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Matrix model superpotentials and ADE singularities

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Abstract

We use F. Ferrari's methods relating matrix models to Calabi–Yau spaces in order to explain much of Intriligator and Wecht's ADE classification of $\mathcal{N} = 1$ superconformal theories which arise as RG fixed points of $\mathcal{N} = 1$ SQCD theories with adjoints. We find that ADE superpotentials in the Intriligator–Wecht classification exactly match matrix model superpotentials obtained from Calabi–Yau with corresponding ADE singularities. Moreover, in the additional $\widehat{O}, \widehat{A}, \widehat{D}$ and $\widehat{E}$ cases we find new singular geometries. These "hat" geometries are closely related to their ADE counterparts, but feature non-isolated singularities. As a byproduct, we give simple descriptions for small resolutions of Gorenstein threefold singularities in terms of transition functions between just two co-ordinate charts. To obtain these results we develop an algorithm for blowing down exceptional $\mathbb{P}^1$, described in the appendix.

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1 Introduction

Duality has long played an important role in string theory. In addition, by relating physical quantities (correlators, partition functions, spectra) between different theories with geometric input, dualities have uncovered many unexpected patterns in geometry. This has led to surprising conjectures (such as mirror symmetry and T-duality) which not only have important implications for physics, but are interesting and meaningful in a purely geometric light.

Recently, there has been a tremendous amount of work surrounding dualities which relate string theories to other classes of theories. Maldacena's 1997 AdS/CFT correspondence is perhaps the most famous example of such a duality [19]. The connection between Chern-Simons gauge theory and string theory was first introduced by Witten in 1992 [21]. In 1999, Gopakumar and Vafa initiated a program to study the relationship between large N limits of Chern-Simons theory (gauge theory) and type IIA topological string theory (geometry) by using ideas originally proposed by 't Hooft in the 1970s [14]. The resulting gauge theory/geometry correspondence led to a conjecture about extremal transitions, often referred to as the "geometric transition conjecture." In the case of conifold singularities, this is more or less understood. The conifold singularity can be resolved in two very different ways: (1) with a traditional blow up in algebraic geometry, in which the singular point gets replaced by an exceptional $\mathbb{P}^1$, or (2) by a deformation of the algebraic equation which replaces the singular point with an S^3 whose size is controlled by the deformation parameter (see figure 1). The physical degrees of freedom associated to D5 branes wrapping the $\mathbb{P}^1$ correspond to a three-form flux through the S^3. The geometric statement is that one can interpolate between the two kinds of resolutions.

In 2002, Dijkgraaf and Vafa expanded this program and proposed new dualities between type IIB topological strings on Calabi–Yau threefolds and matrix models [9, 10]. Due to the symmetry between type IIA and type IIB string theories, this may be viewed as "mirror" to the Gopakumar–Vafa conjecture. By studying the conifold case, they found strong evidence for the matching of the string theory partition function with that of a matrix model whose potential is closely related to the geometry in question. In particular, a dual version of special geometry in Calabi–Yau threefolds is seen in the eigenvalue dynamics of the associated matrix model [9]. The proposed string theory/matrix model duality has led to an explosion of research on matrix models, a topic which had been dormant since the early 1990s, when it was studied in the context of two-dimensional (2D)

$$\xrightarrow{\text{Geometric transition conjecture}}$$

$$
\begin{tikzcd}
\text{D5 Branes wrapping} & \text{singular Calabi–Yau} & \text{3-form flux through} \\
\mathbb{P}^1 \subset \widehat{\mathcal{M}} \arrow[r, "\text{blow down}"] \arrow[d, "\text{geometric engineering}"'] & \mathcal{M}_0 \arrow[r, "\text{deformation}"] & S^3 \subset \mathcal{M} \arrow[d, "?"] \\
\mathcal{N}=1\ \text{SYM} \arrow[r, "\text{RG flow}"'] & \text{strong coupling} \arrow[r, "\text{RG flow}"'] & \mathcal{F} = \text{matrix model} \\
1/g^2_{\mu_{UV}} \sim \text{size}\ (\mathbb{P}^1) & g^2_{\mu_c} = \infty & \text{partition function}
\end{tikzcd}
$$

$$\xrightarrow[\text{Dijkgraaf}-\text{Vafa conjecture}]{}$$

Figure 1: The big picture.

gravity [8]. The connection between string theory and matrix models is of very tangible practical importance, as many quantities which are difficult to compute in string theory are much easier to handle on the matrix model side.

Inspired by these developments (summarized in figure 1), in 2003 F. Ferrari was led to propose a direct connection between matrix models and the Calabi–Yau spaces of their dual string theories [13]. It is well known that the solution to a one-matrix model can be characterized geometrically, in terms of a hyperelliptic curve. The potential for the matrix model serves as direct input into the algebraic equation for the curve, and the vacuum solutions (distributions of eigenvalues) can be obtained from the geometry of the curve and correspond to branch cuts on the Riemann surface. The work of Vafa and collaborators on the strongly coupled dynamics of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories [14, 4, 5, 11] suggests that for multi-matrix models, higher-dimensional Calabi–Yau spaces might be useful. Ferrari pursues this idea in [13], finding evidence that certain multi-matrix models can, indeed, be directly characterized in terms of higher-dimensional (non-compact) Calabi–Yau spaces.

By thinking of the matrix model potential $W(x_1, \ldots, x_M)$ as providing constraints on the deformation space of an exceptional $\mathbb{P}^1$ within a smooth (resolved) Calabi–Yau $\widehat{\mathcal{M}}$, Ferrari outlines a precise prescription for constructing such smooth geometries directly from the potential. Specifically, the resolved geometry $\widehat{\mathcal{M}}$ is given by transition functions

$$\beta = 1/\gamma, \quad v_1 = \gamma^{-n} w_1, \quad v_2 = \gamma^{-m} w_2 + \partial_{w_1} E(\gamma, w_1), \tag{1.1}$$

between two coordinate patches (γ, w_1, w_2) and (β, v_1, v_2), where β and γ are stereographic coordinates over an exceptional $\mathbb{P}^1$. The perturbation comes from the "geometric potential" $E(\gamma, w)$, which is related to the matrix model potential W via

$$W(x_1, \ldots, x_M) = \frac{1}{2\pi i} \oint_{C_0} \gamma^{-M-1} E\left(\gamma, \sum_{i=1}^{M} x_i \gamma^{i-1}\right) d\gamma, \tag{1.2}$$

where $M = n + 1 = -m - 1$. We explain this construction in detail in Section 2.4.

In the absence of the perturbation term $\partial_{w_1} E(\gamma, w_1)$, the transition functions (1.1) simply describe an $\mathcal{O}(M-1) \oplus \mathcal{O}(-M-1)$ bundle over a $\mathbb{P}^1$. The matrix model superpotential W encodes the constraints on the sections $x_1, \ldots, x_M$ of the bundle due to the presence of $\partial_{w_1} E(\gamma, w_1)$. Note that this procedure is also invertible. In other words, given a matrix model superpotential $W(x_1, \ldots, x_M)$, one can find a corresponding geometric potential $E(\gamma, w_1)$. However, not all perturbation terms $\gamma^j w_1^k$ contribute to the superpotential (1.2), so there may be many choices of geometric potential for a given W. Nevertheless, the associated geometry $\widehat{\mathcal{M}}$ is unique [13, p. 634].

In 2000, S. Katz had already shown how to codify constraints on versal deformation spaces of curves in terms of a potential function,[1] in cases such as (1.1) where the constraints are integrable [16]. In 2001, F. Cachazo, S. Katz and C. Vafa constructed $\mathcal{N} = 1$ supersymmetric gauge theories corresponding to D5 branes wrapping two-cycles of ADE fibered threefolds [5]. Ferrari studies the same kinds of geometries in a different context, by interpreting the associated potential as belonging to a matrix model, and proposing that the Calabi–Yau geometry encodes all relevant information about the matrix theory.[2] He is able to verify this in a few examples, and computes known resolvents of matrix models in terms of periods in the associated geometries. The matching results, as well as Ferrari's solution of a previously unsolved matrix model, suggest that not only can matrix models simplify computations in string theory, but associated geometries from string theory can simplify computations in matrix models.

[1] For a rigorous derivation of the D-brane superpotential, see [3].

[2] Specifically, it is the triple of spaces $\widehat{\mathcal{M}}, \mathcal{M}_0$ and $\mathcal{M}$ that are conjectured to encode the matrix model quantities; the blow-down map $\pi : \widehat{\mathcal{M}} \longrightarrow \mathcal{M}_0$ is the most difficult step towards performing the matrix model computations [13].

Many questions immediately arise from Ferrari's construction. In particular, the matrix model resolvents are not directly encoded in the resolved geometry $\widehat{\mathcal{M}}$, but require knowing the corresponding singular geometry $\mathcal{M}_0$ obtained by blowing down the exceptional $\mathbb{P}^1$. It is not clear how to do this blow-down in general. It is also not obvious that a geometry constructed from a matrix model potential in this manner will indeed contain a $\mathbb{P}^1$ which can be blown down to become an isolated singularity.[3] Just which matrix models can be "geometrically engineered" in this fashion? What are the corresponding geometries? Can different matrix model potentials correspond to the same geometry? If so, what common features of those models does the geometry encode? Ferrari asks many such questions at the end of his paper [13], and also wonders whether or not it might be possible to devise an algorithm which will automatically construct the blow-down given the initial resolved space.

Previously established results in algebraic geometry such as Laufer's theorem [18] and the classification of Gorenstein threefold singularities by S. Katz and D. Morrison [17] provide a partial answer to these questions.

Theorem 1.1 (Laufer 1979)**.** *Let $\mathcal{M}_0$ be an analytic space of dimension $D \geq 3$ with an isolated singularity at p. Suppose there exists a non-zero holomorphic D-form Ω on $\mathcal{M}_0 - \{p\}$.*[4] *Let $\pi : \widehat{\mathcal{M}} \longrightarrow \mathcal{M}_0$ be a resolution of $\mathcal{M}_0$. Suppose that the exceptional set $A = \pi^{-1}(p)$ is one-dimensional and irreducible. Then $A \cong \mathbb{P}^1$ and $D = 3$. Moreover, the normal bundle of $\mathbb{P}^1$ in $\widehat{\mathcal{M}}$ must be either $\mathcal{N} = \mathcal{O}(-1) \oplus \mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$.*

Laufer's theorem immediately tells us that we can restrict our search of possible geometries to dimension 3, and that there are only three candidates for the normal bundle to our exceptional $\mathbb{P}^1$. In Ferrari's construction, the bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-2)$ and $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ correspond to zero-, one- and two-matrix models, respectively.[5] Following the Dijkgraaf–Vafa correspondence, this puts a limit of two adjoint fields on the associated gauge theory, which is precisely the requirement for asymptotic freedom in $\mathcal{N}=1$ supersymmetric gauge theories. This happy coincidence is perhaps our first indication that the Calabi–Yau geometry encodes information about the RG flow of its corresponding matrix model or gauge theory.

