]} = GW_{[2,1,1,2]} = -2. \end{aligned} \tag{4.68}$$

Invariants for $R_3-\mathbb{C}_6^3$: The vectors ℓ_a , $a = 1, \dots, 4$ are

$$\begin{aligned} C_1 &: \ell_1 = (1, 0, 1, 0, 0, -2, 0), \\ C_2 &: \ell_2 = (0, 1, 0, 1, -2, 0, 0), \\ C_3 &: \ell_3 = (0, 0, -1, -1, 0, 1, 1), \\ C_4 &: \ell_4 = (0, 0, 1, 0, 1, 0, -2). \end{aligned} \tag{4.69}$$

The selected basis of the cohomology is

$$\begin{aligned} Q_0 &= 1, & Q_1 &= J_1, & Q_2 &= J_2, & Q_3 &= J_3, & Q_4 &= J_4 + C, \\ Q_5 &= C. \end{aligned} \tag{4.70}$$

Making use of the mirror symmetry identification

$$w\left(\vec{x}, \frac{\vec{J}}{2\pi i}\right) = Q_0 1 + \sum_{a=1}^4 Q_a t_a + Q_5 g(t_1, \dots, t_4), \tag{4.71}$$

then we find

$$\begin{aligned} 2\pi i t_1 &= \log x_1 + 2\Psi(x_1), \\ 2\pi i t_2 &= \log x_2 - \Phi(x_2, x_1 x_3^2 x_4) + 2\Phi(x_1 x_3^2 x_4, x_2), \\ 2\pi i t_3 &= \log x_3 - \Psi(x_1) + \aleph(\vec{x}), \\ 2\pi i t_4 &= \log x_4 - \Phi(x_1 x_3^2 x_4, x_2) - 2\aleph(\vec{x}), \end{aligned} \tag{4.72}$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2} \phi(\vec{t}), \tag{4.73}$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{6} - t_4 + t_4^2 + t_2 t_4, \tag{4.74}$$

and

$$\begin{aligned} \phi(\vec{t}) &= 8\aleph^{(1)}(\vec{x}) - 4\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - \aleph^{(4)}(\vec{x}) + \Lambda_6(\vec{x}) - 2\Lambda_7(\vec{x}) + \Lambda_8(\vec{x}) \\ &\quad + \Psi_3(x_2, x_1 x_3^2 x_4) + \Psi_2(x_1 x_3^2 x_4, x_2) + \Psi_3(x_1 x_3^2 x_4, x_2) \\ &\quad - 2\Psi_1(x_1 x_3^2 x_4, x_2) - 4\aleph^2(\vec{x}) - 2\aleph(\vec{x})\Phi(x_2, x_1 x_3^2 x_4) \\ &\quad - \Phi(x_2, x_1 x_3^2 x_4)\Phi(x_1 x_3^2 x_4, x_2) + \Phi^2(x_1 x_3^2 x_4, x_2), \end{aligned} \tag{4.75}$$

with \vec{x} expressed as a function of \vec{t} by inverting system (4.72), is the part corresponding to instantonic contributions. Using (4.49) we find

$$(\partial_{t_3} - 2\partial_{t_4})F(\vec{t}) = g(\vec{t}), \tag{4.76}$$

where F is the prepotential. Setting

$$q_k := e^{2\pi i t_k}, \tag{4.77}$$

we then find

$$\begin{aligned} F(\vec{t}) &= \frac{1}{6}t_3 + \frac{1}{4}t_4^2 - \frac{1}{6}t_4^3 - \frac{1}{4}t_2 t_4^2 + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_1, t_2, 2t_3 + t_4) \\ &\quad + Q_{\text{inst}}(q_1, q_2, q_3^2 q_4). \end{aligned} \tag{4.78}$$

P and Q are the undetermined parts. We list the GW -invariants up to degree six. The curves with degree in the integer cone generated by $[1, 0, 0, 0]$,

$[0, 1, 0, 0]$, $[0, 0, 2, 1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$\begin{aligned}
 GW_{[0,0,1,0]} &= GW_{[0,0,1,1]} = GW_{[0,1,1,1]} \\
 &= GW_{[1,0,1,0]} = GW_{[1,0,1,1]} = GW_{[1,1,1,1]} = 1; \\
 GW_{[0,0,0,1]} &= GW_{[0,1,0,1]} = GW_{[1,1,2,2]} = -2; \\
 GW_{[0,1,1,2]} &= GW_{[1,1,1,2]} = 3; \\
 GW_{[0,1,0,2]} &= -4; \quad GW_{[0,1,1,3]} = GW_{[1,1,1,3]} = GW_{[0,2,1,3]} = 5; \\
 GW_{[0,1,0,3]} &= GW_{[0,2,0,3]} = -6; \quad GW_{[0,1,1,4]} = 7; \quad GW_{[0,1,0,4]} = -8; \\
 GW_{[0,1,0,5]} &= -10; \quad GW_{[0,2,0,4]} = -32.
 \end{aligned}
 \tag{4.79}$$

Invariants for $R_4 - \mathbb{C}_6^3$: The vectors ℓ_a , $a = 1, \dots, 4$ are

$$\begin{aligned}
 C_1 &: \quad \ell_1 = (1, 0, 0, -2, 1, 0, 0), \\
 C_2 &: \quad \ell_2 = (0, 1, 0, 0, -1, 1, -1), \\
 C_3 &: \quad \ell_3 = (0, 0, 1, 0, 1, 0, -2), \\
 C_4 &: \quad \ell_4 = (0, 0, 0, 1, -1, -1, 1).
 \end{aligned}
 \tag{4.80}$$

