

Topological membranes with 3-form H flux on generalized geometries

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Abstract

We construct topological string and topological membrane actions with a nontrivial 3-form flux H in arbitrary dimensions. These models realize Bianchi identities with a nontrivial H flux as consistency conditions. Especially, we discuss the models with a generalized $SU(3)$ structure, a generalized G_2 structure and a generalized $Spin(7)$ structure. These models are constructed from the AKSZ formulation of the Batalin–Vilkovisky formalism.

1 Introduction

Recently, flux vacua in superstring theory have been studied. Some superstring background geometries with a nonzero NS–NS 3-form flux $H = d_M b \neq 0$ are called generalized geometries, which were proposed by Hitchin as a generalization of Calabi–Yau geometry [1]. Also, $N = (2, 2)$ supersymmetric sigma model with a nontrivial NS B -field has a bi-Hermitian structure [2], which was proved to be equivalent to a generalized Kähler structure [3]. By use of topological twisting, topological sigma model with a generalized Calabi–Yau structure was constructed [4–7]. Other types of topological sigma models with a generalized complex structure were proposed and analyzed in [8–15]. Moreover, topological M -theory with a G_2 structure on a 7-manifold [16–19] and topological F -theory with a $\text{Spin}(7)$ structure on an 8-manifold [20] were studied. In [21–23], by studying current algebra, topological membrane models with $\text{SU}(3)$, G_2 and $\text{Spin}(7)$ structures were studied.

As a next step after these good works, we would need to construct topological string and topological membrane theory with $H \neq 0$. In this paper, we construct topological membrane actions on generalized geometries, including topological M - and topological F -theory with H -flux. Then we use the definitions of generalized exceptional structures, which were introduced by Witt [24, 25] as generalizations of G_2 and $\text{Spin}(7)$ structures. Here generalized G_2 and $\text{Spin}(7)$ structures mean to add a B -field. These generalized exceptional structures appear in background geometries of flux compactifications in type II superstring and M -theory [26–28]. In this paper, for examples, we show that generalized $\text{SU}(3)$, G_2 and $\text{Spin}(7)$ structures appear from consistency conditions of our actions. Moreover, dimensional reductions of these models are studied. For example, generalized topological G_2 models can be related to two kinds of generalized topological $\text{SU}(3)$ models, which are called A-type and B-type. Also, Batalin–Vilkovisky (BV) formalism [29, 30] is one of the most general and the most theoretical settings to construct a field theory. Especially, some topological field theories can be systematically formulated by the AKSZ formulation [31] of the BV formalism. In the last part of this paper, we rewrite our actions by using superfield AKSZ formulation of the BV formalism. Then we use an extension of the BV construction of topological theories to any dimensions [32, 33].

This paper is organized as follows. In Section 2, we briefly explain generalized geometry, which means to have an H flux. The definition of generalized structures (2.2) is similar to Bianchi identities for RR fluxes in type II superstring background with an NS–NS flux. (See [34] as a review.) In Section 3, we propose topological sigma models with generalized structures (2.2) on

arbitrary dimensional target spaces M . In Section 4, as some examples, we show that these models have generalized $SU(3)$ structures on 6-manifolds, generalized G_2 structures on 7-manifolds and generalized $Spin(7)$ structures on 8 manifolds. In Section 5, we discuss two kinds of dimensional reductions. Generally, we derive lower dimensional generalized structures from higher dimensional ones. In Section 6, we propose an extension of the AKSZ construction of the BV formalism. In Section 7, we rewrite topological sigma models with generalized structures by use of superfield BV formalism.

2 Geometries with 3-form flux

In this section, we briefly summarize geometries with H flux.

We define a geometry with H flux, which is a closed 3-form. Let M be a manifold in d dimensions with local coordinates $\{\phi^i\}$. Let $\Lambda(M) = \Lambda^*T^*M$ be the space of all exterior differential forms on M . If $\rho \in \Lambda(M)$ on M satisfies the following equations

$$d_M H = 0, \tag{2.1}$$

$$(d_M + H) \wedge \rho = 0, \tag{2.2}$$

where d_M is an exterior differentiation on M , we call a *generalized (geometry) structure*. ρ is expanded as formal sum of p -forms as

$$\rho = \sum_{p=0}^d \rho_p, \tag{2.3}$$

where ρ_p is a p -form on a target space M , i.e., $\rho_p = \frac{1}{p!} \rho_{p,i_1 \dots i_p}(\phi) d\phi^{i_1} \dots d\phi^{i_p}$. After substituting (2.3) by (2.2), equation (2.2) is separated in to two independent sets of equations. The equations for even p -forms are

$$d_M \rho_0 = 0, \quad d_M \rho_2 + H \wedge \rho_0 = 0, \quad d_M \rho_4 + H \wedge \rho_2 = 0, \quad \dots \tag{2.4}$$

The equations for odd p -forms are

$$d_M \rho_1 = 0, \quad d_M \rho_3 + H \wedge \rho_1 = 0, \quad d_M \rho_5 + H \wedge \rho_3 = 0, \quad \dots \tag{2.5}$$

Here we call equations (2.4) *A-type*, and equations (2.5) *B-type*. Also, $\Lambda(M)$ can be decomposed to irreducible representation spaces $\Lambda^{\text{ev}}(M)$ and $\Lambda^{\text{od}}(M)$ for spinor representation. $\Lambda^{\text{ev}}(M)$ is the sum of representation spaces of even forms. $\Lambda^{\text{od}}(M)$ is the sum of representation spaces of odd forms. Here we call $\rho \in \Lambda^{\text{ev}}(M)$ which satisfies (2.2) as *A-type* or even type, and $\rho \in \Lambda^{\text{od}}(M)$ which satisfies (2.2) as *B-type* or odd type. Especially, on a 6-manifold M , we call ($\rho \in \Lambda^{\text{ev}}(M)$) an A-type generalized $SU(3)$ structure, and $\rho \in \Lambda^{\text{od}}(M)$ a B-type generalized $SU(3)$ structure. On a 7-manifold M , we call

$\rho \in \Lambda^{\text{ev}}(M)$ an A-type generalized G_2 structure, and $\rho \in \Lambda^{\text{od}}(M)$ a B-type generalized G_2 structure. On an 8-manifold M , we call $\rho \in \Lambda^{\text{ev}}(M)$ an A-type generalized Spin(7) structure, and $\rho \in \Lambda^{\text{od}}(M)$ a B-type generalized Spin(7) structure. We will see that these examples of generalized structures appear from our models.

3 Topological membranes with 3-form flux

In this section, we propose membrane action whose target space has a generalized structure.