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[3]We will see later that the "hat" potentials from the Intriligator–Wecht classification lead to geometries where an entire family of $\mathbb{P}^1$s is blown down to reveal a space $\mathcal{M}_0$ with non-isolated singularities. It is interesting to wonder what the corresponding "geometric transition conjecture" should be for these cases.

[4]This is the Calabi–Yau condition.

[5]This is because these bundles have zero, one and two independent global sections, respectively.

The condition $M \leq 2$ for the normal bundle $\mathcal{O}(M-1) \oplus \mathcal{O}(-M-1)$ in Laufer's theorem is equivalent to asymptotic freedom, and reflects the fact that only for asymptotically free theories can we expect the $\mathbb{P}^1$ to be exceptional. In considering matrix model potentials with $M \geq 3$ fields, Ferrari points out that the normal bundle to the $\mathbb{P}^1$ changes with the addition of the perturbation $\partial_{w_1} E(\gamma, w_1)$, and makes the following conjecture [13, p. 636].

Conjecture 1.1 (RG flow, Ferrari 2003). *Consider the perturbed geometry for $m = -n-2$ and associated superpotential W. Let $\mathcal{N}$ be the normal bundle of a $\mathbb{P}^1$ that sits at a given critical point of W. Let r be the corank of the Hessian of W at the critical point. Then $\mathcal{N} = \mathcal{O}(r-1) \oplus \mathcal{O}(-r-1)$.*

Ferrari proves the conjecture for $n = 1$, and limits himself to two-matrix models ($M = n+1 = 2$) in the rest of his paper. Our first result gives evidence in support of the RG flow conjecture in a more general setting.

Proposition 1.1. *For $-M \leq r \leq M$, the addition of the perturbation term $\partial_{w_1} E(\gamma, w_1) = \gamma^{r+1} w_1$ in the transition functions*

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-M+1} w_1, \quad v_2 = \gamma^{M+1} w_2 + \gamma^{r+1} w_1,$$

changes the bundle from $\mathcal{O}(M-1) \oplus \mathcal{O}(-M-1)$ to $\mathcal{O}(r-1) \oplus \mathcal{O}(-r-1)$. In particular, the M-matrix model potential

$$W(x_1, \ldots, x_M) = \frac{1}{2} \sum_{i=1}^{M-r} x_i x_{M-r+1-i}, \quad (r \geq 0)$$

is geometrically equivalent[6] *to the r-matrix model potential*

$$W(x_1, \ldots, x_r) = 0.$$

The proof is given in Section 3. The fact that the associated superpotential is purely quadratic is satisfying since for quadratic potentials we can often "integrate out" fields, giving a field-theoretic intuition for why the geometry associated to an M-matrix model can be equivalent to that of an r-matrix model, with $r < M$.

Laufer's theorem constraints the dimension, the exceptional set and its normal bundle, but what are the possible singularity types? In the surface case (complex dimension two), the classification of simple singularities is a classic result.[7] As hypersurfaces in $\mathbb{C}^3$, the distinct geometries

[6] We will call two potentials *geometrically equivalent* if they yield the same geometry under Ferrari's construction.

[7] An excellent reference for this and other results in singularity theory is [2]. For a more applications-oriented treatment (with many cute pictures!) see Arnold's 1991 book [1]. For 15 characterizations of rational double points, see [12].

Table 1: Gorenstein threefold singularities in preferred versal form [17, p. 465].

S	Preferred versal form
$A_{n-1}(n \geq 2)$	$-XY + Z^n + \sum_{i=2}^{n} \alpha_i Z^{n-i}$
$D_n(n \geq 3)$	$X^2 + Y^2 Z - Z^{n-1} - \sum_{i=1}^{n-1} \delta_{2i} Z^{n-i-1} + 2\gamma_n Y$
E_4	$-XY + Z^5 + \varepsilon_2 Z^3 + \varepsilon_3 Z^2 + \varepsilon_4 Z + \varepsilon_5$
E_5	$X^2 + Y^2 Z - Z^4 - \varepsilon_2 Z^3 - \varepsilon_4 Z^2 + 2\varepsilon_5 Y - \varepsilon_6 Z - \varepsilon_8$
E_6	$-X^2 - XZ^2 + Y^3 + \varepsilon_2 YZ^2 + \varepsilon_5 YZ + \varepsilon_6 Z^2 + \varepsilon_8 Y \varepsilon_9 Z + \varepsilon_{12}$
E_7	$-X^2 - Y^3 + 16YZ^3 + \varepsilon_2 Y^2 Z + \varepsilon_6 Y^2 + \varepsilon_8 YZ + \varepsilon_{10} Z^2$ $+ \varepsilon_{12} Y + \varepsilon_{14} Z + \varepsilon_{18}$
E_8	$-X^2 + Y^3 - Z^5 + \varepsilon_2 YZ^3 + \varepsilon_8 YZ^2 + \varepsilon_{12} Z^3 + \varepsilon_{14} YZ$ $+\varepsilon_{18} Z^2 + \varepsilon_{20} Y + \varepsilon_{24} Z + \varepsilon_{30}$

are given by:

$$\begin{array}{ll} A_k \quad : x^2 + y^2 + z^{k+1} & = 0 \\ D_{k+2} : x^2 + y^2 z + z^{k+1} & = 0 \\ E_6 \quad : x^2 + y^3 + z^4 & = 0 \\ E_7 \quad : x^2 + y^3 + yz^3 & = 0 \\ E_8 \quad : x^2 + y^3 + z^5 & = 0 \end{array}$$

In 1992, Katz and Morrison answered this question in dimension 3 when they characterized the full set of Gorenstein threefold singularities with irreducible small resolutions using invariant theory [17]. In order to do the classification, Katz and Morrison find it useful to think of threefolds as deformations of surfaces, where the deformation parameter t takes on the role of the extra dimension. The equations for the singularities can thus be written in so-called *preferred versal form*, as given in table 1. The coefficients $\alpha_i, \delta_i, \gamma_i$ and ε_i are given by invariant polynomials, and are implicity functions of the deformation parameter t. We will also find this representation of the singular threefolds useful in identifying what kinds of singular geometries we get upon blowing down resolved geometries.

In contrast to what one might expect,[8] there are only a finite number of families of Gorenstein threefold singularities with irreducible small

[8]By taking hyperplane sections, one may get surface singularities corresponding to any of the ADE Dynkin diagrams. A priori, this could indicate that there is an infinite number of families of the threefold singularities with irreducible small resolutions. What Katz and Morrison discovered is that only a finite number of Dynkin diagrams can arise from "generic" hyperplane sections.

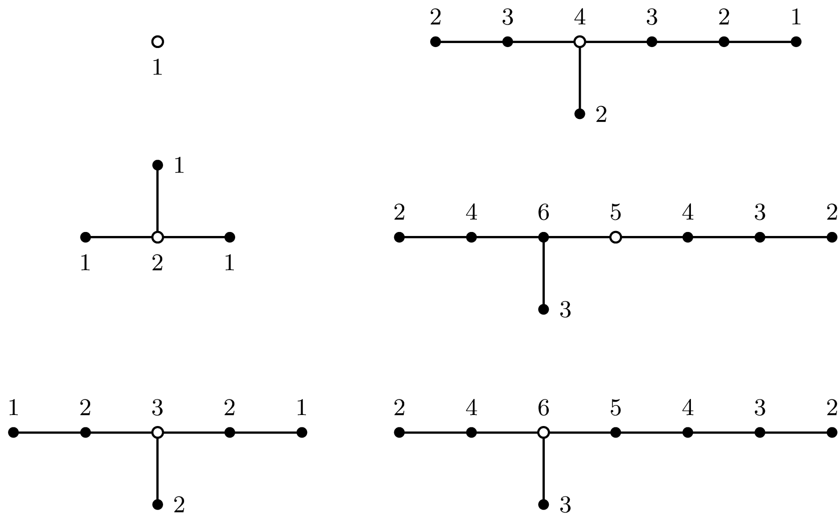

Figure 2: The six types of Gorenstein threefold singularities.

resolutions. They are distinguished by the Kollár "length" invariant,[9] and are resolved via small resolution of the appropriate length node in the corresponding Dynkin diagram. The precise statement of Katz and Morrison's results are given by the following theorem and corollary [17, p. 456].

Theorem 1.2 (Katz & Morrison 1992)**.** *The generic hyperplane section of an isolated Gorenstein threefold singularity which has an irreducible small resolution defines one of the primitive partial resolution graphs in figure* 2*. Conversely, given any such primitive partial resolution graph, there exists an irreducible small resolution $Y \to X$ whose general hyperplane section is described by that partial resolution graph.*

Corollary 1.1. *The general hyperplane section of X is uniquely determined by the length of the singular point P.*

We thus know that there are only a finite number of families of distinct geometries with the desired properties for Ferrari's construction, and they correspond to isolated threefold singularities with small resolutions. While much is known about the resolution of these singularities (they are obtained by blowing up divisors associated to nodes of the appropriate length in the corresponding Dynkin diagram), it is not easy to perform the small blow-up explicitly.

[9]**Definition.** Let $\pi : Y \longrightarrow X$ irreducible small resolution of an isolated threefold singularity $p \in X$. Let $C = \pi^{-1}(p)$ be the exceptional set. The *length* of p is the length at the generic point of the scheme supported on C, with structure sheaf $\mathcal{O}_Y/\pi^{-1}(m_p, x)$.

type	$W(X,Y)$
$\widehat{O}$	0
$\widehat{A}$	$\mathrm{Tr}\, Y^2$
$\widehat{D}$	$\mathrm{Tr}\, XY^2$
$\widehat{E}$	$\mathrm{Tr}\, Y^3$
A_k	$\mathrm{Tr}(X^{k+1} + Y^2)$
D_{k+2}	$\mathrm{Tr}(X^{k+1} + XY^2)$
E_6	$\mathrm{Tr}(Y^3 + X^4)$
E_7	$\mathrm{Tr}(Y^3 + YX^3)$
E_8	$\mathrm{Tr}(Y^3 + X^5)$

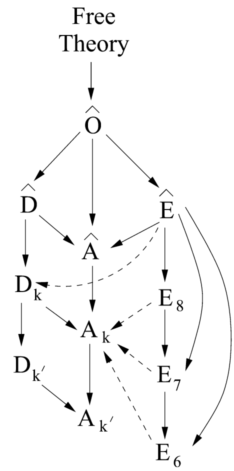

Figure 3: Intriligator–Wecht classification of RG fixed points. The diagram on the right shows the map of possible flows between fixed points. Dotted lines indicate flow to a particular value of k. Note that $k' < k$ [15, pp. 3–4].