The selected basis of the cohomology is

$$\begin{aligned}
 Q_0 &= 1, \quad Q_1 = J_1, \quad Q_2 = J_2, \\
 Q_3 &= J_3 + \frac{1}{2}C, \quad Q_4 = J_4, \quad Q_5 = C.
 \end{aligned}
 \tag{4.81}$$

Via the mirror symmetry identification

$$w \left(\vec{x}, \frac{\vec{J}}{2\pi i} \right) = Q_0 1 + \sum_{a=1}^4 Q_a t_a + Q_5 g(t_1, \dots, t_4),
 \tag{4.82}$$

we get

$$\begin{aligned}
 2\pi i t_1 &= \log x_1 + 2\Phi(x_1, x_2 x_4) - \Phi(x_2 x_4, x_1), \\
 2\pi i t_2 &= \log x_2 + \Phi(x_2 x_4, x_1) - \Psi(x_1 x_3 x_4^2) - \aleph(\vec{x}), \\
 2\pi i t_3 &= \log x_3 - \Phi(x_2 x_4, x_1) - 2\aleph(\vec{x}), \\
 2\pi i t_4 &= \log x_4 + \Psi(x_1 x_3 x_4^2) - \Phi(x_1, x_2 x_4) + \Phi(x_2 x_4, x_1) + \aleph(\vec{x}),
 \end{aligned}
 \tag{4.83}$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2} \phi(\vec{t}),
 \tag{4.84}$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{6} - \frac{1}{2}t_3 + \frac{1}{2}t_3^2 + t_2 t_3,
 \tag{4.85}$$

and

$$\begin{aligned}
 \phi(\vec{t}) = & 8\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 2\aleph^{(3)}(\vec{x}) - 2\aleph^{(5)}(\vec{x}) - \aleph^{(7)}(\vec{x}) \\
 & - \Lambda_9(\vec{x}) + \Lambda_{10}(\vec{x}) - 2\Lambda_{11}(\vec{x}) \\
 & - \Psi_1(x_2x_4, x_1) + \Psi_2(x_2x_4, x_1) + \Psi_5(x_1x_3x_4^2) \\
 & - 4\aleph^2(\vec{x}) - 2\aleph(\vec{x})\Psi(x_1x_3x_4^2) - \aleph(\vec{x})\Phi(x_2x_4, x_1) \\
 & - \Psi(x_1x_3x_4^2)\Phi(x_2x_4, x_1) + \frac{1}{2}\Phi^2(x_2x_4, x_1)
 \end{aligned} \tag{4.86}$$

with \vec{x} expressed as a function of \vec{t} by inverting system (4.83), is the part corresponding to instantonic contributions. Using (4.53) we find

$$(\partial_{t_4} - \partial_{t_2} - 2\partial_{t_3})\bar{F}(\vec{t}) = g(\vec{t}), \tag{4.87}$$

where F is the prepotential. Setting

$$q_k := e^{2\pi i t_k}, \tag{4.88}$$

we then find

$$\begin{aligned}
 F(\vec{t}) = & \frac{1}{6}t_4 + \frac{1}{8}t_3^2 - \frac{1}{12}t_3^3 - \frac{1}{2}t_2^2t_3 + \frac{1}{3}t_2^3 \\
 & + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(t_1, t_4 + t_2, 2t_4 + t_3) \\
 & + Q_{\text{inst}}(q_1, q_4q_2, q_4^2q_3),
 \end{aligned} \tag{4.89}$$

P and Q being the undetermined parts. We list the GW -invariants up to degree six. The curves with degree in the integer cone generated by $[0, 1, 0, 1]$, $[0, 0, 1, 2]$, $[1, 0, 0, 0]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$\begin{aligned}
 & GW_{[0,0,0,1]} = GW_{[0,0,1,1]} = GW_{[0,1,0,0]} \\
 = & GW_{[1,0,0,1]} = GW_{[1,0,1,1]} = GW_{[1,1,1,2]} = 1; \\
 & GW_{[0,0,1,0]} = GW_{[0,1,1,1]} = GW_{[1,1,1,1]} = -2; \\
 & GW_{[0,1,1,0]} = GW_{[1,1,2,2]} = 3; \\
 & GW_{[0,1,2,1]} = GW_{[1,1,2,1]} = -4; \\
 & GW_{[0,1,2,0]} = GW_{[0,2,2,1]} = GW_{[1,2,2,1]} = 5; \\
 & GW_{[0,1,3,1]} = GW_{[0,2,2,0]} = GW_{[1,1,3,1]} = -6; \quad GW_{[0,1,3,0]} = 7; \\
 & GW_{[0,1,4,1]} = -8; \quad GW_{[0,1,4,0]} = 9; \quad GW_{[0,1,5,0]} = 11; \\
 & GW_{[0,3,3,0]} = 27; \\
 & GW_{[0,2,3,0]} = -32; \quad GW_{[0,2,3,1]} = 35; \quad GW_{[0,2,4,0]} = -110.
 \end{aligned} \tag{4.90}$$

Invariants for $R_5 - \mathbb{C}_6^3$: The vectors ℓ_a , $a = 1, \dots, 4$ are

$$C_1 \quad : \quad \ell_1 = (1, 0, 0, -2, 1, 0, 0),$$

$$\begin{aligned}
 C_2 & : \quad \ell_2 = (0, -1, 0, 0, 1, -1, 1), \\
 C_3 & : \quad \ell_3 = (0, 1, 1, 0, 0, 1, -3), \\
 C_4 & : \quad \ell_4 = (0, 1, 0, 1, -2, 0, 0).
 \end{aligned}
 \tag{4.91}$$