We consider a smooth map $\{\phi^i\}$ from an n -dimensional worldvolume Σ_n to a target manifold M in d dimensions. Let ρ be a formal sum of forms on M . Then ρ can be expanded for the form degrees as $\rho = \sum_{p=0}^d \rho_p$, where ρ_p is a p -form on M . By use of local coordinates, ρ_p can be represented as $\rho_p = \frac{1}{p!} \rho_{p,i_1 \dots i_p}(\phi) d\phi^{i_1} \dots d\phi^{i_p}$, where d is an exterior differentiation on Σ_n . Also, let $H = H_{ijk} d\phi^i d\phi^j d\phi^k$ be an NS 3-form which satisfies $d_M H = 0$. From closeness of H , H can locally be expressed by a 2-form B -field b as $H = \frac{1}{2} d_M b$, where $b = b_{ij} d\phi^i d\phi^j$.

Here, in order to construct sigma models, we introduce an auxiliary field Y_2 , which is a 2-form on Σ_n and a scalar on M , i.e., $Y_2 \in \wedge^2 T^* \Sigma_n$. Then we propose following action:

$$S = \int_{\Sigma_n} e^{Y_2} e^{1/2b} \rho = \sum_{p=0}^d \frac{1}{p!} \int_{\Sigma_n} e^{Y_2} e^{1/2b} \rho_{p,i_1 \dots i_p}(\phi) d\phi^{i_1} \dots d\phi^{i_p}. \tag{3.1}$$

Note that enough higher orders of the exponentials vanish because Σ_n has a finite number of dimensions. Also, Y_2 and b are even forms, so if n is even or odd, only even or odd parts of ρ_p remain. This means that the part $\int_{\Sigma_n} e^{Y_2}$ projects ρ ($\in \Lambda(M)$) to $\Lambda^{\text{od}}(M)$ or $\Lambda^{\text{ev}}(M)$. This action can be topological if enough gauge symmetry exists to exclude all degree freedom of ϕ^i .

Next we consider this condition that gauge symmetry remains. First, we define canonical conjugate momenta for ϕ^i and Y_2 :

$$\begin{aligned} \pi_{\phi^i} &= e^{Y_2} e^{1/2b} \left(b_{ij} d\phi^j \rho + \sum_{p=1}^d \frac{1}{(p-1)!} \rho_{p,i j_2 \dots j_p}(\phi) d\phi^{j_2} \dots d\phi^{j_p} \right), \\ \pi_{Y_2} &= 0. \end{aligned} \tag{3.2}$$

We define gauge symmetry generators G_i :

$$G_i = \pi_{\phi_i} - e^{Y_2} e^{1/2b} \left(b_{ij} d\phi^j \rho + \sum_{p=1}^d \frac{1}{(p-1)!} \rho_{p,i j_2 \dots j_p}(\phi) d\phi^{j_2} \dots d\phi^{j_p} \right),$$

$$G_{Y_2} = \pi_{Y_2}. \quad (3.3)$$

Then this system has the constraints $G_i \approx 0$ and $G_{Y_2} \approx 0$ on the phase space $(\phi^i, Y_2, \pi_{\phi_i}, \pi_{Y_2})$. We require that these constraints should be the first class in order to make a consistent theory, i.e., $\{G_I, G_J\} \approx 0$, where $\{\cdot, \cdot\}$ is the Poisson bracket on the phase space. Here G_I is (G_i, G_{Y_2}) , and \approx means that the equation is satisfied on the constraint surface. This condition is

$$\frac{\partial \rho_{p+2, i_1 \dots i_{p+2}}(\phi)}{\partial \phi^j}(\phi) + \frac{3!}{(p+2)(p+1)} H_{[j i_{p+1} i_{p+2}}(\phi) \rho_{p, i_1 \dots i_p]}(\phi) = 0. \quad (3.4)$$

Note that this condition is just equal to the generalized structure condition (2.2). The gauge transformation is derived from the Poisson bracket with G_i . The number of G_i is equal to the number of ϕ^i . If the gauge symmetry exists, this model is topological. Therefore, from the condition that this action is topological, we can get the generalized structure condition on the target geometry with 3-form H flux. We can construct topological membrane theory on generalized geometry.

Moreover, we can rewrite this action using a first-order formalism. We introduce four auxiliary fields. The first one is a 1-form A_1^i such that $A_1^i = d\phi^i$. The second is an $n-1$ -form field $B_{n-1, i}$ as a Lagrange multiplier field. The others are an $n-2$ -form $B_{n-2, i}$ and an $n-3$ -form Z_{n-3} , which are introduced in order to make kinetic terms for A_1^i and Y_2 , like $(-1)^{n-1} B_{n-2, i} dA_1^i - Y_2 dZ_{n-3}$. These kinetic terms are consistent with the equations of motion, and do not change constraints (3.4). Then we can write the following first-order action:

$$S = \int_{\Sigma_n} (-1)^n B_{n-1, i} d\phi^i + (-1)^{n-1} B_{n-2, i} dA_1^i - Y_2 dZ_{n-3} + (-1)^{n-1} B_{n-1, i} A_1^i$$

$$+ \exp(Y_2) \exp\left(\frac{1}{2}(b_{ij} A_1^i A_1^j)\right) \left(\sum_{p=0}^d \frac{1}{p!} \rho_{p, i_1 \dots i_p}(\phi) A_1^{i_1} \dots A_1^{i_p} \right), \quad (3.5)$$

where $(-1)^n$ is a convention for sign factor. We will analyze this first-order action by use of the BV formalism in the latter part.

4 Topological string, topological M and topological F theories on generalized geometries

In this section, as examples of our models, we consider the cases that the target spaces have six, seven and eight dimensions. We call these topological models as topological string (membrane), topological M - or topological F -theories, respectively.

4.1 Topological strings and membranes on generalized $SU(3)$ -manifolds

First as the first example, we consider the case of topological membrane theory with a generalized $SU(3)$ structure on a 6-dimensional target space M . Let M be a 6-dimensional manifold.

We shortly review A model and B model in the case of $H = 0$. The A model is constructed from a Kähler structure ρ_2 on M . The A model action is

$$S = \int_{\Sigma_2} \frac{1}{2!} \rho_{2,ij} d\phi^i d\phi^j. \quad (4.1)$$

The first-class constraint is $d_M \rho_2 = 0$. That is, we can see that ρ_2 is a symplectic structure on M . Also, B model is constructed from a holomorphic 3-form ρ_3 on 6-dimensional manifold M , which defines a complex structure on M . Let Σ_3 be a worldvolume in three dimensions. The 2-brane action of B model is

$$S = \int_{\Sigma_3} \frac{1}{3!} \rho_{3,ijk} d\phi^i d\phi^j d\phi^k. \quad (4.2)$$

The first-class constraint $d_M \rho_3 = 0$ is the integrability condition of the holomorphic form. This action is a dual membrane formulation of the B model [22].