The major obstacle in identifying which matrix model corresponds to each of the candidate singular geometries from [17] is the absence of a simple description in the form of (1.1) for their small resolutions. This frustration is also expressed in [5], where the same geometries are used to construct $\mathcal{N}=1$ ADE quiver theories.[10] In the case where the normal bundle to the exceptional $\mathbb{P}^1$ is $\mathcal{O}(1) \oplus \mathcal{O}(-3)$, only Laufer's example [18] and its extension by Pinkham and Morrison [20, p. 368] was known. For us, the resolution to this problem came from a timely, albeit surprising, source.

In September 2003, Intriligator and Wecht posted their results on RG fixed points of $\mathcal{N}=1$ SQCD with adjoints [15]. Using "a-maximization" and doing a purely field theoretic analysis, they classified all relevant adjoint superpotential deformations for 4d $\mathcal{N}=1$ SQCD with $\mathcal{N}_f$ fundamentals and $\mathcal{N}_a = 2$ adjoint matter chiral superfields, X and Y. The possible RG fixed points, together with the map of possible flows between fixed points, are summarized in figure 3.

Due to the form of the polynomials, Intriligator and Wecht named the relevant superpotential deformations according to the famous ADE classification of singularities in dimensions 1 and 2 (see equation (1.3)). There is

[10] "... the gauge theory description suggests a rather simple global geometric description of the blown up $\mathbb{P}^1$ for all cases. However, such a mathematical construction is not currently known in the full generality suggested by the gauge theory. Instead only some explicit blown up geometries are known in detail ..." [5, p. 35].

no geometry in their analysis, however, and they seem surprised to uncover a connection to these singularity types.[11]

Naively, one may speculate that this is the answer.[12] We make the following conjecture.

Conjecture 1.2. *The superpotentials in Intriligator and Wecht's ADE classification for $\mathcal{N}=1$ gauge theories (equivalently, the polynomials defining simple curve singularities), are precisely the matrix model potentials which yield small resolutions of Gorenstein threefold singularities using Ferrari's construction* (1.1)*.*

Armed with this new conjecture, we may now run the classification program backwards. Starting from the resolved space $\widehat{\mathcal{M}}$ given by transition functions over the exceptional $\mathbb{P}^1$, we can verify the correspondence by simply performing the blow down and confirming that the resulting geometry has the right singularity type. In particular, the matrix model superpotentials (if correct) give us simple descriptions for the small resolutions of Gorenstein threefold singularities in terms of transition functions as in (1.1). Like other geometric insights stemming from dualities in string theory, such a result is of independent mathematical interest.

This still leaves us with some major challenges. As Ferrari pointed out, there was no known systematic way of performing the blow downs, and our first task was to devise an algorithm to do so [7]. The algorithm can be implemented by the computer,[13] and searches for global holomorphic functions which can be used to construct the blow down. Any global holomorphic function on the resolved geometry $\widehat{\mathcal{M}}$ is necessarily constant on the exceptional $\mathbb{P}^1$, so these functions are natural candidates for coordinates on the blown down geometry $\mathcal{M}_0$, since the $\mathbb{P}^1$ must collapse to a point. The algorithm finds all (independent) global holomorphic functions which can be built from a specified list of monomials. Because such a list can never be exhaustive, the resulting singular space $\mathcal{M}_0$, whose defining equations are obtained by finding relations among the global holomorphic functions, must be checked. We can verify that we do, in fact, recover the original smooth space by inverting the blow down and performing the small resolution of the singular point. Once we have shown that the collection of global

[11] "On the face of it, this has no obvious connection to any of the other known ways in which Arnold's singularities have appeared in mathematics or physics" [15, p. 3].

[12] In particular, if the Dijkgraaf–Vafa conjecture holds, we should expect any classification of $\mathcal{N}=1$ gauge theories to have a matrix model counterpart. Verifying such a correspondence thus provides a non-trivial consistency check on the proposed string theory/matrix model duality.

[13] See [7] for Maple code.

holomorphic functions gives us the right blow-down map, it does not really matter how we found them. Because it may be used more generally for finding blow-down maps (in particular, for resolved geometries corresponding to potentials we have not considered here), we include a description of the algorithm in the appendix.

We find that this program works perfectly in the A_k (length 1) and D_{k+2} (length 2) cases, lending credence to the idea that the Intriligator–Wecht classification is, indeed, the right answer. In the exceptional cases, however, a few mysteries arise. We are only able to find the blow down for the Intriligator–Wecht superpotential E_7, and the resulting singular space has a length 3 singularity. We summarize these results in the following theorem.

Theorem 1.3. *Consider the two-matrix model potentials* $W(x,y)$ *in table* 2*, with corresponding resolved geometries* $\widehat{\mathcal{M}}$*,*

$$\beta = 1/\gamma, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \partial_{w_1} E(\gamma, w_1),$$

given by perturbation terms $\partial_{w_1} E(\gamma, w_1)$*. Blowing down the exceptional* $\mathbb{P}^1$ *in each* $\widehat{\mathcal{M}}$ *yields the singular geometries* $\mathcal{M}_0$ *given in table* 2*.*

By comparing the above singular geometries with the equations in preferred versal form (table 1), we immediately identify the A_k and D_{k+2} superpotentials as corresponding to lengths 1 and 2 threefold singularities, respectively. For the E_7 potential, we first note that the polynomial $X^2 - Y^3 + Z^5 + 3TYZ^2 + T^3 Z$ is in the preferred versal form for E_8 (with

Table 2: Superpotentials corresponding to length 1, length 2, and length 3 singularities.

Type	$W(x,y)$	$\partial_{w_1} E(\gamma, w_1)$	Singular geometry $\mathcal{M}_0$
A_k	$\frac{1}{k+1} x^{k+1} + \frac{1}{2} y^2$	$\gamma^2 w_1^k + w_1$	$XY - T(Z^k - T) = 0$
D_{k+2}	$\frac{1}{k+1} x^{k+1} + xy^2$	$\gamma^2 w_1^k + w_1^2$	$X^2 - Y^2 Z + T(Z^{k/2} - T)^2 = 0$, k even $X^2 - Y^2 Z - T(Z^k - T^2) = 0$, k odd
E_7	$\frac{1}{3} y^3 + yx^3$	$\gamma^{-1} w_1^2 + \gamma w_1^3$	$X^2 - Y^3 + Z^5 + 3TYZ^2 + T^3 Z = 0.$

$\varepsilon_8 = -3T$ and $\varepsilon_{24} = -T^3$).[14] On the other hand, using Proposition 4 in the proof of the Katz–Morrison classification [17, pp. 499–500], we see that the presence of the monomial T^3Z constrains the threefold singularity type to length 3. We thus have the following corollary.

Corollary 1.2. *The resolved geometries $\widehat{\mathcal{M}}$ given by the two-matrix model potentials A_k, D_{k+2} and E_7 in table* 2 *are small resolutions for length 1, length 2 and length 3 singularities, respectively.*

Although simple descriptions of the form (1.1) were previously known for small resolutions of length 1 and length 2 Gorenstein threefold singularities (Laufer's example [18] in the length 2 case), it is striking that in no other case such a concrete representation for the blow-up was known. Theorem 1.3, together with its corollary, show a length 3 example where the small resolution also has an extremely simple form:

$$\beta = 1/\gamma, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^{-1}w_1^2 + \gamma w_1^3.$$

The proof of Theorem 1.3 is given in Section 4. Missing are examples of length 4, length 5 and length 6 singularity types. For the moment, we are skeptical about whether or not these are describable using geometries that are simple enough to fit into Ferrari's framework.

In some sense the Intriligator–Wecht classification does not contain enough superpotentials; only length 1, 2, and 3 singularities appear to be included. On the other hand, there are too many: the additional superpotentials $\widehat{O}, \widehat{A}, \widehat{D}$ and $\widehat{E}$ have no candidate geometries corresponding to the Katz–Morrison classification of Gorenstein threefold singularities! What kind of geometries do these new cases correspond to? What (if any) is their relation to the original ADE classification? Using Ferrari's framework and our new algorithmic blow down methods we are able to identify the geometries corresponding to these extra "hat" cases. We summarize the results in the following theorem.

Theorem 1.4. *The singular geometries corresponding to the $\widehat{O}, \widehat{A}, \widehat{D}$ and $\widehat{E}$ cases in the Intriligator–Wecht classification of superpotentials are given in table* 3.

The proof of Theorem 1.4 is the content of Section 5. We find that the resolved geometries have full families of $\mathbb{P}^1$s which are blown down, and the resulting singular spaces have interesting relations to the ADE cases. The $\widehat{A}$ geometry is a curve of A_1 singularities, while the equation for $\widehat{D}$

[14] $T = 0$ yields a hyperplane section with E_8 surface singularity.

Table 3: Geometries for the "hat" cases.

Type	$W(x,y)$	$\partial_{w_1} E(\gamma, w_1)$	Singular geometry $\mathcal{M}_0$
$\widehat{O}$	0	0	$\mathbb{C}^3/\mathbb{Z}_3$
$\widehat{A}$	$\frac{1}{2}y^2$	$\gamma^2 w_1$	$\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$
$\widehat{D}$	xy^2	γw_1^2	$X^2 + Y^2 Z - T^3 = 0$
$\widehat{E}$	$\frac{1}{3}y^3$	$\gamma^2 w_1^2$	$\mathrm{Spec}(\mathbb{C}[a,b,u,v]/\mathbb{Z}_2)/(b^4 - u^2 - av)$

looks like the equations for D_{k+2} where the k-dependent terms have been dropped. The identification of new, related geometries obtained by combining Ferrari's framework with the Intriligator–Wecht classification turns out to be one of the most interesting parts of our story. The presence of these extra geometries may have implications for the relevant string dualities; perhaps the geometric transition conjecture can be expanded beyond isolated singularities.

It is surprising that even in the $\widehat{O}$ case, with $W(x,y) = 0$ superpotential, the geometry is highly non-trivial. In fact, we find that it is the A_1, A_k and $\widehat{A}$ cases which are, in some sense, the "simplest." Although the descriptions for the resolved geometries $\widehat{\mathcal{M}}$ in these cases make it appear as though the normal bundles to the exceptional $\mathbb{P}^1$s are all $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ [as required by a two-matrix model potential $W(x,y)$], these geometries can all be described with fewer fields. A straightforward application of Proposition 1.1 shows that the $\widehat{A}$ case is equivalent to a one-matrix model with $W(x) = 0$ superpotential. Similarly, we will see in Section 4.2 that

$$W(x,y) = \frac{1}{k+1}x^{k+1} + \frac{1}{2}y^2 \quad \text{and} \quad W(x) = \frac{1}{k+1}x^{k+1}$$

are geometrically equivalent, so the A_k (length 1) cases are also seen to correspond to one-matrix models, where the y field has been "integrated out." This is a relief because we know that the exceptional $\mathbb{P}^1$ after blowing up an A_k singularity should have normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$. When $k = 1$, Proposition 1.1 further reduces the geometry to that of a zero-matrix model (no superpotential possible!), showing that A_1 is the most trivial case, with

Table 4: Geometrically equivalent superpotentials.