The selected basis of the cohomology is

$$\begin{aligned}
 Q_0 &= 1, \quad Q_1 = J_1, \quad Q_2 = J_2, \\
 Q_3 &= J_3 + \frac{1}{2}C, \quad Q_4 = J_4, \quad Q_5 = C.
 \end{aligned}
 \tag{4.92}$$

If we make this change of basis and use the mirror symmetry identification

$$w \left(\vec{x}, \frac{\vec{J}}{2\pi i} \right) = Q_0 1 + \sum_{a=1}^4 Q_a t_a + Q_5 g(t_1, \dots, t_4),
 \tag{4.93}$$

we get

$$\begin{aligned}
 2\pi i t_1 &= \log x_1 + 2\Phi(x_1, x_4) - \Phi(x_4, x_1), \\
 2\pi i t_2 &= \log x_2 + \Psi(x_1 x_2^3 x_3 x_4^2) - \Phi(x_4, x_1) - \aleph(\vec{x}), \\
 2\pi i t_3 &= \log x_3 - \Psi(x_1 x_2^3 x_3 x_4^2) + 3\aleph(\vec{x}), \\
 2\pi i t_4 &= \log x_4 - \Phi(x_1, x_4) + 2\Phi(x_4, x_1),
 \end{aligned}
 \tag{4.94}$$

and

$$g(\vec{t}) = P_2(\vec{t}) + \frac{1}{(2\pi i)^2} \phi(\vec{t}),
 \tag{4.95}$$

where P_2 is the degree two polynomial part

$$P_2(\vec{t}) = \frac{1}{4} - \frac{1}{2}t_3 + \frac{1}{2}t_3^2,
 \tag{4.96}$$

and

$$\begin{aligned}
 \phi(\vec{t}) &= 9\aleph^{(1)}(\vec{x}) - 3\aleph^{(2)}(\vec{x}) - 3\aleph^{(3)}(\vec{x}) - 3\aleph^{(5)}(\vec{x}) + \Lambda_{12}(\vec{x}) + \Lambda_{13}(\vec{x}) \\
 &\quad + \Psi_4(x_1 x_2^3 x_3 x_4^2) + \Psi_5(x_1 x_2^3 x_3 x_4^2) \\
 &\quad - \frac{1}{2}\Psi^2(x_1 x_2^3 x_3 x_4^2) - \frac{9}{2}\aleph^2(\vec{x}) + 3\Psi(x_1 x_2^3 x_3 x_4^2)\aleph(\vec{x})
 \end{aligned}
 \tag{4.97}$$

with \vec{x} expressed as a function of \vec{t} by inverting system (4.94), is the part corresponding to instantonic contributions. Using (4.57) we find

$$(\partial_{t_1} - 3\partial_{t_3})F(\vec{t}) = g(\vec{t}),
 \tag{4.98}$$

where F is the prepotential. Setting

$$q_k := e^{2\pi i t_k}, \tag{4.99}$$

we then find

$$F(\vec{t}) = \frac{1}{4}t_1 + \frac{1}{12}t_2^2 - \frac{1}{18}t_3^3 + F_{\text{inst}}(\vec{q}) + P_{\text{class}}(3t_1 + t_3, t_2, t_4) + Q_{\text{inst}}(q_1^3 q_3, q_2, q_4). \tag{4.100}$$

We list the GW -invariants up to degree six. The curves with degree in the integer cone generated by $[0, 3, 1, 0]$, $[1, 0, 0, 0]$, $[0, 0, 0, 1]$ must be excluded, because corresponding to the undetermined part of the prepotential. The only nonvanishing invariants in the considered range are

$$\begin{aligned} GW_{[0,1,0,0]} &= GW_{[0,1,0,1]} = GW_{[1,2,1,1]} = GW_{[1,2,1,2]} = GW_{[1,1,0,1]} = 1; \\ GW_{[0,1,1,0]} &= GW_{[1,1,1,1]} = -2; \quad GW_{[0,0,1,0]} = 3; \\ GW_{[0,2,2,1]} &= GW_{[1,2,2,1]} = -4; \\ GW_{[0,1,2,0]} &= GW_{[0,1,2,1]} = GW_{[1,1,2,1]} = 5; \\ GW_{[0,0,2,0]} &= GW_{[0,1,2,0]} = -6; \quad GW_{[0,2,3,0]} = 7; \quad GW_{[0,0,3,0]} = 27; \\ GW_{[0,1,3,0]} &= GW_{[0,1,4,1]} = GW_{[0,1,4,0]} = -32; \quad GW_{[0,2,3,1]} = 35; \\ GW_{[0,2,4,0]} &= -110; \quad GW_{[0,0,4,0]} = -192; \\ GW_{[0,1,4,0]} &= GW_{[0,1,4,1]} = 286; \\ GW_{[0,0,5,0]} &= 1695; \quad GW_{[0,1,5,1]} = 3038; \quad GW_{[0,0,6,0]} = -17,064. \end{aligned} \tag{4.101}$$

5 Conclusions

As stated in the introduction, this paper is the first one of a short series devoted to a detailed analysis of some aspects of local (homological) mirror symmetry in relation to its physical meaning. As much of such a project results to be quite technical, we preferred to begin with a preparatory article, where we mainly fix our notations and the objects of study. For this reason we tried here to be as elementary as possible and included an introductory section which obviously does not pretend to be neither exhaustive nor self-contained. However, we also included the first step of our analysis, that is the application of local mirror symmetry to the construction of a prepotential accounting for the lower genus Gopakumar–Vafa invariants (which we simply called the Gromov–Witten invariants or GW -invariants). In particular, for this purpose, we have adopted a particularly elegant way introduced by Hosono in [30] and which we dubbed the “Hosono conjecture”. Our results can be thus interpreted also as a positive (partial) check of the Hosono conjecture for the case of an orbifold with multiple resolutions.