As the first example of our model, we consider A model with $H \neq 0$. We choose $n = 2$. Our action (3.1) is

$$S = \int_{\Sigma_2} e^{Y_2} e^{1/2b} \rho = \int_{\Sigma_2} \left[Y_2 \rho_0 + \frac{1}{2!} (\rho_{2,ij} + b_{ij} \rho_0) d\phi^i d\phi^j \right]. \quad (4.3)$$

Here we will check that ρ is a generalized structure (2.2). The consistency conditions (3.4) for $\rho \in \Lambda^{\text{ev}}(M)$ are

$$\begin{aligned} d_M \rho_0 &= 0, \\ d_M \rho_2 + H \wedge \rho_0 &= 0. \end{aligned} \quad (4.4)$$

From these two conditions and $d_M H = 0$, we obtain the equation $d_M(H \wedge \rho_2) = 0$. Therefore, from the Poincaré Lemma, at least locally ρ_4 exists, s.t.

$$d_M \rho_4 + H \wedge \rho_2 = 0. \tag{4.5}$$

Also, the equation

$$d_M \rho_6 + H \wedge \rho_4 = 0, \tag{4.6}$$

is trivially satisfied in six dimensions. Therefore, A-type conditions (4.4), (4.5) and (4.6) for a generalized SU(3) structure are satisfied on the 6-manifold.

Second, we consider B model with $H \neq 0$. We choose $n = 3$ in (3.1). The action is

$$S = \int_{\Sigma_3} e^{Y_2} e^{1/2b} \rho = \int_{\Sigma_3} \left[Y_2 \rho_{1,i} d\phi^i + \frac{1}{3!} (\rho_{3,ijk} + b_{[ij} \rho_{1,k]}) d\phi^i d\phi^j d\phi^k \right]. \tag{4.7}$$

The first-class constraints (3.4) for $\rho \in \Lambda^{\text{od}}(M)$ are

$$\begin{aligned} d_M \rho_1 &= 0, \\ d_M \rho_3 + H \wedge \rho_1 &= 0. \end{aligned} \tag{4.8}$$

Then since $d_M(H \wedge \rho_3) = 0$, at least locally ρ_5 exists s.t.

$$d_M \rho_5 + H \wedge \rho_3 = 0. \tag{4.9}$$

The conditions (4.8) and (4.9) are B-type conditions for the generalized SU(3) structure on 6-manifold.

Also, we have other realizations of generalized structures on 6-manifold. In the cases $n = 4$ and $n = 6$, we can similarly derive A-type generalized SU(3) conditions on M . In the cases $n = 5$, we can derive B-type generalized SU(3) conditions on M .

4.2 Topological membranes on generalized G_2 -manifolds

Next we consider the case of topological membrane theory with a generalized G_2 structure on a 7-dimensional manifold M .

First, we consider a G_2 structure on the 7-manifold M , namely $H = 0$. A G_2 structure is defined from a 3-form ρ_3 , which satisfies $d_M \rho_3 = 0$ and $d_M * \rho_3 = 0$. A G_2 structure is defined as the subgroup of GL(7) to preserve ρ_3 . A manifold with a G_2 structure is called a G_2 manifold [35]. The

worldvolume action of the topological M -theory, which has a G_2 structure, was proposed as a topological 2-brane [22]. The action is

$$S = \int_{\Sigma_3} \frac{1}{3!} \rho_{3,ijk} d\phi^i d\phi^j d\phi^k, \tag{4.10}$$

which is called self-dual membrane [36, 37]. If a target space M is a G_2 manifold, $d_M \rho_3 = 0$ can be derived from the action (4.10) as the first-class constraint.

We consider the $n = 3$ case in the action (3.1):

$$S = \int_{\Sigma_3} e^{Y_2} e^{1/2b} \rho = \int_{\Sigma_3} \left[Y_2 \rho_{1,i} d\phi^i + \frac{1}{3!} (\rho_{3,ijk} + b_{[ij} \rho_{1,k]}) d\phi^i d\phi^j d\phi^k \right]. \tag{4.11}$$

This action gives B-type generalized G_2 conditions. In fact, the consistency conditions (3.4) are

$$\begin{aligned} d_M \rho_1 &= 0, \\ d_M \rho_3 + H \wedge \rho_1 &= 0. \end{aligned} \tag{4.12}$$

From these two equations, we obtain $d_M(H \wedge \rho_3) = 0$. Using the Poincaré Lemma, at least locally ρ_5 exists

$$d_M \rho_5 + H \wedge \rho_3 = 0. \tag{4.13}$$

Also, the equation

$$d_M \rho_7 + H \wedge \rho_5 = 0, \tag{4.14}$$

is trivially satisfied in seven dimensions. Therefore we obtain (4.12), (4.13) and (4.14) as the B-type conditions for the generalized G_2 structure.

Next we consider a model with an A-type condition for a generalized G_2 structure. For $\hat{\rho} \in \Lambda^{\text{ev}}(M)$, A-type condition is $(d_M + H) \wedge \hat{\rho} = 0$ [24]. Here, we use only the fact that $\hat{\rho}$ satisfies the condition $(d_M + H) \wedge \hat{\rho} = 0$. The model can be constructed as follows. We choose an $n = 4$ worldvolume Σ_4 in (3.1). Here $n = 4$ means to be dual to $n = 3$ in seven dimensions. The action (3.1) is

$$\begin{aligned} S &= \int_{\Sigma_4} e^{Y_2} e^{1/2b} \hat{\rho} \\ &= \int_{\Sigma_4} \left[Y_2^2 \hat{\rho}_0 + \frac{1}{2!} Y_2 (\hat{\rho}_{2,ij} + b_{ij} \hat{\rho}_0) d\phi^i d\phi^j \right. \\ &\quad \left. + \frac{1}{4!} (\hat{\rho}_{4,ijkl} + b_{[ij} \hat{\rho}_{2,kl]}) d\phi^i d\phi^j d\phi^k d\phi^l \right]. \end{aligned} \tag{4.15}$$

The first-class constraints for $\hat{\rho} \in \Lambda^{\text{ev}}(M)$ in (3.4) are

$$\begin{aligned} d_M \hat{\rho}_0 &= 0, \\ d_M \hat{\rho}_2 + H \wedge \hat{\rho}_0 &= 0, \\ d_M \hat{\rho}_4 + H \wedge \hat{\rho}_2 &= 0. \end{aligned} \tag{4.16}$$

Since $d_M(H \wedge \hat{\rho}_4) = 0$ in seven dimensions, we can locally derive the equation

$$d_M \hat{\rho}_6 + H \wedge \hat{\rho}_4 = 0. \tag{4.17}$$

The conditions (4.16) and (4.17) are the A-type conditions for the generalized G_2 structure. If $(d_M + H) \wedge \hat{\rho} = 0$, the action is consistent.

Also, we have other realizations of generalized G_2 structures. If we choose $n = 5$ and $n = 7$ in the action (3.1), we can also derive B-type generalized G_2 conditions on M . If we consider $n = 2$ and $n = 6$ in the action (3.1), we can also derive A-type generalized G_2 conditions on M .