Type	*Two*-matrix model	*One*-matrix model	*Zero*-matrix model
$\widehat{O}$	$W(x,y)=0$		
$\widehat{A}$	$W(x,y)=\frac{1}{2}y^2$	$W(x)=0$	
A_k	$W(x,y)=\frac{1}{k+1}x^{k+1}+\frac{1}{2}y^2$	$W(x)=\frac{1}{k+1}x^{k+1}$	
A_1	$W(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2$	$W(x)=\frac{1}{2}x^2$	$W=0$

normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$.[15] These results are summarized in table 4, and can be understood as evidence for Ferrari's RG conjecture. (Compare with Intriligator and Wecht's map of possible RG flows in figure 3.)

Our analysis also indicates that the names are well-chosen: the $\widehat{A}$ and $\widehat{D}$ geometries are closely related to their A_k and D_{k+2} counterparts, and the same might be true for the $\widehat{E}$ case. The relationship between the $\widehat{A},\widehat{D}$ and $\widehat{E}$ geometries and the ADE singularities is worth exploring for purely geometric reasons. To summarize, string dualities have told us to enlarge the class of geometries considered in [17], and have pointed us to closely related "limiting cases" of these geometries with non-isolated singularities.

2 The geometric framework

Here we review Ferrari's construction of non-compact Calabi–Yau's from matrix model superpotentials, following section 3 of [13]. The main idea behind the geometric setup is that deformations of the exceptional $\mathbb{P}^1$ in a resolved geometry $\widehat{\mathcal{M}}$ correspond to adjoint fields in the gauge theory [5]. Alternatively, the deformation space for a $\mathbb{P}^1$ wrapped by D-branes can be thought of in terms of matrix models. The number N of D-branes wrapping the $\mathbb{P}^1$ gives the size of the matrices ($N\times N$), while the number

[15]In contrast, the normal bundle in the $\widehat{O}$ case is truly $\mathcal{O}(1)\oplus\mathcal{O}(-3)$, showing that this geometry requires a two-matrix model description.

M of independent sections of the $\mathbb{P}^1$ normal bundle gives the number of matrices (an M-matrix model).

Inspired by the string theory dualities, Ferrari develops a recipe to go straight from the matrix model to a Calabi–Yau space. If the dualities hold, all of the matrix model quantities should be computable from the corresponding geometry. In this way, Ferrari's prescription provides a non-trivial consistency check on the Gopakumar–Vafa and Dijkgraaf–Vafa conjectures. Moreover, such a Calabi–Yau space provides a natural higher-dimensional analogue for the spectral (hyperelliptic) curve which encodes the solution to the hermitian one-matrix model.

For each theory, there are three relevant Calabi–Yau spaces: the resolved Calabi–Yau $\widehat{\mathcal{M}}$, the singular Calabi–Yau $\mathcal{M}_0$ and the smooth deformed space $\mathcal{M}$. In short, Ferrari's game consists of the following steps:

1. Start with an M-matrix model matrix model superpotential $W(x_1, \ldots, x_M)$ and construct a smooth Calabi–Yau $\widehat{\mathcal{M}}$. The details of this construction are presented subsequently.
2. Identify the exceptional $\mathbb{P}^1$ in the resolved space $\widehat{\mathcal{M}}$.
3. Blow down the exceptional $\mathbb{P}^1$ to get the singular $\mathcal{M}_0$. The blow down map is $\pi : \widehat{\mathcal{M}} \longrightarrow \mathcal{M}_0$.
4. Perturb the algebraic equation for $\mathcal{M}_0$ to get the smooth deformed space $\mathcal{M}$.
5. From the triple of geometries, compute matrix model quantities (resolvents).
6. Use standard matrix model techniques (loop equations) to check answers in cases where the matrix model solution is known.

In this framework, the matrix model superpotential is encoded in the transition functions defining the resolved geometry $\widehat{\mathcal{M}}$. Ferrari shows that a wide variety of matrix model superpotentials arise in this fashion, and that matrix model resolvents can be computed directly from the geometry. In other words, the solution to the matrix model is encoded in the corresponding triple of Calabi–Yau's.

The bottleneck to this program is Step 3, the construction of the blow-down map. While Ferrari's ad-hoc methods for constructing the blow-down are successful in his particular examples, he does not know how to construct the blow down in general. Moreover, it seems the calculation of the blow down map π is essentially equivalent in solving the associated matrix model, and hence it would be very useful to have an algorithm which computes π [13, p. 655].

In this paper, we are mainly concerned with Steps 1–3. Our goal is to show how to compute the blow-down map in a large class of examples, and therefore to understand better which singular geometries $\mathcal{M}_0$ arise from matrix models in Ferrari's framework. In the future, it would be nice to implement the deformation to $\mathcal{M}$ and also to compute matrix model quantities for our examples (Steps 4–6). For the present, this is beyond our scope.

2.1 Step 1: Construction of resolved Calabi–Yau

We now turn to Step 1, the construction of the "upstairs" resolved space $\widehat{\mathcal{M}}$ given the matrix model superpotential $W(x_1, \ldots, x_k)$. $\widehat{\mathcal{M}}$ is given the transition functions between just two coordinate charts over an exceptional $\mathbb{P}^1$: (β, v_1, v_2) in the first chart, and (γ, w_1, w_2) in the second chart. β and γ should be thought of as stereographic coordinates for the $\mathbb{P}^1$, with $\beta = \gamma^{-1}$. The other coordinates v_1, v_2 and w_1, w_2 span the normal directions to the $\mathbb{P}^1$, and have non-trivial transition functions.

We first discuss the case where $W = 0$, in which the Calabi–Yau is the total space of a vector bundle over the exceptional $\mathbb{P}^1$. We then show how a simple deformation of the transition functions leads to constraints on the sections of the bundle. The independent sections $x_1, \ldots, x_k$ correspond to matrix degrees of freedom (k independent sections for a k-matrix model). The constraints can be encoded in a potential $W(x_1, \ldots, x_k)$. When W is non-zero, our geometry $\widehat{\mathcal{M}}$ is no longer a vector bundle — if the total space were a vector bundle, the sections $x_1, \ldots, x_k$ would be allowed to move freely and therefore satisfy no constraints. We shall refer to geometries with $W \neq 0$ as "deformed" or "constrained" bundles.

Pure $\mathcal{O}(n) \oplus \mathcal{O}(m)$ bundle. Consider the following $\widehat{\mathcal{M}}$ geometry for $n \geq 0$ and $m < 0$:

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-n} w_1, \quad v_2 = \gamma^{-m} w_2.$$

There is an $(n+1)$-dimensional family of $\mathbb{P}^1$s that sit at

$$w_1(\gamma) = \sum_{i=1}^{n+1} x_i \gamma^{i-1}, \quad w_2(\gamma) = 0.$$

We have no freedom in the w_2 coordinate because $m < 0$ precludes $v_2(\beta)$ from being holomorphic whenever $w_2(\gamma)$ is. $w_1(\gamma)$ and $w_2(\gamma)$ define globally holomorphic sections, and in the β coordinate patch become

$$v_1(\beta) = \sum_{i=1}^{n+1} x_i \beta^{n-i+1}, \quad v_2(\beta) = 0.$$

The parameters x_i are precisely the fields in the associated superpotential, and they span the versal deformation space of the $\mathbb{P}^1$s. In this case there are no constraints on the x_is, which correspond to the fact that the superpotential is

$$W(x_1, \ldots, x_{n+1}) = 0.$$

The geometry $\widehat{\mathcal{M}}$ is the total space of a vector bundle, which we might refer to as a "free" bundle because it is not constrained.

Deformed bundle; enter superpotential. Now consider the deformed geometry (with $n \geq 0$ and $m < 0$):

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-n} w_1, \quad v_2 = \gamma^{-m} w_2 + \partial_{w_1} E(\gamma, w_1),$$

where $E(\gamma, w_1)$ is a function of two complex variables which can be Laurent expanded in terms of entire functions E_i,

$$E(\gamma, w_1) = \sum_{i=-\infty}^{\infty} E_i(w_1)\gamma^i.$$

We call E the "geometric potential." The most general holomorphic section $(w_1(\gamma), w_2(\gamma))$ of the normal bundle $\mathcal{N}$ to the $\mathbb{P}^1$s still has

$$w_1(\gamma) = \sum_{i=1}^{n+1} x_i \gamma^{i-1}, \quad v_1(\beta) = \sum_{i=1}^{n+1} x_i \beta^{n-i+1},$$

but in order to ensure that $v_2(\beta)$ is holomorphic, the x_is will have to satisfy some constraints. Since a holomorphic $w_2(\gamma)$ can only cancel poles in β^{-j} for $j \geq |m|$, the x_is must satisfy $|m| - 1$ constraints in order to cancel remaining lower-order poles introduced by the perturbation. Hence the versal deformation space of the $\mathbb{P}^1$ is spanned by $n+1$ parameters x_i satisfying $|m| - 1 = -m - 1$ constraints.

For the $\mathbb{P}^1$ to be isolated we need $n + 1 = -m - 1$, and we denote this quantity (the number of fields) by M. The constraints are integrable, and equivalent to the extremization $\delta W = 0$ of the corresponding superpotential $W(x_1, \ldots, x_M)$. The $\mathbb{P}^1$s then sit at the critical points of the superpotential, in the sense that for critical values of the x_is, the pair $(w_1(\gamma), w_2(\gamma))$ will be a global holomorphic section defining a $\mathbb{P}^1$.

Summary: General transition functions for $\widehat{\mathcal{M}}$. The resolved geometry $\widehat{\mathcal{M}}$ is described by two coordinate patches (γ, w_1, w_2) and (β, v_1, v_2), with transition functions

$$\beta = 1/\gamma, \quad v_1 = \gamma^{-n} w_1, \quad v_2 = \gamma^{-m} w_2 + \partial_{w_1} E(\gamma, w_1).$$

In the absence of the $\partial_{w_1} E(\gamma, w_1)$ term, this would simply be an $\mathcal{O}(n) \oplus \mathcal{O}(m)$ bundle over the $\mathbb{P}^1$ parametrized by the stereographic coordinates γ and β. The perturbation comes from the "geometric potential" $E(\gamma, w)$, which can be expanded as

$$E(\gamma, w) = \sum_{i=-\infty}^{+\infty} E_i(w) \gamma^i.$$

The superpotential. The matrix model superpotential encodes the constraints on the sections $x_1, \ldots, x_M$ due to the presence of the perturbation term $\partial_{w_1} E(\gamma, w_1)$ in the defining transition functions for $\widehat{\mathcal{M}}$. It can be obtained directly from the geometric potential via

$$W(x_1, \ldots, x_M) = \frac{1}{2\pi i} \oint_{C_0} \gamma^{-M-1} E\left(\gamma, \sum_{i=1}^{M} x_i \gamma^{i-1}\right) \delta\gamma,$$

where

$$M = n + 1 = -m - 1.$$

The contour integral is meant as a bookkeeping device; C_0 should be taken as a small loop encircling the origin. The integral is a compact notation used by Ferrari to encode all of the constraints at once. The general method for going from transition function perturbation (geometric potential) to superpotential was first presented in [16].