Indeed, we applied the Hosono conjecture to an orbifold model admitting five distinct crepant resolutions, showing that, for each resolution, it partially determines a prepotential encoding information about the Gromov–Witten invariants. As we seen, not all GW -invariant are determined. Indeed, it is not even clear how they could be defined as some ambiguity is introduced by noncompactness of the varieties considered. However, we can note that for all resolutions, the only noncomputable invariants are the one associated to curves having zero intersection with the compact divisor D_7 . Curves having negative intersection with D_7 cannot deform out of D_7 . When they have nonnegative intersection with all the other (noncompact) divisors, then we expect for the invariant numbers to count the number of deformations in D_7 . However, when the intersection with some of the noncompact divisors is negative, then the deformations are constrained on the intersection between the divisors and we expect for the G – W numbers to vanish.

The invariants predicted by the prepotential agree with the ones computed directly by means of the methods described in [9]. In place of repeating such computations here, we will simply compare some of the invariants of our examples with the ones provided in [9]. Let us start with resolution five. It contains a \mathbb{P}^2 associated the compact divisor D_7 and the curves C_{27} , C_{37} and C_{67} , all equivalent. Then, let us fix $b := C_3 \equiv C_{67}$. It has intersection -3 with D_7 so that it is the null section of the normal bundle of D_7 in X . It also has intersection numbers 1 with D_3 , D_5 , D_6 and 0 with the other noncompact divisors. Thus it freely deforms out from all the noncompact divisors, in the sense discussed above, and then one expects that the number of its deformations is just the number of deformations inside \mathbb{P}^2 . From the list (4.101) we see that, up to degree six, the corresponding numbers are $GW[d] = GW[0, 0, d, 0]$ with

$$\begin{aligned} GW[1] &= 3, & GW[2] &= -6, & GW[3] &= 27, \\ GW[4] &= -192, & GW[5] &= 1695, & GW[6] &= -17,064, \end{aligned} \tag{5.1}$$

which indeed coincide with the GW -numbers of $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$, see table 1 in [9].

Let us move to the fourth resolution. It contains the Hirzebruch surface \mathbb{F}_1 associated to D_7 and the curves C_{27} , C_{37} , C_{57} , C_{67} . The independent curves are $b := C_2 \equiv C_{57}$ and $f := C_3 \equiv C_{67}$ which define the base and the fibre of the Hirzebruch fibration. Note that b has intersection -1 with D_5 so that we expect for its eventual deformations in \mathbb{F}_1 to be constrained. This does not happens for f or for all combinations $[d_B, d_F] \equiv [0, d_B, d_F, 0]$ with $d_F \geq d_B$ which have negative intersections with D_7 only. Thus we again

expect for the GW -invariants corresponding to $[d_B, d_F]$ to be the same as in F_1 . Indeed from table 10 in [9] we see that deformations appears for $K_{\mathbb{F}_1}$ only for $d_F \geq d_B$ (apart from the case $[1, 0]$). We can see that our results, as listed in (4.90), are in perfect agreement

$$\begin{aligned} GW[1, 0] &= 1, & GW[0, 1] &= -2, & GW[1, 1] &= 3, & GW[1, 2] &= 5, \\ GW[2, 2] &= -6, & GW[1, 3] &= 7, & GW[1, 4] &= 9, & GW[1, 5] &= 9, \\ GW[3, 3] &= 27, & GW[2, 3] &= -32, & GW[2, 4] &= -110, & & \end{aligned} \tag{5.2}$$

and $GW[i, j] = GW[0, k] = 0$ for $j < i$, $i > 2$, $i + j \leq 6$ and $k = 2, 3, 4, 5$.

A similar comparison can be done for resolution three. In that case, D_7 and the curves C_{27} , C_{37} , C_{47} and C_{57} define an Hirzebruch surface \mathbb{F}_2 with base $b := C_2 \equiv C_{57}$ and fibre $f := C_4 \equiv C_{47}$. Note that b has intersection 0 with D_7 so that curves $[d_B, 0]$ are not countable. These correspond to the first column of table 11 in [9]. Next curves $[d_B, d_F] = [0, d_B, 0, d_F]$ with positive d_F are computable in D_7 , but only for $d_F > d_B$ their intersections with D_i are negative only when they intersect the compact divisor. We then expect for the curves $[d_B, d_B + 1 + k]$ to determine the same numbers as for $K_{\mathbb{F}_2}$, whereas for $d_F \leq d_B$ they can be constrained by the fact they have negative intersection with D_5 also. However, as follows from table 11 in [9] all such numbers vanish (excluding the case $[d_B, d_F] = [1, 1]$) and again our results, collected in (4.79), agree with the numbers of $K_{\mathbb{F}_2}$

$$\begin{aligned} GW[1, 1] &= -2, & GW[1, 2] &= -4, & GW[1, 3] &= GW[2, 3] &= -6, \\ GW[1, 4] &= -8, & GW[1, 5] &= -10, & GW[2, 4] &= -32, & \end{aligned} \tag{5.3}$$

and $GW[i, j] = 0$ for all the remaining ones up to degree 6 (and with $j \neq 0$).