4.3 Topological membranes on generalized Spin(7)-manifolds

Let M be an 8-dimensional manifold. A Spin(7) structure is defined by a Spin(7) 4-form ρ_4 , which satisfies $d_M \rho_4 = 0$ and self-dual condition $\rho_4 = *\rho_4$. A Spin(7) structure is defined as the subgroup of $\text{GL}(8)$ to preserve the Spin(7) form ρ_4 [35]. We can define a consistent topological 3-brane model in a 4-dimensional worldvolume. The action is

$$S = \int_{\Sigma_4} \frac{1}{4!} \rho_{4,ijkl} d\phi^i d\phi^j d\phi^k d\phi^l, \tag{4.18}$$

where ρ_4 is a Spin(7) form. The first-class constraints are equal to the Spin(7) condition $d_M \rho_4 = 0$. This action is considered as a worldvolume description of topological F -theory.

Topological membrane action on a generalized Spin(7) structures is constructed as follows. There are two types of generalized Spin(7) structures, namely A-type and B-type. First we consider an A-type Spin(7) structure, which is defined by the sum of even forms $\rho \in \Lambda^{\text{ev}}(M)$. By setting $n = 4$ in (3.1), topological 3-brane action is

$$\begin{aligned} S &= \int_{\Sigma_4} e^{Y_2} e^{1/2b} \rho \\ &= \int_{\Sigma_4} \left[Y_2 Y_2 \rho_0 + \frac{1}{2!} Y_2 (\rho_{2,ij} + b_{ij} \rho_0) d\phi^i d\phi^j \right. \\ &\quad \left. + \frac{1}{4!} (\rho_{4,ijkl} + b_{[ij} \rho_{2,kl]}) d\phi^i d\phi^j d\phi^k d\phi^l \right]. \end{aligned} \tag{4.19}$$

The first-class constraints are

$$\begin{aligned} d_M \rho_0 &= 0, \\ d_M \rho_2 + H \wedge \rho_0 &= 0, \\ d_M \rho_4 + H \wedge \rho_2 &= 0. \end{aligned} \tag{4.20}$$

From these conditions, we can locally obtain the equations:

$$\begin{aligned} d_M \rho_6 + H \wedge \rho_4 &= 0, \\ d_M \rho_8 + H \wedge \rho_6 &= 0. \end{aligned} \tag{4.21}$$

The conditions (4.20) and (4.21) give an A-type generalized Spin(7) structure.

Next we consider a B-type generalized Spin(7) structure. In the previous subsection, a generalized G_2 structure is realized by use of topological 2-brane and topological 3-brane in the action (3.1). As we will discuss later, we can assume that that the method of dimensional reduction from a topological F -theory in eight dimensions to a topological M -theory in seven dimensions suggests two types of Spin(7) structures using a topological 3-brane and a topological 4-brane. Therefore, we choose $n = 5$ for a B-type generalized Spin(7) structure in (3.1). The action is

$$\begin{aligned} S &= \int_{\Sigma_5} e^{Y_2} e^{1/2b} \rho \\ &= \int_{\Sigma_5} \left[Y_2 Y_2 \rho_{1,i} d\phi^i + \frac{1}{3!} Y_2 (\rho_{3,ijk} + b_{[ij} \rho_{1,k]}) d\phi^i d\phi^j d\phi^k \right. \\ &\quad \left. + \frac{1}{5!} (\rho_{5,ijklm} + b_{[ij} \rho_{3,klm]} + b_{[ij} b_{kl} \rho_{1,m]}) d\phi^i d\phi^j d\phi^k \right]. \end{aligned} \tag{4.22}$$

The first-class constraints for $\rho \in \Lambda^{\text{od}}(M)$ are

$$\begin{aligned} d_M \rho_1 &= 0, \\ d_M \rho_3 + H \wedge \rho_1 &= 0, \\ d_M \rho_5 + H \wedge \rho_3 &= 0. \end{aligned} \tag{4.23}$$

From these conditions, we obtain the equation

$$d_M \rho_7 + H \wedge \rho_5 = 0. \tag{4.24}$$

The conditions (4.23) and (4.24) give a B-type generalized Spin(7) structure.

Also, we have other realizations of generalized Spin(7) structures. When we choose $n = 2$, $n = 6$ or $n = 8$ in the action (3.1), we can derive A-type generalized Spin(7) conditions on M . When we choose $n = 3$ or $n = 7$ in the action (3.1), we can derive B-type generalized Spin(7) conditions on M .

5 Dimensional reduction

Here we consider dimensional reduction of our models from compactification of a target space M . Our generalized topological membranes models are related to each other. There are two types of reductions. One is a wrapping membrane, and the other is a longitudinal membrane. These reductions from higher dimensional topological generalized membranes give lower dimensional A-type and B-type topological generalized membranes. Here we consider 1-dimensional compactification $M_d = M_{d-1} \times S^1$, where M_d is a d -dimensional manifold and M_{d-1} is a $d - 1$ -dimensional manifold.

5.1 Wrapping membranes

First we consider topological $n - 1$ -brane wrapping to S^1 . Let $(\sigma^1, \dots, \sigma^n)$ be a local coordinate on an $n - 1$ -brane Σ_n . σ^n is chosen as a local coordinate of wrapping direction S^1 . We use indices $i, j, k = 1, \dots, d$ and $r, s = 1, \dots, d - 1$. Local coordinates of the wrapping membrane are

$$\phi^i = \begin{cases} \phi^r(\sigma^1, \dots, \sigma^{n-1}) & \text{if } r = 1, \dots, d - 1, \\ \sigma^n & \text{if } r = d \end{cases} \tag{5.1}$$

and

$$b_{ij} = \begin{cases} b_{rs}(\sigma^1, \dots, \sigma^{n-1}) & \text{if } r, s = 1, \dots, d - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_2 = Y_2(\sigma^1, \dots, \sigma^{n-1}). \tag{5.2}$$

Then the action (3.1) is reduced to

$$\begin{aligned} S &= \int_{\Sigma_n} e^{Y_2} e^{1/2b} \rho \\ &= \sum_{p=0}^d \frac{1}{p!} \int_{\Sigma_n} e^{Y_2} e^{1/2b} \rho_{p, i_1 \dots i_p}(\phi) d\phi^{i_1} \dots d\phi^{i_p}, \\ &= \sum_{p=1}^d \frac{1}{(p - 1)!} \left[\int_{\Sigma_{n-1}} e^{Y_2} e^{1/2b} \rho_{p, i_1 \dots i_{p-1} d}(\phi) d\phi^{i_1} \dots d\phi^{i_{p-1}} \right] \int_{S^1} d\sigma^n, \end{aligned} \tag{5.3}$$

where $\Sigma_n = \Sigma_{n-1} \times S^1$. We can see that a generalized structure on M_d reduces to a generalized structure on M_{d-1} . Also, this reduction changes an A-type generalized structure and a B-type generalized structure. For examples, a generalized A-type (B-type) G_2 membrane is reduced to a generalized B-type (A-type) $SU(3)$ string(membrane). A generalized A-type

(B-type) Spin(7) membrane is reduced to a generalized B-type (A-type) G_2 membrane.