This procedure is also invertible. In other words, given a matrix model superpotential $W(x_1, \ldots, x_M)$ one can find a corresponding geometric potential $E(\gamma, w_1)$, and therefore construct the associated geometry. E is not in general unique; from the expression for W one can see that terms can always be added to the geometric potential which will not contribute to the residue of the integrand, and hence will not affect the superpotential. Such terms have no effect on the geometry, however.

Going from W to E is essentially the implementation of Step 1 in Ferrari's game. We now turn to Step 2, which is to locate the exceptional $\mathbb{P}^1$s.

2.2 Step 2: Locating the $\mathbb{P}^1$s

The first task in constructing the blow-down maps is figuring out where the $\mathbb{P}^1$s that we want to blow down are located. We will mostly be interested in the $M = 2$ case,

$$\beta = \gamma^{-1}, \qquad v_1 = \gamma^{-1} w_1, \qquad v_2 = \gamma^3 w_2 + \partial_{w_1} E(\gamma, w_1),$$

where we always have

$$w_1(\gamma) = x + \gamma y, \qquad v_1(\beta) = \beta x + y,$$

with x and y critical points of $W(x, y)$ at the $\mathbb{P}^1$s. Depending on the form of the perturbation $\partial_{w_1} E(\gamma, w_1)$, $w_2(\gamma)$ will be chosen to cancel poles of order ≥ 3. The requirement that $v_2(\beta)$ be holomorphic will fix x and y values to be the same as for the critical points of $W(x, y)$.

2.3 Step 3: Finding the blow down map in Ferrari's examples

As previously mentioned, Ferrari has no systematic way of constructing the blow-down map

$$\pi : \widehat{\mathcal{M}} \longrightarrow \mathcal{M}_0.$$

He successfully finds the blow-down in several examples, however, through clever but ad-hoc methods. To see how Ferrari finds the blow-down, see the main examples from his paper [13].

2.4 Example: A_k

We now illustrate Steps 1–3 in a simple example. Consider the matrix model potential

$$W(x) = \frac{1}{k+1} x^{k+1}.$$

Since there is only one field, the resolved geometry $\widehat{\mathcal{M}}$ is given by transition functions

$$\beta = \gamma^{-1}, \quad v_1 = w_1, \quad v_2 = \gamma^2 w_2 + \gamma w_1^k,$$

for an $\mathcal{O} \oplus \mathcal{O}(-2)$ bundle over the exceptional $\mathbb{P}^1$. To locate the $\mathbb{P}^1$, we first note that

$$w_1(\gamma) = x = v_1(\beta)$$

are the only holomorphic sections for the $\mathcal{O}$ line bundle. Substituting this into the transition function for v_2 yields

$$v_2(\beta) = \beta^{-2} w_2(\gamma) + \beta^{-1} x^k,$$

which is only holomorphic if $x^k = w_2(\gamma) = 0$. Therefore, we have a single $\mathbb{P}^1$ located at

$$w_1(\gamma) = w_2(\gamma) = 0, \quad v_1(\beta) = v_2(\beta) = 0.$$

Note that the position of the $\mathbb{P}^1$ corresponds exactly to the critical point of the superpotential:

$$\delta W = x^k dx = 0 \Longrightarrow x^k = 0.$$

The blow-down. To find the blow down map π, we must look for global holomorphic functions (which will necessarily be constant on the $\mathbb{P}^1$). We can immediately write down

$$\pi_1 = v_1 = w_1,$$
$$\pi_2 = v_2 = \gamma^2 w_2 + \gamma w_1^k,$$

which are independent. Moreover, notice the combination $\beta v_2 - v_1^k = \gamma w_2$. This gives us

$$\pi_3 = \beta v_2 - v_1^k = \gamma w_2,$$
$$\pi_4 = \beta^2 v_2 - \beta v_1^k = w_2.$$

Since $\beta = \pi_4/\pi_3$, we have immediately from the definition of π_3 the relation

$$\mathcal{M}_0 : \quad \pi_3^2 = \pi_4 \pi_2 - \pi_3 \pi_1^k.$$

This corresponds to an A_k (length 1) singularity!

The blow-up. We check our computation by inverting the blow-down. If we define

$$v_3 = \beta v_2 - v_1^k, \quad \text{and} \quad w_3 = \gamma w_2 + w_1^k,$$

we can write

$$\begin{aligned} \pi_1 &= \ v_1 \ = \ w_1, \\ \pi_2 &= \ v_2 \ = \gamma w_3, \\ \pi_3 &= \ v_3 \ = \gamma w_2, \\ \pi_4 &= \beta v_3 = \ w_2. \end{aligned}$$

In particular

$$\beta = \frac{\pi_4}{\pi_3} = \gamma^{-1}.$$

This suggests that to recover the small resolution $\widehat{\mathcal{M}}$ we should blow up

$$\pi_3 = \pi_4 = 0 \quad \text{in} \quad \mathcal{M}_0 : \pi_3^2 = \pi_4 \pi_2 - \pi_3 \pi_1^k.$$

Denoting the $\mathbb{P}^1$ coordinates by $[\beta : \gamma]$ and imposing the relation

$$\beta \pi_3 = \gamma \pi_4$$

in the blow-up, we find in each chart

$$\begin{array}{c|c} (\gamma = 1) & (\beta = 1) \\ \hline \\ \pi_4 = \beta\pi_3 & \pi_3 = \gamma\pi_4 \\ \pi_3 = \beta\pi_2 - \pi_1^k & \gamma^2\pi_4 = \pi_2 - \gamma\pi_1^k \\ \\ (\beta, \pi_1, \pi_2) & (\gamma, \pi_1, \pi_4) \end{array}$$

The transition functions between the two charts are easily found to be

$$\beta = \gamma^{-1}, \quad \pi_1 = \pi_1, \quad \pi_2 = \gamma^2\pi_4 + \gamma\pi_1^k.$$

Identifying with the original coordinates, we find

$$\beta = \gamma^{-1}, \quad v_1 = w_1, \quad v_2 = \gamma^2 w_2 + \gamma w_1^k.$$

This is exactly what we started with!

Note that for $k = 1$, we have a bundle-changing superpotential (see Section 3), and the normal bundle to the exceptional $\mathbb{P}^1$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ instead of $\mathcal{O} \oplus \mathcal{O}(-2)$.

3 Superpotentials which change bundle structure

In this section we describe a family of superpotentials which change the underlying bundle structure, thus proving Proposition 1.1.

Proposition 1.1. *For $-M \leq r \leq M$, the addition of the perturbation term $\partial_{w_1} E(\gamma, w_1) = \gamma^{r+1} w_1$ in the transition functions*

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-M+1} w_1, \quad v_2 = \gamma^{M+1} w_2 + \gamma^{r+1} w_1,$$

changes the bundle from $\mathcal{O}(M-1) \oplus \mathcal{O}(-M-1)$ to $\mathcal{O}(r-1) \oplus \mathcal{O}(-r-1)$. In particular, the M-matrix model potential

$$W(x_1, \ldots, x_M) = \frac{1}{2} \sum_{i=1}^{M-r} x_i x_{M-r+1-i}, \quad (r \geq 0)$$

is geometrically equivalent[16] *to the r-matrix model potential*

$$W(x_1, \ldots, x_r) = 0.$$

[16] We will call two potentials *geometrically equivalent* if they yield the same geometry under Ferrari's construction.

Consider an $\mathcal{O}(n) \oplus \mathcal{O}(m)$ normal bundle over a $\mathbb{P}^1$ with geometric potential

$$E(\gamma, w_1) = \frac{1}{2}\gamma^k w_1^2, \quad k \in \mathbb{Z}.$$

The perturbed transition functions are

$$\beta = 1/\gamma, \quad v_1 = \gamma^{-n} w_1, \quad v_2 = \gamma^{-m} w_2 + \gamma^k w_1.$$

If (n, m, k) satisfies

$$-n \leq k \leq -m,$$

we can perform the following change of coordinates:

$$\begin{array}{ll} \widetilde{w_1} = w_1 + \gamma^{-m-k} w_2, & \widetilde{v_1} = v_2, \\ \widetilde{w_2} = w_2, & \widetilde{v_2} = -v_1 + \beta^{n+k} v_2. \end{array}$$

Notice that

$$\begin{aligned} \widetilde{v_1} &= \gamma^{-m} w_2 + \gamma^k w_1 = \gamma^k \widetilde{w_1}, \\ \widetilde{v_2} &= -\gamma^{-n} w_1 + \gamma^{-n-k}(\gamma^{-m} w_2 + \gamma^k w_1) = \gamma^{-n-m-k} \widetilde{w_2}, \end{aligned}$$

and so the new transition functions are

$$\beta = 1/\gamma, \quad \widetilde{v_1} = \gamma^k \widetilde{w_1}, \quad \widetilde{v_2} = \gamma^{-n-m-k} \widetilde{w_2}.$$

The geometric potential has changed our $\mathcal{O}(n) \oplus \mathcal{O}(m)$ bundle into an $\mathcal{O}(-k) \oplus \mathcal{O}(n+m+k)$ bundle, with no superpotential. The corresponding superpotential can be computed for $n + m = -2$, and depends on the number of fields $M = n + 1 = -m - 1$:

$$\begin{aligned} W(x_1, \ldots, x_M) &= \frac{1}{2\pi i} \oint_{C_0} \gamma^{-M-1} E\left(\gamma, \sum_{i=1}^{M} x_i \gamma^{i-1}\right) \delta\gamma \\ &= \frac{1}{2} \sum_{i=1}^{M} x_i x_{M-k-i+2}. \end{aligned}$$

Note that all of these bundle-changing superpotentials are purely quadratic! In the cases of interest, where $n + m = -2$, the condition for the change of coordinates to be valid becomes

$$-n \leq k \leq n + 2.$$

For allowed pairs (n, k) we can thus get

$$\mathcal{O}(n) \oplus \mathcal{O}(-n-2) \longrightarrow \mathcal{O}(-k) \oplus \mathcal{O}(k-2)$$

by means of the perturbation. Alternatively, we can think of these examples as "true" $\mathcal{O}(-k) \oplus \mathcal{O}(k-2)$ bundles which can be rewritten to "look like" $\mathcal{O}(n) \oplus \mathcal{O}(-n-2)$ plus a superpotential term.