All these are only a part of the numbers predicted by means of the Hosono construction. Indeed, these are the ones corresponding to curves having negative intersection number with the compact divisor and thus admitting a representant contained in it. However, as yet remarked, the constructed potential results to determine much more numbers and in particular we note that, for all cases, the only noncomputable numbers are the ones associated to curves having null intersection with the compact divisor. Actually, the true meaning of these facts are not completely clear to us and deserve a deeper analysis, which is left as part of a future work. One way to proceed in such a direction is to search for an extended GKZ system whose solutions permit to extend the computation of the invariants to all curves, as proposed for example in [17, 18]. This also should provide a slight improvement of the

Hosono conjecture. Such analysis are actually under investigation. Furthermore, Hosono conjecture goes beyond the determination of the prepotential (or the central charge), involving the monodromy properties of the hypergeometric components and a concrete determination of the mirror map at list at the K -theoretical level, and partial information on the homological mirror map Mir . The multiple resolutions of our example, corresponding to a single mirror family, are related by flop transformations and must be related by Fourier–Mukai transforms at the level of derived categories (see Section 4.2). In this contest it could be helpful to find the solutions of our GKZ system in the full B -moduli space using the approach of [2, 6] and [28]. This is the deeper aspect of the conjecture and will be discussed in the third paper.

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Appendix A The cohomology valued hypergeometric series

Here we compute the coefficient hypergeometric series.

A.1 Computation of the coefficients

First note that $J_i^3 = 0$ so that we need the terms up to order two. Also at order zero is survives only the term with $\vec{m} = \vec{0}$, because for nonpositive integer argument the Γ function diverges. Thus, at order zero $w = 1$.

A.1.1 Some properties of the Gamma function

The Euler Gamma function has integral representation

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{Re}(z) > 0, \quad (\text{A.1})$$

and admits analytical continuation to the whole complex plane excluding the nonpositive integers. Indeed, it admit the very useful Weierstrass representation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}, \tag{A.2}$$

where

$$\gamma = \lim_{N \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} - \log(N + 1)\right) \sim 0.5772156649 \dots \tag{A.3}$$

is the Euler–Mascheroni constant.

From (A.2) it follows easily the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + 1/2). \tag{A.4}$$

Another useful function is the Psi function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{A.5}$$

From (A.2)

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)} \Rightarrow \psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \zeta_z(2). \tag{A.6}$$

Here $\zeta_a(z) = \zeta(a; z)$ is the usual Hurwitz Zeta function. In particular $\psi'(1) = \pi^2/6$ and, if N is a nonnegative integer, using these relations we have

(1)

$$\frac{1}{\Gamma(N+1)} = \frac{1}{N!}, \quad \frac{1}{\Gamma(-N)} = 0;$$

(2)

$$\partial_\rho \frac{1}{\Gamma(1 + N + a\rho)} \Big|_{\rho=0} = \frac{a}{N!} \psi(N + 1) = \frac{a}{N!} \left(\gamma - 1 - \frac{1}{2} - \dots - \frac{1}{N} \right)$$

if $N \neq 0$, and

$$\partial_\rho \frac{1}{\Gamma(1 + a\rho)} \Big|_{\rho=0} = a\gamma;$$

also

$$\partial_\rho \frac{1}{\Gamma(-N + \rho)} \Big|_{\rho=0} = (-1)^N N!.$$

(3)

$$\partial_\rho \partial_\sigma \frac{1}{\Gamma(1 + N + a\rho + b\sigma)} \Big|_{\rho=\sigma=0} = \frac{ab}{N!} (\psi(N + 1)^2 - \psi'(N + 1));$$

In particular for $N = 0$

$$\partial_\rho \partial_\sigma \frac{1}{\Gamma(1 + a\rho + b\sigma)} \Big|_{\rho=\sigma=0} = ab \left(\gamma^2 - \frac{\pi^2}{6} \right);$$

(4)

$$\partial_\rho \partial_\sigma \frac{1}{\Gamma(-N + a\rho + b\sigma)} \Big|_{\rho=\sigma=0} = -2ab(-1)^N N! \psi(N + 1).$$

A.1.2 Order one

To compute the coefficients at order one, we can distinguish three cases:

- *The derivative acts on the numerator:* This gives a term $\log x$ only, because the sum contributes only with the term $\vec{m} = 0$;
- *The derivative acts on a factor of the form $1/\Gamma(N + 1)$:* Then the remaining factors force again \vec{m} to zero and by (2) we see that it contribute with a factor $a\gamma$. There is one such factor for any Gamma factor, and all sum up to zero. This is due to the fact that for any fixed ℓ the sum of its components vanishes;
- *The main contributions come out when the derivative act on a factor $1/\Gamma(-N + 1)$:* In this case such factor does not contribute to limiting the allowed values for \vec{m} , which is no more constrained to zero.

In conclusion, all the results can be expressed in terms of the following functions:

$$\Psi(x) = \sum_{n=1}^{\infty} \frac{(2n-1)!}{(n!)^2} x^n, \tag{A.7}$$

$$\Phi(x, y) = \sum_{\substack{(m, k) \in \mathbb{Z}_{\geq} \\ (m, k) \neq (0, 0)}} \frac{(2k+3m-1)!}{m!(k+2m)!k!} (-x)^m y^{k+2m}, \tag{A.8}$$

$$\begin{aligned} \mathfrak{K}(\vec{x}) = - \sum_{\substack{\vec{n} \in \mathbb{Z}_{\geq}^4 \\ \vec{n} \neq \vec{0}}} & \frac{(6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)!}{n_1!n_2!n_3!n_4!(2n_1 + n_2 + n_4)!(3n_1 + 2n_2 + n_3 + n_4)!} \\ & \cdot (x_1 x_2^2 x_3^3 x_4^4)^{n_1} (x_2 x_3^2 x_4^2)^{n_2} (x_3 x_4)^{n_3} (-x_2 x_3 x_4^2)^{n_4}. \end{aligned} \tag{A.9}$$