5.2 Longitudinal membranes

Next we consider topological membrane longitudinal to a compactified direction S^1 . Local coordinates of the longitudinal membrane are

$$\phi^i = \begin{cases} \phi^r(\sigma^1, \dots, \sigma^n) & \text{if } r = 1, \dots, d - 1, \\ 0 & \text{if } r = d \end{cases} \tag{5.4}$$

and

$$\begin{aligned} b_{ij} &= \begin{cases} b_{rs}(\sigma^1, \dots, \sigma^n) & \text{if } r, s = 1, \dots, d - 1, \\ 0 & \text{otherwise,} \end{cases} \\ Y_2 &= Y_2(\sigma^1, \dots, \sigma^n). \end{aligned} \tag{5.5}$$

Then the action (3.1) is reduced to

$$\begin{aligned} S &= \int_{\Sigma_n} e^{Y_2} e^{1/2b} \rho = \sum_{p=0}^d \frac{1}{p!} \int_{\Sigma_n} e^{Y_2} e^{1/2b} \rho_{p,i_1 \dots i_p}(\phi) d\phi^{i_1} \dots d\phi^{i_p}, \\ &= \sum_{p=0}^d \frac{1}{p!} \int_{\Sigma_n} e^{Y_2} e^{1/2b_{rs} d\phi^r d\phi^s} \rho_{p,r_1 \dots r_p}(\phi) d\phi^{r_1} \dots d\phi^{r_p}, \end{aligned} \tag{5.6}$$

where $r, s = 1, \dots, d - 1$. A generalized structure on M_d is reduced to a generalized structure on M_{d-1} . This reduction does not change A-types and B-types. For examples, a generalized A-type (B-type) G_2 membrane is reduced to a generalized A-type (B-type) SU(3) membrane. A generalized A-type (B-type) Spin(7) membrane is reduced to a generalized A-type (B-type) G_2 membrane.

6 AKSZ formulation of BV formalism

In order to construct and analyze models systematically, it is useful to use the BV formalism. In the case of topological field theories, the AKSZ formulation [31] is a theoretical and general setting of the BV formalism.

6.1 BV structures on graded vector bundles

We explain a general setting of the AKSZ formulation of the BV formalism for a general graded bundle [33].

Let M be a smooth manifold in d dimensions. We define a *supermanifold* ΠT^*M . Mathematically, ΠT^*M , whose bosonic part is M , is defined as a cotangent bundle with reversed parity of the fiber. That is, a base manifold M has Grassman even coordinates and the fiber of ΠT^*M has Grassman odd coordinates. We introduce a grading called *total degrees*. The coordinates of the base manifold have grade zero and the coordinates of the fiber have grade one. Similarly, we can define ΠTM for a tangent bundle TM .

We must consider more general assignments for the degree of the fibers of T^*M or TM . For an integer p , we define $T^*[p]M$, which is called a graded cotangent bundle. $T^*[p]M$ is a cotangent bundle, whose fiber has the degree p . The coordinates of the base manifold have the total degree zero and the coordinates of the fiber have the total degree p . If p is odd, the fiber is Grassman odd, and if p is even, the fiber is Grassman even. We define a graded tangent bundle $T[p]M$ in the same way. If we consider a general vector bundle E , a graded vector bundle $E[p]$ is defined in the similar way. $E[p]$ is a vector bundle whose fiber has a shifted degree by p . Note that only the degree of fiber is shifted, and the degree of base space is not shifted.

We consider a Poisson manifold N with a Poisson bracket $\{*,*\}$. If we shift the total degree, we can construct a graded manifold \tilde{N} from N . Then the Poisson structure $\{*,*\}$ shifts to a graded Poisson structure by grading of \tilde{N} . The graded Poisson bracket is called an *antibracket* and denoted by $(*,*)$. $(*,*)$ is graded symmetric and satisfies the graded Leibniz rule and the graded Jacobi identity with respect to grading of the manifold. The antibracket $(*,*)$ with the total degree $-n + 1$ satisfies the following identities:

$$\begin{aligned}
 (F, G) &= -(-1)^{(|F|+1-n)(|G|+1-n)}(G, F), \\
 (F, GH) &= (F, G)H + (-1)^{(|F|+1-n)|G|}G(F, H), \\
 (FG, H) &= F(G, H) + (-1)^{|G|(|H|+1-n)}(F, H)G, \\
 (-1)^{(|F|+1-n)(|H|+1-n)}(F, (G, H)) &+ \text{cyclic permutations} = 0, \tag{6.1}
 \end{aligned}$$

where F, G and H are functions on \tilde{N} , and $|F|, |G|$ and $|H|$ are total degrees of the functions, respectively. The graded Poisson structure is also called *P-structure*.

Typical examples of Poisson manifold N are a cotangent bundle T^*M and a vector bundle $E \oplus E^*$. First we consider a cotangent bundle T^*M . Since T^*M has a symplectic structure, we can define a Poisson bracket induced from the natural symplectic structure. If we take local coordinates ϕ^i on M and local coordinates B_i of the fiber, we can define a Poisson bracket as follows:

$$\{F, G\} \equiv F \overleftarrow{\frac{\partial}{\partial \phi^i}} \overrightarrow{\frac{\partial}{\partial B_i}} G - F \overleftarrow{\frac{\partial}{\partial B_i}} \overrightarrow{\frac{\partial}{\partial \phi^i}} G, \tag{6.2}$$

where F and G are functions on T^*M , and $\overleftarrow{\partial}/\partial\varphi$ and $\overrightarrow{\partial}/\partial\varphi$ are the right and left differentiations with respect to φ , respectively. Here we shift the degree of fiber by p , i.e., the space $T^*[p]M$. Then the Poisson structure shifts to a graded Poisson structure. The corresponding graded Poisson bracket is called *antibracket*, $(*, *)$. Let ϕ^i be local coordinates of M and $\mathbf{B}_{n-1,i}$ a basis of the fiber of $T^*[p]M$. The antibracket $(*, *)$ on a cotangent bundle $T^*[p]M$ is expressed as:

$$(F, G) \equiv F \overleftarrow{\frac{\partial}{\partial \phi^i}} \overrightarrow{\frac{\partial}{\partial \mathbf{B}_{p,i}}} G - F \overleftarrow{\frac{\partial}{\partial \mathbf{B}_{p,i}}} \overrightarrow{\frac{\partial}{\partial \phi^i}} G. \tag{6.3}$$

The total degree of the antibracket $(*, *)$ is $-p$. This antibracket satisfies the property (6.1) for $-p = -n + 1$.