RG conjecture. In order to make contact with Ferrari's RG conjecture (see Section 1), we change notation a bit from the previous discussion:

$$M = n + 1, \quad r = k - 1.$$

The perturbation term in the following transition functions

$$\beta = \gamma^{-1}, \quad w_1' = \gamma^{-M+1} w_1, \quad w_2' = \gamma^{M+1} w_2 + \gamma^{r+1} w_1,$$

changes the bundle

$$\mathcal{O}(M-1) \oplus \mathcal{O}(-M-1) \longrightarrow \mathcal{O}(r-1) \oplus \mathcal{O}(-r-1), \quad \text{for } -M \le r \le M.$$

The change of coordinates:

$$\begin{array}{ll} v_1 = w_1 + \gamma^{M-r} w_2, & v_1' = w_2', \\ v_2 = w_2, & v_2' = -w_1' + \beta^{M+r} w_2', \end{array}$$

yields new transition functions

$$\beta = \gamma^{-1}, \quad v_1' = \gamma^{r+1} v_1, \quad v_2' = \gamma^{1-r} v_2.$$

The superpotential. The superpotential corresponding to the perturbation

$$\partial_{w_1} E(\gamma, w_1) = \gamma^{r+1} w_1$$

is given by

$$W_r(x_1, \ldots, x_M) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{i=1}^{M-r} x_i x_{M-r+1-i}, & \text{for } r \ge 0, \\[2ex] \dfrac{1}{2} \displaystyle\sum_{i=1-r}^{M} x_i x_{M-r+1-i}, & \text{for } r \le 0. \end{cases}$$

This completes the proof of Proposition 1.1.

Notice that the case $r = M$ is not interesting, as the bundle remains unchanged and the superpotential vanishes. Moreover, the symmetry $r \mapsto -r$ in the bundle expression interchanges two different superpotentials, but this amounts to a simple change of coordinates. To see this, first note that:

$$r \geq 0 : W_r(x_1, \dots, x_M) = \frac{1}{2} \sum_{i=1}^{M-|r|} x_i x_{M-|r|+1-i}$$

$$r \leq 0 : W_r(x_1, \dots, x_M) = \frac{1}{2} \sum_{i=1+|r|}^{M} x_i x_{M+|r|+1-i}$$

$$= \frac{1}{2} \sum_{i=1}^{M-|r|} x_{i+|r|} x_{M+1-i}$$

The direction of the coordinate shift depends on the sign of r:

$$r \geq 0 : \quad r \mapsto -r \text{ is equivalent to } x_i \mapsto x_{i+|r|}$$

$$r \leq 0 : \quad r \mapsto -r \text{ is equivalent to } x_i \mapsto x_{i-|r|}$$

i.e. $r \mapsto -r$ on the bundle side is equivalent to a simple coordinate change for the corresponding superpotential.

We summarize the first few examples in the following table.

r	-3	-2	-1	0	1	2	3
$M=1$				$\frac{1}{2}x_1^2$			
$M=2$			$\frac{1}{2}x_2^2$	x_1x_2	$\frac{1}{2}x_1^2$		
$M=3$		$\frac{1}{2}x_3^2$	x_2x_3	$x_1x_3+\frac{1}{2}x_2^2$	x_1x_2	$\frac{1}{2}x_1^2$	
$M=4$	$\frac{1}{2}x_4^2$	x_3x_4	$x_2x_4+\frac{1}{2}x_3^2$	$x_1x_4+x_2x_3$	$x_1x_3+\frac{1}{2}x_2^2$	x_1x_2	$\frac{1}{2}x_1^2$

The Hessian. We compute the partial derivatives of our bundle-changing superpotentials:

$$r \geq 0 : \frac{\partial W_r}{\partial x_j} = x_{M-r+1-j}, \frac{\partial^2 W_r}{\partial x_k \partial x_j} = \delta_{k,M-r+1-j} \quad \text{for } 1 \leq j \leq M-r.$$

$$r \leq 0 : \frac{\partial W_r}{\partial x_j} = x_{M-r+1-j}, \frac{\partial^2 W_r}{\partial x_k \partial x_j} = \delta_{k,M-r+1-j} \quad \text{for } 1-r \leq j \leq M.$$

In each case, there is only one k for every j which yields a non-zero second-partial. This means the Hessian matrix has at most one non-zero entry in each row and in each column. The co-rank of the Hessian is thus easy to compute, and is equal to the number of rows (or columns) comprised entirely of zeroes. In both the $r \geq 0$ and $r \leq 0$ cases, the co-rank of the Hessian is r (see the ranges for j values). This is consistent with what we expect from Ferrari's RG conjecture.

4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by: (1) computing the resolved geometry $\widehat{\mathcal{M}}$ corresponding to the Intriligator–Wecht superpotentials, (2) finding global holomorphic functions (*ghf*'s) to define a blow down map $\pi : \widehat{\mathcal{M}} \to \mathcal{M}_0$, (3) determining the geometry of the blow down $\mathcal{M}_0$ by finding relations among the *ghf*'s and (4) blowing up the singular point in $\mathcal{M}_0$ in order to check that we do, indeed, recover the original resolved space. The global holomorphic functions are found using the algorithm described in the Appendix (see [7] for a detailed implementation). Techniques for performing the blow ups can be found in [6, 7].

4.1 The case A_k

In this section we prove the first part of Theorem 1.3: *The resolved geometry corresponding to the Intriligator–Wecht superpotential*

$$W(x,y) = \frac{1}{k+1}x^{k+1} + \frac{1}{2}y^2,$$

corresponds to the singular geometry

$$XY - T(Z^k - T) = 0.$$

In the next section, we will also discover that this potential is geometrically equivalent to

$$W(x) = \frac{1}{k+1}x^{k+1}.$$

The resolved geometry $\widehat{\mathcal{M}}$. From the Intriligator–Wecht superpotential

$$W(x,y) = \frac{1}{k+1}x^{k+1} + \frac{1}{2}y^2,$$

we compute the resolved geometry $\widehat{\mathcal{M}}$ in terms of transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1.$$

To find the $\mathbb{P}^1$s, we substitute $w_1(\gamma) = x + \gamma y$ into the v_2 transition function

$$v_2(\beta) = \beta^{-3}w_2 + \beta^{-2}(x+\beta^{-1}y)^k + x + \beta^{-1}y.$$

If we choose

$$w_2(\gamma) = \frac{x^k - (x+\gamma y)^k}{\gamma},$$

in the β chart the section is

$$v_1(\beta) = \beta x + y, \quad v_2(\beta) = \beta^{-2}x^k + \beta^{-1}y + x.$$

This is only holomorphic if

$$x^k = y = 0,$$

and so we have a single $\mathbb{P}^1$ located at

$$w_1(\gamma) = w_2(\gamma) = 0, \quad v_1(\beta) = v_2(\beta) = 0.$$

This is exactly what we expect from computing critical points of the superpotential

$$\delta W = x^k dx + ydy = 0.$$

Global holomorphic functions. The transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1,$$

are quasi-homogeneous if we assign the weights

$$\begin{array}{c|c|c|c|c|c} \beta & v_1 & v_2 & \gamma & w_1 & w_2 \\ k-1 & k+1 & 2 & 1-k & 2 & 3k-1 \end{array}.$$

The global holomorphic functions will thus necessarily be quasi-homogeneous in these weights. We find the following global holomorphic functions:

$$\begin{array}{c|l} 2 & y_1 = v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1 \\ k+1 & y_2 = \beta v_2 - v_1 = \gamma^2 w_2 + \gamma w_1^k \\ 2k & y_3 = \beta^2 v_2 - \beta v_1 = \gamma w_2 + w_1^k \\ 3k-1 & y_4 = v_2^{k-1} v_1 - \beta^3 v_2 + \beta^2 v_1 = \end{array}$$

These are the first four "distinct" functions produced by our algorithm, in the sense that none is contained in the ring generated by the other three.

The singular geometry $\mathcal{M}_0$. We conjecture that the ring of global holomorphic functions is generated by y_1, y_2, y_3 and y_4, subject to the degree $4k$

relation

$$\mathcal{M}_0: \quad y_2y_4 + y_3^2 + y_2^2y_1^{k-1} - y_3y_1^k = 0.$$

The functions y_i give us a blow-down map whose image $\mathcal{M}_0$ has an isolated A_k singularity. To see this, consider the change of variables

$$\widetilde{y}_4 = y_4 + y_2y_1^{k-1} = \beta v_2^k - \beta^3 v_2 + \beta^2 v_1.$$

Note that like y_4, $\widetilde{y}_4$ is also quasi-homogeneous of degree $3k-1$. The functions y_1, y_2, y_3 and $\widetilde{y}_4$ now satisfy the simpler relation

$$\mathcal{M}_0: \quad y_2\widetilde{y}_4 + y_3(y_3 - y_1^k) = 0.$$

The blow-up. We now verify that we have identified the right singular space $\mathcal{M}_0$ by inverting the blow-down. In the β and γ charts we find

$$\begin{array}{l|l}
\beta = y_3/y_2 & \gamma = y_2/y_3 \\
v_1 = \beta y_1 - y_2 = (y_3y_1 - y_2^2)/y_2 & w_1 = y_1 - \gamma y_2 = (y_1y_3 - y_2^2)/y_3 \\
v_2 = y_1 & w_2 = \beta(y_3 - w_1^k) \\
 & \quad = -\widetilde{y}_4 + \dfrac{y_1^k - (y_1 - \gamma y_2)^k}{\gamma}.
\end{array}$$

This suggests that we should blow up

$$y_2 = y_3 = 0,$$

for the small resolution of $\mathcal{M}_0$. We introduce $\mathbb{P}^1$ coordinates $[\beta : \gamma]$ such that $\beta y_2 = \gamma y_3$. The blow up in each chart is then

$$\begin{array}{c|c}
(\gamma = 1) & (\beta = 1) \\
 & \\
y_3 = \beta y_2 & y_2 = \gamma y_3 \\
\widetilde{y}_4 = \beta(y_1^k - \beta y_2) & y_3 = y_1^k - \gamma\widetilde{y}_4 \\
 & \\
\text{co-ords}: \ (y_1, y_2, \beta) & \text{co-ords}: \ (y_1, \widetilde{y}_4, \gamma)
\end{array}$$

Transition functions. The transition functions between the β and γ charts are

$$\beta = \gamma^{-1}, \quad y_1 = y_1, \quad y_2 = \gamma(y_1^k - \gamma\widetilde{y}_4) = -\gamma^2\widetilde{y}_4 + \gamma y_1^k.$$

Note that for $k > 1$, this is an $\mathcal{O} \oplus \mathcal{O}(-2)$ bundle over the exceptional $\mathbb{P}^1$, and corresponds to a superpotential with a single field ($M = 1$):

$$W(x) = \frac{1}{k+1}x^{k+1}.$$

(For $k = 1$ the bundle is actually $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $W = 0$.)