A.1.3 Order two

The second order term is obtained applying the second order operator

$$O_2 = \frac{1}{2} \partial_{\rho_3}^2 + \partial_{\rho_4}^2 + \partial_{\rho_2 \rho_3}^2 + \partial_{\rho_2 \rho_4}^2 + 2 \partial_{\rho_3 \rho_4}^2 \tag{A.10}$$

at $\vec{\rho} = 0$. In this case there are several contributions:¹³

- both derivatives acts on the numerator in the terms of the series. This gives rise to terms of the form

$$(\log x_i)^2, \quad \log x_i \log x_j;$$

- one derivative acts on the numerator and the other one acts on the Gamma factors. This gives terms of the form

$$\log x_i w_j^{(1)},$$

where $w_j^{(1)}$ is one of the first order terms computed before;

- both derivatives acts on two regular Gamma factors. These can be two distinct factors or the same factor. In both the cases it contributes only the $\vec{m} = 0$ term. For two distinct factors, we will find a contribution proportional to γ^2 and for the same factor one finds a term proportional to $\gamma^2 - \pi^2/6$. A simple argumentation similar to the first order case

¹³For simplicity we will call *regular* the Gamma factors with positive argument, and *singular* the Gamma factors with negative argument.

shows that the terms in γ^2 sum up to zero. Thus we expect only a term proportional to π^2 ;

- one derivative acts on a regular term and the other one acts on a singular term. This gives rise to a contribution very similar to (A.7), (A.8), where the terms of the series are corrected by a multiplicative factor of the form $\psi(N + 1)$;
- both derivatives acts on the same singular term. This gives a contribution very similar to the previous point;
- the derivatives acts on two distinct singular Gamma terms. These give the more complicated series, because there are minimal constrictions for the range of \vec{m} in the sums.

A.1.4 Second order functions

Here we collect the functions which appear in the second order terms of the cohomological hypergeometric functions. When the ranges $\vec{m} \in \mathbb{Z}_{>}^4$ are intended to be restricted to the subsets where all factorials and psi functions are well defined. The results can then be expressed in terms of the following 26 functions:

$$\Psi_1(x, y) = \sum_{\substack{(m, k) \in \mathbb{Z}_{\geq} \\ (m, k) \neq (0, 0)}} \frac{(2k + 3m - 1)!}{m!(k + 2m)!k!} [\psi(2k + 3m) - \psi(1)](-x)^m y^{k+2m}; \tag{A.11}$$

$$\Psi_2(x, y) = \sum_{\substack{(m, k) \in \mathbb{Z}_{\geq} \\ (m, k) \neq (0, 0)}} \frac{(2k + 3m - 1)!}{m!(k + 2m)!k!} [\psi(1 + k + 2m) - \psi(1)](-x)^m y^{k+2m}; \tag{A.12}$$

$$\Psi_3(x, y) = \sum_{\substack{(m, k) \in \mathbb{Z}_{\geq} \\ (m, k) \neq (0, 0)}} \frac{(2k + 3m - 1)!}{m!(k + 2m)!k!} [\psi(1 + k) - \psi(1)](-x)^m y^{k+2m}; \tag{A.13}$$

$$\Psi_4(x) = \sum_{m=1}^{\infty} \frac{(2m - 1)!}{(m!)^2} [\psi(2m) - \psi(1)]x^m; \tag{A.14}$$

$$\Psi_5(x) = \sum_{m=1}^{\infty} \frac{(2m - 1)!}{(m!)^2} [\psi(m + 1) - \psi(1)]x^m; \tag{A.15}$$

$$\Psi_6(x, y) = \sum_{\substack{(m, n) \in \mathbb{Z}_{\geq}^2 \\ 2n - m > 0 \\ 2m - n > 0}} \frac{(2n - m - 1)!(2m - n - 1)!}{m!n!} (-x)^m (-y)^n; \tag{A.16}$$

$$\mathcal{N}^{(i)}(\vec{x}) = \sum_{\substack{\vec{n} \in \mathbb{Z}_{\geq}^4 \\ \vec{n} \neq 0}} \frac{(x_1 x_2^2 x_3^3 x_4^4)^{n_1} (x_2 x_3^2 x_4^2)^{n_2} (x_3 x_4)^{n_3} (-x_2 x_3 x_4^2)^{n_4}}{n_1! n_2! n_3! n_4! (2n_1 + n_2 + n_4)! (3n_1 + 2n_2 + n_3 + n_4)!} \chi_{\vec{n}}^{(i)}, \quad i = 1, \dots, 7; \tag{A.17}$$

$$\chi_{\vec{n}}^{(1)} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(6n_1 + 4n_2 + 2n_3 + 3n_4) - \psi(1)]; \tag{A.18}$$

$$\chi_{\vec{n}}^{(2)} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + 3n_1 + 2n_2 + n_3 + n_4) - \psi(1)]; \tag{A.19}$$

$$\chi_{\vec{n}}^{(3)} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + 2n_1 + n_2 + n_4) - \psi(1)]; \tag{A.20}$$

$$\chi_{\vec{n}}^{(4)} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_2) - \psi(1)]; \tag{A.21}$$

$$\chi_{\vec{n}}^{(5)} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_4) - \psi(1)]; \tag{A.22}$$

$$\chi_{\vec{n}}^{(6)} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_1) - \psi(1)]; \tag{A.23}$$

$$\chi_{\vec{n}}^{(7)} = (6n_1 + 4n_2 + 2n_3 + 3n_4 - 1)! [\psi(1 + n_3) - \psi(1)]; \tag{A.24}$$