Next, we consider a vector bundle $E \oplus E^*$. There is a natural Poisson structure on the fiber of $E \oplus E^*$ induced from a pairing of E and E^* . If we take local coordinates A^a on the fiber of E and B_a on the fiber of E^* , we can define

$$\{F, G\} \equiv F \overleftarrow{\frac{\partial}{\partial A^a}} \overrightarrow{\frac{\partial}{\partial B_a}} G - F \overleftarrow{\frac{\partial}{\partial B_a}} \overrightarrow{\frac{\partial}{\partial A^a}} G, \tag{6.4}$$

where F and G are functions on $E \oplus E^*$. We shift the degrees of fibers of E and E^* like $E[p] \oplus E^*[q]$, where p and q are positive integers. The Poisson structure changes to a graded Poisson structure $(*, *)$. Let \mathbf{A}_p^a be a basis of the fiber of $E[p]$ and $\mathbf{B}_{q,a}$ a basis of the fiber of $E^*[q]$. The antibracket is represented as

$$(F, G) \equiv F \overleftarrow{\frac{\partial}{\partial \mathbf{A}_p^a}} \overrightarrow{\frac{\partial}{\partial \mathbf{B}_{q,a}}} G - (-1)^{pq} F \overleftarrow{\frac{\partial}{\partial \mathbf{B}_{q,a}}} \overrightarrow{\frac{\partial}{\partial \mathbf{A}_p^a}} G. \tag{6.5}$$

The total degree of the antibracket $(*, *)$ is $-p - q$. This antibracket satisfies the property (6.1) for $-p - q = -n + 1$.

We define a *Q-structure*. A *Q-structure* is a function S on a graded manifold \tilde{N} which satisfies the classical master equation $(S, S) = 0$. S is

called a *BV action*. We require that S satisfy the compatibility condition

$$S(F, G) = (SF, G) + (-1)^{|F|+1}(F, SG), \tag{6.6}$$

where F and G are arbitrary functions, and $|F|$ is the total degree of F . $(S, F) = \delta F$ generates an infinitesimal transformation. We call this a *BRST transformation*, which coincides with the gauge transformation of the theory.

The AKSZ formulation of the BV formalism is defined as a *P-structure* and a *Q-structure* on a *graded manifold*.

6.2 BV structures of topological sigma models

In this subsection, we explain BV structures of topological sigma models. Let X be a base manifold in n dimensions, with or without boundary, and M be a target manifold in d dimensions. We denote ϕ a smooth map from X to M .

We consider a *supermanifold* $\Pi T X$, whose bosonic part is X . $\Pi T X$ is defined as a tangent bundle with reversed parity of the fiber. We extend a smooth function ϕ to a function on the supermanifold $\phi : \Pi T X \rightarrow M$. ϕ is an element of $\Pi T^* X \otimes M$. The *total degree* defined in the previous section is a grading with respect to M . We introduce a new nonnegative integer grading on $\Pi T^* X$. A coordinate on a base manifold has zero and a coordinate on the fiber has one. This grading is called *form degrees*. We denote $\deg F$ the form degree of the function F . $\text{gh } F = |F| - \deg F$ is called *ghost number*.

First we consider a *P-structure* on $T^*[p]M$. We take $p = n - 1$ to construct a BV structure in a topological sigma model on a general n -dimensional worldvolume. We consider $T^*[n - 1]M$ for an n -dimensional base manifold X . Let ϕ^i be local coordinates of $\Pi T^* X \otimes M$, where i, j, k, \dots are indices of the local coordinates on M . Let $\mathbf{B}_{n-1,i}$ be a basis of sections of $\Pi T^* X \otimes \phi^*(T^*[n - 1]M)$. As we discussed in the previous subsection, we can define an *antibracket* $(*, *)$ on a cotangent bundle $T^*[n - 1]M$ as

$$(F, G) \equiv F \overleftarrow{\frac{\partial}{\partial \phi^i}} \overrightarrow{\frac{\partial}{\partial \mathbf{B}_{n-1,i}}} G - F \overleftarrow{\frac{\partial}{\partial \mathbf{B}_{n-1,i}}} \overrightarrow{\frac{\partial}{\partial \phi^i}} G, \tag{6.7}$$

where F and G are functions of ϕ^i and $\mathbf{B}_{n-1,i}$. The total degree of the antibracket is $-n + 1$. If F and G are functionals of ϕ^i and $\mathbf{B}_{n-1,i}$, we

understand an antibracket is defined as

$$(F, G) \equiv \int_X F \frac{\overleftarrow{\partial}}{\partial \phi^i} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-1,i}} G - F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-1,i}} \frac{\overrightarrow{\partial}}{\partial \phi^i} G, \tag{6.8}$$

where the integration over X picks up only the n -form part of the integrand. Through this article, we always understand an antibracket on two functionals in a similar manner and abbreviate this notation.

Next we consider a P -structure on $E \oplus E^*$. We assign the total degrees p and q such that $p + q = n - 1$. That is, we consider $E[p] \oplus E^*[n - p - 1]$, where $-1 \leq p \leq n - 1$. Then we can construct a topological sigma model.

Let $\mathbf{A}_p^{a_p}$ be a basis of sections of $\Pi T^*X \otimes \phi^*(E[p])$ and \mathbf{B}_{n-p-1, a_p} a basis of the fiber of $\Pi T^*X \otimes \phi^*(E^*[n - p - 1])$. From (6.5), we can define the antibracket as

$$(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} G - (-1)^{np} F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} G. \tag{6.9}$$

We need to consider various grading assignments for $E \oplus E^*$, because each assignment induces different BV structures. In order to consider all independent assignments, we define the following bundle. Let E_p be n series of vector bundles, where $-1 \leq p \leq n - 1$. We consider a direct sum of each bundle $E_p[p] \oplus E_p^*[n - p - 1]$:

$$\sum_{p=-1}^{n-1} E_p[p] \oplus E_p^*[n - p - 1]. \tag{6.10}$$

We define a P -structure on the graded vector bundle

$$\left(\sum_{p=-1}^{n-1} E_p[p] \oplus E_p^*[n - p - 1] \right) \oplus T^*[n - 1]M. \tag{6.11}$$

A local (Darboux) coordinate expression for the antibracket (\cdot, \cdot) is a sum of (6.7) and (6.9):

$$(F, G) \equiv \sum_{p=-1}^{n-1} F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} G - (-1)^{np} F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-p-1, a_p}} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_p^{a_p}} G. \tag{6.12}$$

where $p = 0$ component is the antibracket (6.7) on the graded cotangent bundle $T^*[n - 1]M$, and $\mathbf{A}_0^{a_0} = \phi^i$. Note that all terms of the antibracket have the total degree $-n + 1$, and we can confirm that the antibracket (6.12) satisfies the identity (6.1).