In terms of the original coordinates, the transition functions become

$$\beta = \gamma^{-1}, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1, \quad \beta v_2 - v_1 = \gamma^2 w_2 + \gamma w_1^k.$$

Substituting the second transition function into the third reveals $v_1 = \gamma^{-1} w_1$, and so we recover our original transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1,$$

which define an $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ bundle deformed by the two field ($M = 2$) superpotential

$$W(x, y) = \frac{1}{k+1} x^{k+1} + \frac{1}{2} y^2.$$

4.2 A puzzle

The problem. We saw the A_k case in Section 2.4, with geometry $\widehat{\mathcal{M}}$ given by the superpotential

$$W(x) = \frac{1}{k+1} x^{k+1},$$

and hence corresponding to a deformed $\mathcal{O} \oplus \mathcal{O}(-2)$ bundle over the $\mathbb{P}^1$, with one field. However, Intriligator and Wecht identify

$$W(x, y) = \frac{1}{k+1} x^{k+1} + \frac{1}{2} y^2,$$

as corresponding to an A_k-type singularity, with an extra field y which requires that the transition functions look like

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1.$$

In particular, the geometry $\widehat{\mathcal{M}}$ looks like that of an $\mathcal{O}(1) \oplus \mathcal{O}(-3)$ bundle over the exceptional $\mathbb{P}^1$! What's going on here?

Resolution of the problem. For $n = 1$ and $k = 0$, Proposition 1.1 tells us that the superpotential

$$W(x, y) = \frac{1}{2} y^2$$

changes the bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(-3) \longrightarrow \mathcal{O} \oplus \mathcal{O}(-2).$$

Hence the extra field y from the Intriligator–Wecht potential (with purely quadratic contribution to the superpotential) can be "integrated out." Its effect is to change the bundle for A_k from $\mathcal{O}(1) \oplus \mathcal{O}(-3)$, which is necessary for a two-field description, to reveal the true underlying $\mathcal{O} \oplus \mathcal{O}(-2)$ structure. In other words,

$$W(x,y) = \frac{1}{k+1}x^{k+1} + \frac{1}{2}y^2 \quad \text{and} \quad W(x) = \frac{1}{k+1}x^{k+1}$$

are geometrically equivalent.

Beyond Proposition 1.1. Note that this is not just a straightforward application of Proposition 1.1, which implies that $W(x,y) = \frac{1}{2}y^2$ and $W(x) = 0$ are geometrically equivalent. Beginning with transition functions for the Intriligator–Wecht A_k superpotential $W(x,y) = \frac{1}{k+1}x^{k+1} + \frac{1}{2}y^2$,

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1,$$

the change of coordinates suggested in the proof of Proposition 1.1

$$\begin{aligned} \widetilde{v_1} &= v_2, & \widetilde{w_1} &= w_1 + \gamma^3 w_2, \\ \widetilde{v_2} &= -v_1 + \beta v_2, & \widetilde{w_2} &= w_2, \end{aligned}$$

does not yield the appropriate new transition functions. Instead, the more complicated change of coordinates

$$\begin{aligned} \widetilde{v_1} &= v_2, & \widetilde{w_1} &= w_1 + \gamma^3 w_2 + \gamma^2 w_1^k, \\ \widetilde{v_2} &= -v_1 + \beta v_2, & \widetilde{w_2} &= w_2 - \gamma^{-1}\left[(\gamma^3 w_2 + \gamma^2 w_1^k + w_1)^k - w_1^k\right], \end{aligned}$$

is needed to give new transition functions

$$\beta = \gamma^{-1}, \quad \widetilde{v_1} = \widetilde{w_1}, \quad \widetilde{v_2} = \gamma^2 \widetilde{w_2} + \gamma \widetilde{w_1}^k,$$

corresponding to the superpotential $W(x) = \frac{1}{k+1}x^{k+1}$.

It would be interesting to try to generalize Proposition 1.1 to include examples such as this, where there are additional terms in the superpotential besides the quadratic pieces which suggest a change in bundle structure. Trying to understand what makes the change of coordinates possible in this case may give hints as to how the geometric picture for RG flow might be extended. The ultimate goal would be to understand how "integrating out" the y coordinate in a potential of the form $W(x,y) = f(x,y) + y^2$ affects other terms involving y.

4.3 The case D_{k+2}

In this section we prove the second part of Theorem 1.3: *The resolved geometry corresponding to the Intriligator–Wecht superpotential*

$$W(x,y) = \frac{1}{k+1}x^{k+1} + xy^2$$

corresponds to the singular geometry

$$X^2 - Y^2Z + T(Z^{k/2} - T)^2 = 0 \quad (k \text{ even}),$$

$$X^2 - Y^2Z - T(Z^k - T^2) = 0 \quad (k \text{ odd}).$$

The resolved geometry $\widehat{\mathcal{M}}$. From the Intriligator–Wecht superpotential

$$W(x,y) = \frac{1}{k+1}x^{k+1} + xy^2,$$

we compute the resolved geometry $\widehat{\mathcal{M}}$ in terms of transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1^2.$$

To find the $\mathbb{P}^1$s, we substitute $w_1(\gamma) = x + \gamma y$ into the v_2 transition function

$$v_2(\beta) = \beta^{-3}w_2 + \beta^{-2}(x + \beta^{-1}y)^k + (x + \beta^{-1}y)^2.$$

If we choose

$$w_2(\gamma) = \frac{x^k - (x+\gamma y)^k}{\gamma},$$

in the β chart the section is

$$v_1(\beta) = \beta x + y, \quad v_2(\beta) = \beta^{-2}(x^k + y^2) + \beta^{-1}(2xy) + x^2.$$

This is only holomorphic if

$$x^k + y^2 = 2xy = 0,$$

and so we have a single $\mathbb{P}^1$ located at

$$w_1(\gamma) = w_2(\gamma) = 0, \quad v_1(\beta) = v_2(\beta) = 0.$$

This is exactly what we expect from computing critical points of the superpotential

$$\delta W = (x^k + y^2)dx + 2xy\, dy = 0.$$

Global holomorphic functions. The transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1^2,$$

are quasi-homogeneous if we assign the weights

$$\begin{array}{c|c|c|c|c|c} \beta & v_1 & v_2 & \gamma & w_1 & w_2 \\ k-2 & k & 4 & 2-k & 2 & 3k-2 \end{array}.$$

The global holomorphic functions will thus necessarily be quasi-homogeneous in these weights. We find the following global holomorphic functions for k even:

$$\begin{array}{c|l} 4 & Z = v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1^2 \\ 2k & T = \beta^2 v_2 - v_1^2 = \gamma w_2 + w_1^k \\ 3k-2 & Y = \beta(Z^{k/2} - T) = \gamma^{-1}(Z^{k/2} - T) \\ 3k & X = v_1(Z^{k/2} - T) = \gamma^{-1} w_1 (Z^{k/2} - T), \end{array}$$

and a similar set of global holomorphic functions for k odd:

$$\begin{array}{c|l} 4 & Z = v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1^2 \\ 2k & T = \beta^2 v_2 - v_1^2 = \gamma w_2 + w_1^k \\ 3k-2 & Y = v_1 Z^{(k-1)/2} - \beta T = \gamma^{-1}(w_1 Z^{(k-1)/2} - T) \\ 3k & X = \beta Z^{(k+1)/2} - v_1 Y = \gamma^{-1}(Z^{(k+1)/2} - w_1 T). \end{array}$$

The singular geometry $\mathcal{M}_0$. We conjecture that the ring of global holomorphic functions is generated by X, Y, Z and T, subject to the degree $6k$ relation

$$\mathcal{M}_0 : X^2 - ZY^2 + T(Z^{k/2} - T)^2 = 0, \ k \text{ even},$$
$$\mathcal{M}_0 : X^2 - ZY^2 - T(Z^k - T^2) = 0, \ k \text{ odd}.$$

The functions X, Y, Z and T give us a blow-down map whose image $\mathcal{M}_0$ has an isolated D_{k+2} singularity.

Review of length 2 blow-up. Before doing the blow-up to show that we have the right blow down, we review some results from [7, Chap. 4]. There we found small resolutions of length 2 singularities by using deformations of matrix factorizations for D_{n+2} surface singularities.

The deformed D_{n+2} equation was given by

$$0 = X^2 + Y^2 Z - h'(ZP''^2 + Q''^2) + 2\delta'(YQ'' + (-1)^{m+1} XP''),$$

where for $t = 0$ we have $\delta' = 0$ and

$$\begin{array}{r|lll} m \text{ even} & h'(Z) = Z^{n-m}, & P''(Z) = i^m Z^{m/2}, & Q''(Z) = 0. \\ m \text{ odd} & h'(Z) = Z^{n-m}, & P''(Z) = 0, & Q''(Z) = i^{m+1} Z^{(m+1)/2}. \end{array}$$

The blow-up was given by the equation for the Grassmannian $G(2,4) \subseteq \mathbb{P}^5$

$$\alpha^2 - \varphi^2 Z + (-1)^{m+1} h' \varepsilon^2 + 2i^{m+1}\delta' \varepsilon\varphi = 0,$$

in terms of Plücker coordinates

$$\begin{aligned}\alpha &= iXY - i^{2m+1} h' P'' Q'' \\ \varepsilon &= i^{m+3} X P'' + i^{-m+1} Y Q'' \\ \varphi &= Y^2 - h' P''^2.\end{aligned}$$

The interesting charts were $\varphi = 1$ and $\varepsilon = 1$, with transition functions

$$\varphi_2 = \varepsilon_1^{-1}, \tag{4.1}$$
$$\alpha_2 = \varepsilon_1^{-1}\alpha_1, \tag{4.2}$$
$$Z = (-1)^{m+1}\varepsilon_1^2 h'(Z,t) + \alpha_1^2 + 2i^{m+1}\delta'\varepsilon_1, \tag{4.3}$$
$$t = t. \tag{4.4}$$

The blow-up. For k even, the equation for $\mathcal{M}_0$ corresponds to

$$\delta' = 0, \quad h' = T, \quad P'' = 0, \quad Q'' = i^k(Z^{k/2} - T), \quad n = m = k - 1,$$

and Plücker coordinates

$$\alpha = iXY, \quad \varepsilon = -Y(Z^{k/2} - T), \quad \varphi = Y^2.$$

The connection with the original transition function coordinates is

$$\begin{array}{c|l} 4 & Z = v_2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1^2 \\ 2k & T = \beta^2 v_2 - v_1^2 = \gamma w_2 + w_1^k \\ 3k-2 & Y = \beta(Z^{k/2} - T) = \gamma^{-1}(Z^{k/2} - T) \\ 3k & iX = v_1(Z^{k/2} - T) = \gamma^{-1} w_1 (Z^{k/2} - T). \end{array}$$

Note that

$$\begin{aligned}\beta &= \frac{Y}{Z^{k/2} - T} = -\frac{\varphi}{\varepsilon} = -\varphi_2, \\ v_1 &= \frac{iX}{Z^{k/2} - T} = -\frac{\alpha}{\varepsilon} = -\alpha_2, \\ v_2 &= Z, \\ \gamma &= \frac{Z^{k/2} - T}{Y} = -\frac{\varepsilon}{\varphi} = -\varepsilon_1,\end{aligned}$$

$$w_1 = \frac{iX}{Y} = \frac{\alpha}{\varphi} = \alpha_1,$$
$$w_2 = \beta(T - w_1^k) = -\varepsilon_2^{-1}(T - \alpha_2^k).$$

In particular, the transition functions (4.1) become

$$\beta = \gamma^{-1},$$
$$v_1 = \gamma^{-1} w_1,$$
$$v_2 = \gamma^2 T + w_1^2 = \gamma^3 w_2 + \gamma^2 w_1^k + w_1^2.$$

The blow-up for k odd is similar. For more details about the Grassmann blow up for singularities of type D_n see [6, pp. 21–23].