$$\Lambda_1(\vec{x}) = \sum_{\substack{\vec{n} \in \mathbb{Z}_{\geq}^4 \\ n_1 + n_3 \neq 0}} \frac{n_1! n_2! n_3! n_4! (n_1 + 2n_2 + n_3 + n_4)!}{(n_1 + 3n_2 + 2n_4)! (n_1 + 2n_3)! x_1^{n_1}}. \tag{A.25}$$

$$\Lambda_2(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{\geq}^4} \frac{(m_1 - m_2 - m_3 + m_4 - 1)! (m_3 + m_4 - m_1 - 1)!}{m_1! m_2! m_3! (m_4 - 2m_2)! (m_1 - m_3 - m_4)!} \cdot (-x_1^2 x_2 x_4^2)^{m_2} (x_1 x_3)^{m_3} (x_1 x_4)^{m_4}; \tag{A.26}$$

$$\Lambda_3(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{\geq}^4} \frac{(m_1 + m_3 - m_4 - 1)! (m_3 + m_4 - m_1 - 1)!}{m_1! m_2! m_3! (m_2 + m_3 - m_1 - m_4)! (m_4 - 2m_2)!} \cdot x_1^{m_1} (-x_2)^{m_2} x_3^{m_3} x_4^{m_4}; \tag{A.27}$$

$$\Lambda_4(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_1 - m_3 - m_4 - 1)!(m_1 - 1)!}{m_2!m_3!(m_1 + m_2 - 2m_4)!(m_4 - 2m_2)!(m_1 - 2m_3)!} \cdot (A.28)$$

$$\cdot x_1^{m_1} x_2^{m_2} (-x_3)^{m_3} (-x_4)^{m_4};$$

$$\Lambda_5(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^3} \frac{(2m_3 - m_1 - 1)!(2m_2 - 1)!}{m_1!m_2!(m_2 + m_3)!(m_3 - 2m_1)!} \cdot (A.29)$$

$$\cdot (-x_2)^{m_1} x_3^{m_2} (-x_4)^{m_3};$$

$$\Lambda_6(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_3 - m_4 - m_1 - 1)(m_3 - m_2 - 1)!}{m_1!m_2!(m_4 - 2m_2)!(m_3 - 2m_2)!(m_3 - 2m_4)!} \cdot (A.30)$$

$$\cdot (-x_1)^{m_1} (-x_2)^{m_2} x_3^{m_3} (-x_4)^{m_4};$$

$$\Lambda_7(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_3 - m_2 - 1)!(2m_4 - m_3 - 1)!}{m_1!m_2!(m_1 - m_3 + m_4)!(m_4 - 2m_2)!(m_3 - 2m_2)!} \cdot (A.31)$$

$$\cdot x_1^{m_1} (-x_2)^{m_2} x_3^{m_3} x_4^{m_4};$$

$$\Lambda_8(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_3 - m_2 - 1)!(2m_2 - m_4 - 1)!}{m_1!m_2!(m_1 - m_3 + m_4)!(m_3 - 2m_2)!(m_3 - 2m_4)!} \cdot (A.32)$$

$$\cdot x_1^{m_1} (-x_2)^{m_2} (-x_3)^{m_3} (-x_4)^{m_4};$$

$$\Lambda_9(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_4 - m_3 + m_2 - m_1 - 1)!(2m_3 + m_2 - m_4 - 1)!}{m_1!m_2!m_3!(m_4 - 2m_1)!(m_2 - m_4)!} \cdot (A.33)$$

$$\cdot (-x_1)^{m_1} x_2^{m_2} (-x_3)^{m_3} x_4^{m_4};$$

$$\Lambda_{10}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_4 - m_3 + m_2 - m_1 - 1)!(m_4 - m_2 - 1)!}{m_1!m_2!m_3!(m_4 - 2m_1)!(m_4 - 2m_3 - m_2)!} \cdot (A.34)$$

$$\cdot (-x_1)^{m_1} x_2^{m_2} (-x_3)^{m_3} x_4^{m_4};$$

$$\Lambda_{11}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_4 - m_2 - 1)!(2m_3 + m_2 - m_4 - 1)!}{m_1!m_2!m_3!(m_4 - 2m_1)!(m_1 + m_3 - m_2 - m_4)!} \cdot (A.35)$$

$$\cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4};$$

$$\Lambda_{12}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_2 - m_3 - m_4 - 1)!(m_2 - m_3 - 1)!}{m_1!m_3!(m_4 - 2m_1)!(m_1 + m_2 - 2m_4)!(m_2 - 3m_3)!} \cdot (A.36)$$

$$\cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} (-x_4)^{m_4};$$

$$\Lambda_{13}(\vec{x}) = \sum_{\vec{m} \in \mathbb{Z}_{>}^4} \frac{(m_2 - m_3 - 1)!(3m_3 - m_2 - 1)!}{m_1!m_3!(m_3 + m_4 - m_2)!(m_4 - 2m_1)!(m_1 + m_2 - 2m_4)!} \cdot (A.37)$$