7 BV formulation of topological membranes with generalized structures

We consider the target graded bundle $(T[1]M \oplus T^*[n-2]M) \oplus (M \times \mathbf{R}[n-3] \oplus M \times \mathbf{R}[2]) \oplus T^*[n-1]M$. That is, We take $E_1 = TM$, $E_{n-3} = M \times \mathbf{R}$ and $E_p = 0$ for the other p in (6.11). We introduce the following superfields which are sections on the above total bundle:

$$\begin{aligned} \phi : \Pi T X \rightarrow M, \quad \mathbf{B}_{n-1,i} \in \Gamma(\Pi T^* X \otimes \phi^*(T^*[n-1]M)), \mathbf{A}_1^i \in \Gamma(\Pi T^* X \\ \otimes \phi^*(T[1]M)), \mathbf{B}_{n-2,i} \in \Gamma(\Pi T^* X \otimes \phi^*(T^*[n-2]M)), \mathbf{Z}_{n-3} \in \Pi T^* X \\ \otimes \phi^*(M \times \mathbf{R}[n-3]) \quad \text{and} \quad \mathbf{Y}_2 \in \Gamma(\Pi T^* X \otimes \phi^*(M \times \mathbf{R}[2])). \end{aligned}$$

The antibracket is derived from (6.12) as:

$$\begin{aligned} (F, G) \equiv & F \frac{\overleftarrow{\partial}}{\partial \phi^i} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-1,i}} G - F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-1,i}} \frac{\overrightarrow{\partial}}{\partial \phi^i} G \\ & + F \frac{\overleftarrow{\partial}}{\partial \mathbf{A}_1^i} \frac{\overrightarrow{\partial}}{\partial \mathbf{B}_{n-2,i}} G - (-1)^n F \frac{\overleftarrow{\partial}}{\partial \mathbf{B}_{n-2,i}} \frac{\overrightarrow{\partial}}{\partial \mathbf{A}_1^i} G \\ & + F \frac{\overleftarrow{\partial}}{\partial \mathbf{Z}_{n-3}} \frac{\overrightarrow{\partial}}{\partial \mathbf{Y}_2} G - F \frac{\overleftarrow{\partial}}{\partial \mathbf{Y}_2} \frac{\overrightarrow{\partial}}{\partial \mathbf{Z}_{n-3}} G. \end{aligned} \tag{7.1}$$

This defines a total degree $-n + 1$ P -structure.

We remember the first-order formalism action (3.5). By the superfield extension of the action (3.5), we can propose the following first-order BV action

$$\begin{aligned} S = \int_{\Sigma_n} & (-1)^n \mathbf{B}_{n-1,i} d\phi^i + (-1)^{n-1} \mathbf{B}_{n-2,i} d\mathbf{A}_1^i - \mathbf{Y}_2 d\mathbf{Z}_{n-3} + (-1)^{n-1} \\ & \times \mathbf{B}_{n-1,i} \mathbf{A}_1^i + \exp(\mathbf{Y}_2) \exp\left(\frac{1}{2} b_{ij} \mathbf{A}_1^i \mathbf{A}_1^j\right) \left(\sum_{p=0}^d \frac{1}{p!} \rho_{p,i_1 \dots i_p}(\phi) \mathbf{A}_1^{i_1} \dots \mathbf{A}_1^{i_p}\right), \end{aligned} \tag{7.2}$$

where $b_{ij}(\phi)$ and $\rho_{p,i_1 \dots i_p}(\phi)$ are functions of ϕ . Then the classical master equation is

$$(S, S) = 0. \tag{7.3}$$

This provides the BV structure in this model. This master equation is

$$\frac{\overrightarrow{\partial}}{\partial \phi^j} \rho_{p+2,i_1 \dots i_{p+2}}(\phi) + \frac{3!}{(p+2)(p+1)} H_{[j i_{p+1} i_{p+2}}(\phi) \rho_{p,i_1 \dots i_p]}(\phi) = 0. \tag{7.4}$$

This equation is nothing but the equation (3.4) for $b_{ij}(\phi)$ and $\rho_{p,i_1\dots i_p}(\phi)$. Therefore this topological sigma model (7.2) defines the generalized geometry structure on the target space M as the BV structure.

7.1 BV actions with generalized SU(3) structures

We consider a BV construction of the models with a generalized SU(3) structure, which we proposed in Section 4.1. First we consider the case of $n = 2$ in (7.2). Σ_2 is a 2-dimensional worldsheet. This model is a topological string. The action is

$$\begin{aligned}
 S &= \int_{\Sigma_2} \mathbf{B}_{1,i}d\phi^i - \mathbf{B}_{0,i}d\mathbf{A}_1^i - \mathbf{Y}_2d\mathbf{Z}_{-1} - \mathbf{B}_{1,i}\mathbf{A}_1^i \\
 &\quad + \exp(\mathbf{Y}_2) \exp\left(\frac{1}{2}b_{ij}\mathbf{A}_1^i\mathbf{A}_1^j\right) \left(\rho_0(\phi) + \frac{1}{2!}\rho_{2,ij}(\phi)\mathbf{A}_1^i\mathbf{A}_1^j\right) \\
 &= \int_{\Sigma_2} \mathbf{B}_{1,i}d\phi^i - \mathbf{B}_{0,i}d\mathbf{A}_1^i - \mathbf{Y}_2d\mathbf{Z}_{-1} - \mathbf{B}_{1,i}\mathbf{A}_1^i \\
 &\quad + \left(\rho_0\mathbf{Y}_2 + \frac{1}{2!}(\rho_{2,ij} + b_{ij}\rho_0)\mathbf{A}_1^i\mathbf{A}_1^j\right). \tag{7.5}
 \end{aligned}$$

This action is a BV construction of the action (4.3) and has an A-type condition for a generalized SU(3) structure.

Also, this action is an extension of the A model by use of the AKSZ superfield formulation. The A model action based on a symplectic form Q_{ij} is defined as

$$S_A = \int_{\Sigma} \mathbf{B}_{1,i}d\phi^i - \mathbf{B}_{0,i}d\mathbf{A}_1^i - \mathbf{B}_{1,i}\mathbf{A}_1^i + \frac{1}{2}Q_{ij}(\phi)\mathbf{A}_1^i\mathbf{A}_1^j. \tag{7.6}$$

We can see that the classical master equation $(S_A, S_A) = 0$ is satisfied, if and only if the 2-form $Q = Q_{ij}d\phi^i d\phi^j$ on the target space is a symplectic form $d_M Q = 0$, i.e., $\partial_k Q_{ij} + \partial_i Q_{jk} + \partial_j Q_{ki} = 0$. When we set $b_{ij} = 0$, \mathbf{Y}_2 and \mathbf{Z}_{-1} decouple from the other fields, and the remaining part of the action (7.5) coincides with A model (7.6). Equation (7.5) is a B-field deformation of A model.

Next we derive a B-type condition for a generalized SU(3) structure. Let a target space M have even dimensions, especially six dimensions. If we set

$n = 3$ in (7.2), we obtain the BV action of the action (4.7):

$$\begin{aligned}
 S = & \int_{\Sigma_3} -\mathbf{B}_{2,i}d\phi^i + \mathbf{B}_{1,i}d\mathbf{A}_1^i - \mathbf{Y}_2d\mathbf{Z}_0 + \mathbf{B}_{2,i}\mathbf{A}_1^i + \exp(\mathbf{Y}_2) \\
 & \times \exp\left(\frac{1}{2}b_{ij}\mathbf{A}_1^i\mathbf{A}_1^j\right) \left(\sum_{q=0}^1 \frac{1}{(2q+1)!} \rho_{2q+1,i_1\dots i_{2q+1}}(\phi)\mathbf{A}_1^{i_1} \dots \mathbf{A}_1^{i_{2q+1}}\right). \tag{7.7}
 \end{aligned}$$

This action is a B-type membrane action with a generalized $SU(3)$ structure.

7.2 BV actions for generalized G_2 structures

Let a target space M have seven dimensions. If we set $n = 3$ in the action (7.2), we obtain the BV action for the action (4.11), which has a B-type generalized G_2 structure:

$$\begin{aligned}
 S = & \int_{\Sigma_3} -\mathbf{B}_{2,i}d\phi^i + \mathbf{B}_{1,i}d\mathbf{A}_1^i - \mathbf{Y}_2d\mathbf{Z}_0 + \mathbf{B}_{2,i}\mathbf{A}_1^i + \exp(\mathbf{Y}_2) \\
 & \times \exp\left(\frac{1}{2}b_{ij}\mathbf{A}_1^i\mathbf{A}_1^j\right) \left(\sum_{q=0}^1 \frac{1}{(2q+1)!} \rho_{2q+1,i_1\dots i_{2q+1}}(\phi)\mathbf{A}_1^{i_1} \dots \mathbf{A}_1^{i_{2q+1}}\right). \tag{7.8}
 \end{aligned}$$

When we set $b_{ij} = 0$, \mathbf{Y}_2 and \mathbf{Z}_0 decouple from the other fields, and the remaining part of the action (7.8) is

$$S = \int_{\Sigma_3} -\mathbf{B}_{2,i}d\phi^i + \mathbf{B}_{1,i}d\mathbf{A}_1^i + \mathbf{B}_{2,i}\mathbf{A}_1^i + \frac{1}{3!}\rho_{3,ijk}(\phi)\mathbf{A}_1^i\mathbf{A}_1^j\mathbf{A}_1^k. \tag{7.9}$$

The master equation $(S, S) = 0$ is equivalent to the G_2 condition $d_M\rho_3 = 0$. The action (7.9) is a BV action for a G_2 topological membrane. The action (7.8) is a B-field deformation of G_2 topological membrane.

Also, if we consider worldvolume Σ_n in even dimensions, we obtain an A-type generalized G_2 structure. When we set $n = 4$, we obtain the BV action for the action (4.15):

$$\begin{aligned}
 S = & \int_{\Sigma_4} \mathbf{B}_{3,i}d\phi^i - \mathbf{B}_{2,i}d\mathbf{A}_1^i - \mathbf{Y}_2d\mathbf{Z}_1 - \mathbf{B}_{3,i}\mathbf{A}_1^i \\
 & + \exp(\mathbf{Y}_2) \exp\left(\frac{1}{2}b_{ij}\mathbf{A}_1^i\mathbf{A}_1^j\right) \left(\sum_{q=0}^2 \frac{1}{(2q)!} \rho_{2q,i_1\dots i_{2q}}(\phi)\mathbf{A}_1^{i_1} \dots \mathbf{A}_1^{i_{2q}}\right). \tag{7.10}
 \end{aligned}$$

This action is an A-type membrane action with a generalized G_2 structure.

7.3 BV actions for generalized Spin(7) structures

Let a target space M have eight dimensions. If we set $n = 4$ in the general action (7.2), we obtain the BV action for the action (4.19):

$$\begin{aligned}
 S = & \int_{\Sigma_4} \mathbf{B}_{3,i} d\phi^i - \mathbf{B}_{2,i} d\mathbf{A}_1^i - \mathbf{Y}_2 d\mathbf{Z}_1 - \mathbf{B}_{3,i} \mathbf{A}_1^i \\
 & + \exp(\mathbf{Y}_2) \exp \frac{1}{2} (b_{ij} \mathbf{A}_1^i \mathbf{A}_1^j) \left(\sum_{q=0}^2 \frac{1}{(2q)!} \rho_{2q,i_1 \dots i_{2q}}(\phi) \mathbf{A}_1^{i_1} \dots \mathbf{A}_1^{i_{2q}} \right). \tag{7.11}
 \end{aligned}$$

This action is a BV action with an A-type generalized Spin(7) structure.

When we set $b_{ij} = 0$, \mathbf{Y}_2 and \mathbf{Z}_1 decouple from the other fields, and the remaining part of the action (7.11) is

$$S = \int_{\Sigma_4} \mathbf{B}_{3,i} d\phi^i - \mathbf{B}_{2,i} d\mathbf{A}_1^i - \mathbf{B}_{3,i} \mathbf{A}_1^i + \frac{1}{4!} \rho_{4,ijkl}(\phi) \mathbf{A}_1^i \mathbf{A}_1^j \mathbf{A}_1^k \mathbf{A}_1^l. \tag{7.12}$$

This action is a BV action of Spin(7) topological membrane. The master equation $(S, S) = 0$ of the action (7.12) is satisfied if ρ_4 is a Spin(7)-form.

Also, when we consider worldvolume Σ_5 and a target space M with eight dimensions, the BV action for the action (4.22) is

$$\begin{aligned}
 S = & \int_{\Sigma_5} -\mathbf{B}_{4,i} d\phi^i + \mathbf{B}_{3,i} d\mathbf{A}_1^i - \mathbf{Y}_2 d\mathbf{Z}_2 + \mathbf{B}_{4,i} \mathbf{A}_1^i + \exp(\mathbf{Y}_2) \\
 & \times \exp \left(\frac{1}{2} b_{ij} \mathbf{A}_1^i \mathbf{A}_1^j \right) \left(\sum_{q=0}^2 \frac{1}{(2q+1)!} \rho_{2q+1,i_1 \dots i_{2q+1}}(\phi) \mathbf{A}_1^{i_1} \dots \mathbf{A}_1^{i_{2q+1}} \right). \tag{7.13}
 \end{aligned}$$

This action is a B-type membrane action with a generalized Spin(7) structure.

8 Conclusions and Discussions

We have proposed topological string and topological membrane actions, which realize generalized geometries with nontrivial 3-form flux H . The constraints of these actions are first class constraints if and only if the condition (2.2) is satisfied on the target manifolds M . Especially, as examples, we have considered target manifolds M with generalized SU(3), G_2 and Spin(7) structures. Also, we have considered wrapping and longitudinal dimensional reductions. Then in our models, lower dimensional A-type and B-type

generalized structures appear from higher dimensional A-type and B-type generalized structures. Finally, we have rewritten these actions by use of the AKSZ formulation of the BV formalism.

In this paper, we have considered only classical theory. It would be an interesting work to quantize these theories as a quantization of generalized geometries. Also, it is known that some topological theories can be related to nontopological physical theories. For example, topological string theory on Calabi–Yau manifolds can be used to know physical information in type II superstring theory on Calabi–Yau backgrounds. In order to derive physical information in superstring theory on NS–NS H flux background from topological string theory with H flux, it would be a useful thing to find a relation with these.

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