$$\cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}.$$

References

- [1] P. S. Aspinwall, *D-branes on Calabi–Yau manifolds*, arXiv:hep-th/0403166.
- [2] M. Aganagic, V. Bouchard and A. Klemm, *Topological strings and (almost) modular forms*, Commun. Math. Phys. **277** (2008), 771 [arXiv:hep-th/0607100].
- [3] V. Bouchard and R. Cavalieri, *On the mathematics and physics of high genus invariants of $[C^3/Z_3]$* , arXiv:0709.3805 [math.AG].
- [4] T. Bridgeland, *Stability conditions on a non-compact Calabi–Yau three-fold*, Commun. Math. Phys. **266** (2006), 715.
- [5] T. Bridgeland, A. King, and M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), 535–554.
- [6] A. Brini and A. Tanzini, *Exact results for topological strings on resolved $Y(p, q)$ singularities*, arXiv:0804.2598 [hep-th].
- [7] P. Candelas, X. C. De la Ossa, P. S. Green and L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi–Yau manifolds*, Phys. Lett. B **258** (1991), 118.
- [8] A. Canonaco, *Exceptional sequences and derived autoequivalences*, arXiv:0801.0173.
- [9] T.-T. Chiang, A. Klemm, S.-T. Yau and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, Adv. Theor. Math. Phys. **3** (1999), 495.
- [10] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), 17–50.
- [11] A. Craw, *An explicit construction of the McKay correspondence for A -Hilb*, J. Algebra **285**, (2005), 682–705.
- [12] A. Craw and A. Ishii, *Flops of G -Hilb and equivalences of derived categories by variation of GIT quotient*, Duke Math. J. **124** (2004), 259–307.
- [13] A. Craw and M. Reid, *How to calculate A -Hilb \mathbb{C}^3* , Geometry of toric varieties, 129–154, Smin. Congr., **6**, Soc. Math. France, Paris, 2002.
- [14] A. Degeratu, *Flops of crepant resolutions*, Turkish J. Math. **28** (1), (2004), 23–40.
- [15] X. De la Ossa, B. Florea and H. Skarke, *D-branes on noncompact Calabi–Yau manifolds: K -theory and monodromy*, Nucl. Phys. B **644** (2002), 170.
- [16] M. R. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, J. Math. Phys. **42** (2001), 2818.

- [17] B. Forbes and M. Jinzenji, *Extending the Picard–Fuchs system of local mirror symmetry*, J. Math. Phys. **46** (2005), 082302.
- [18] B. Forbes and M. Jinzenji, *Prepotentials for local mirror symmetry via Calabi–Yau fourfolds*, JHEP **0603** (2006), 061.
- [19] K. Fukaya, *Morse homotopy, $A1$ -category and Floer homologies*, MSRI preprint No.020-94, 1993.
- [20] W. Fulton, *Intersection theory*, 2nd ed., Springer-Verlag, Berlin, 1998.
- [21] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, **131**, Princeton University Press, Princeton, NJ, 1993.
- [22] I. M. Gelfand, A. V. Zelevinski and M. M. Kapranov, Equations of hypergeometric type and toric varieties, *Funktsional Anal. i. Prilozhen.* **23** (1989), 12–26; English transl. Functional Anal. Appl. **23** (1989), 94–106.
- [23] R. Gopakumar and C. Vafa, *M-theory and topological strings. I*, arXiv:hep-th/9809187.
- [24] R. Gopakumar and C. Vafa, *M-theory and topological strings. II*, arXiv:hep-th/9812127.
- [25] M. B. Green, J. A. Harvey and G. W. Moore, *I-brane inflow and anomalous couplings on D-branes*, Class. Quant. Grav. **14** (1997), 47.
- [26] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, arXiv:hep-th/0005247.
- [27] K. Hori *et al.*, *Mirror symmetry*, AMS, Providence, USA, 2003, 929p.
- [28] P. Horja, *Hypergeometric functions and mirror symmetry in toric varieties*, [math.AG/9912109].
- [29] S. Hosono, *GKZ system, Gröbner fans, and moduli spaces of Calabi–Yau hypersurfaces*, in Topological field theory, primitive forms and related topics, Birkhäuser, Boston 1998.
- [30] S. Hosono, *Central charges, symplectic forms, and hypergeometric series in local mirror symmetry*, arXiv:hep-th/0404043.
- [31] S. Hosono, *Local mirror symmetry and type IIA monodromy of Calabi–Yau manifolds*, Adv. Theor. Math. Phys. **4** (2000), 335 [arXiv:hep-th/0007071].
- [32] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, *Mirror symmetry, mirror map and applications to Calabi–Yau hypersurfaces*, Commun. Math. Phys. **167** (1995), 301.
- [33] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi–Yau spaces*, Nucl. Phys. B **433** (1995), 501.

- [34] S. Hosono, B. H. Lian and S. T. Yau, *GKZ generalized hypergeometric systems in mirror symmetry of Calabi–Yau hypersurfaces*, Commun. Math. Phys. **182** (1996), 535.
- [35] S. Hosono, B. H. Lian and S. T. Yau, *Maximal degeneracy points of GKZ systems*, arXiv:alg-geom/9603014.
- [36] Y. Ito and H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, Topology **39** (2000), 1155–1191.
- [37] B. Iversen, *Local Chern classes*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 155–169.
- [38] R. L. Karp, *On the C^n/Z_m fractional branes*, arXiv:hep-th/0602165.
- [39] M. Kontsevich, *Homological algebra of mirror symmetry*, alg-geom/9411018.
- [40] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, Springer-Verlag, Berlin, 1988.
- [41] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Nonperturbative results on the point particle limit of $N = 2$ heterotic string compactifications*, Nucl. Phys. B **459** (1996), 537.
- [42] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. P. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory*, Nucl. Phys. B **477** (1996) 746.
- [43] A. Sardo-Infirri, *Resolutions of orbifold singularities and the transportation problem on the McKay quiver*, math.AG/-9610005, 1996.
- [44] A. Strominger, *Special geometry*, Commun. Math. Phys. **133** (1990), 163.
- [45] E. Witten, *D-branes and K-theory*, JHEP **9812** (1998), 019.