4.4 The case E_7

In this section we prove the third part of Theorem 1.3: *The resolved geometry corresponding to the Intriligator–Wecht superpotential*

$$W(x,y) = \frac{1}{3}y^3 + yx^3$$

corresponds to the singular geometry

$$X^2 - Y^3 + Z^5 + 3TYZ^2 + T^3 Z = 0.$$

The resolved geometry $\widehat{\mathcal{M}}$. From the Intriligator–Wecht superpotential

$$W(x,y) = \frac{1}{3}y^3 + yx^3,$$

we compute the resolved geometry $\widehat{\mathcal{M}}$ in terms of transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma w_1^3 + \gamma^{-1} w_1^2.$$

To find the $\mathbb{P}^1$s, we substitute $w_1(\gamma) = x + \gamma y$ into the v_2 transition function

$$v_2(\beta) = \beta^{-3} w_2 + \beta^{-1}(x + \beta^{-1} y)^3 + \beta(x + \beta^{-1} y)^2.$$

If we choose

$$w_2(\gamma) = \frac{x^3 + 3\gamma x^2 y - (x + \gamma y)^3}{\gamma^2},$$

in the β chart the section is

$$v_1(\beta) = \beta x + y, \quad v_2(\beta) = \beta^{-2}(3x^2 y) + \beta^{-1}(x^3 + y^2) + 2xy + \beta x^2.$$

This is only holomorphic if

$$3x^2y = x^3 + y^2 = 0,$$

and so we have a single $\mathbb{P}^1$ located at

$$w_1(\gamma) = w_2(\gamma) = 0, \quad v_1(\beta) = v_2(\beta) = 0.$$

This is exactly what we expect from computing critical points of the superpotential

$$\delta W = 3x^2y\, dx + (y^2 + x^3)dy = 0.$$

Global holomorphic functions. The transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma w_1^3 + \gamma^{-1}w_1^2,$$

are quasi-homogeneous if we assign the weights

$$\begin{array}{c|c|c|c|c|c} \beta & v_1 & v_2 & \gamma & w_1 & w_2 \\ 1 & 3 & 5 & -1 & 2 & 8. \end{array}$$

The global holomorphic functions will thus necessarily be quasi-homogeneous in these weights. We find the following global holomorphic functions:

$$\begin{array}{r|l} 6 & X = \beta v_2 - v_1^2 \\ 8 & Y = v_1 v_2 - \beta^2 X \\ 10 & Z = v_2^2 - \beta v_1 X \\ 15 & F = v_2^3 - 2v_1^3 X + (\beta^3 - 3v_1)X^2. \end{array}$$

The singular geometry $\mathcal{M}_0$. We conjecture that the ring of global holomorphic functions is generated by X, Y, Z and F, subject to the degree 30 relation

$$\mathcal{M}_0 : \quad F^2 - Z^3 + X^5 + 3X^2YZ + XY^3 = 0.$$

Do the functions X, Y, Z and F give us a blow-down map whose image $\mathcal{M}_0$ has an isolated E_7 singularity?

The blow up. We now verify that we have identified the right singular space $\mathcal{M}_0$ by inverting the blow-down. In the β and γ charts we find

$$\begin{array}{l|l} \beta = \dfrac{Y^2+XZ}{F} & \gamma = \dfrac{F}{Y^2+XZ} \\ v_1 = \dfrac{X^3+YZ}{F} & w_1 = \dfrac{X^3+YZ}{Y^2+XZ} \\ v_2 = \dfrac{Z^2-X^2Y}{F} & w_2 = \dfrac{X^4-Y^3}{Y^2+XZ} = w_1X - Y. \end{array}$$

This suggests that we should rewrite the equation for $\mathcal{M}_0$ as

$$\mathcal{M}_0: \quad F^2 + XY(Y^2+XZ) + X^2(X^3+YZ) - Z(Z^2-X^2Y) = 0,$$

and that we can obtain $\widehat{\mathcal{M}}$ by blowing up

$$F = Y^2+XZ = X^3+YZ = Z^2-X^2Y = 0.$$

The locus $\mathcal{C}$. Let $\mathcal{S}$ denote the surface

$$\mathcal{S}: \quad F = Y^2+XZ = 0.$$

Our $\widehat{M}$ coordinate patches (β, v_1, v_2) and (γ, w_1, w_2) cover everything except the locus

$$\mathcal{C} = \mathcal{S} \cap \mathcal{M}_0.$$

The intersection of $\mathcal{S}$ with the threefold $\mathcal{M}_0$ yields the new equation

$$X^5 + 2X^2YZ - Z^3 = 0.$$

(This was obtained by finding the Groebner basis for the ideal generated by F, Y^2+XZ, and the equation for $\mathcal{M}_0$.)

For $X \neq 0$ we can write

$$Z = -\frac{Y^2}{X},$$

and so the equations for $\mathcal{C}$ become

$$\mathcal{C}: \quad F = Y^2+XZ = (X^4-Y^3)^2 = 0 \quad (X \neq 0).$$

We can parametrize this curve by

$$\begin{aligned} X &= t^3, \\ Y &= t^4, \\ Z &= -t^5, \\ F &= 0. \end{aligned}$$

From this we see that blowing up $\mathcal{C} \subset \mathcal{M}_0$ is equivalent to blowing up

$$F = Y^2 + XZ = X^3 + YZ = Z^2 - X^2Y = X^4 - Y^3 = 0.$$

In this case we would have additional coordinates v_3, w_3 for the blow-up:

$$\begin{array}{l|l}
\beta = \dfrac{Y^2+XZ}{F} & \gamma = \dfrac{F}{Y^2+XZ} \\
v_1 = \dfrac{X^3+YZ}{F} & w_1 = \dfrac{X^3+YZ}{Y^2+XZ} \\
v_2 = \dfrac{Z^2-X^2Y}{F} & w_2 = \dfrac{X^4-Y^3}{Y^2+XZ} \\
v_3 = \dfrac{X^4-Y^3}{F} & w_3 = \dfrac{Z^2-X^2Y}{Y^2+XZ}.
\end{array}$$

Note that both v_3 and w_3 add nothing new, as we can solve for them in terms of the other coordinates:

$$\begin{aligned}
v_3 &= \gamma^{-1}w_2 = \beta^4 v_2 - \beta^3 v_1^2 - v_1^3, \\
w_3 &= \beta^{-1}v_2 = \gamma^4 w_2 + \gamma^2 w_1^3 + w_1^2.
\end{aligned}$$

These are precisely the additional coordinates we introduced in our resolution of the ideal sheaf (see Section 3.5). Finally, with these identifications we find transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma w_1^3 + \gamma^{-1}w_1^2,$$

which are exactly the ones we started with.

5 Proof of Theorem 1.4

In this section we analyze the extra 'hat' cases in the Intriligator–Wecht classification of superpotentials. In particular, we prove Theorem 1.4 by finding singular geometries corresponding to each of the $\widehat{O}$, $\widehat{A}$, $\widehat{D}$ and $\widehat{E}$ cases. As in the case of Theorem 1.3, we first find global holomorphic functions using the algorithm described in the Appendix, and then verify the resulting singular space by blowing back up to recover our original transition functions.

5.1 The case $\widehat{O}$

The resolved geometry $\widehat{\mathcal{M}}$. From the Intriligator–Wecht superpotential

$$W(x, y) = 0,$$

we compute the resolved geometry $\widehat{\mathcal{M}}$ in terms of transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2.$$

To find the $\mathbb{P}^1$s, we substitute $w_1(\gamma) = x + \gamma y$ into the v_2 transition function

$$v_2(\beta) = \beta^{-3} w_2.$$

If we choose

$$w_2(\gamma) = 0,$$

in the β chart the section is

$$v_1(\beta) = \beta x + y, \quad v_2(\beta) = 0.$$

This is holomorphic for all x and y, and so we have a two-parameter family of $\mathbb{P}^1$s located at

$$w_1(\gamma) = x + \gamma y, w_2(\gamma) = 0, \quad v_1(\beta) = \beta x + y, v_2(\beta) = 0.$$

This is exactly what we expect from computing critical points of the superpotential

$$\delta W = 0.$$

Global holomorphic functions. The transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2$$

are quasi-homogeneous if we assign the weights

$$\begin{array}{c|c|c|c|c|c} \beta & v_1 & v_2 & \gamma & w_1 & w_2 \\ 1 & d+1 & e-3 & -1 & d & e. \end{array}$$

Notice the freedom in choosing d and e: there is a two-dimensional lattice of possible weight assignments. The global holomorphic functions will necessarily be quasi-homogeneous in these weights. We find global holomorphic functions:

$$X_{ij} = \beta^i v_1^j v_2 = \gamma^{3-i-j} w_1^j w_2, \quad i, j \geq 0, \quad i + j \leq 3.$$

The singular geometry $\mathcal{M}_0$. If we rewrite our functions in a homogeneous manner as

$$\widetilde{X}_{ij} = a^{3-i-j} b^i c^j, \quad i, j \geq 0, \quad i + j \leq 3,$$

we can now identify the ring of global holomorphic functions as homogeneous polynomials of degree 3 in three variables. In other words, the ring is isomorphic to

$$\mathbb{C}[a,b,c]^{\mathbb{Z}_3},$$

and our singular variety is simply

$$\mathcal{M}_0: \quad \mathbb{C}^3/\mathbb{Z}_3.$$

The blow-up. We now verify that we have identified the right singular space $\mathcal{M}_0$ by inverting the blow-down. In the β and γ charts we find

$$\begin{array}{l|l} \beta = X_{10}/X_{00} = \widetilde{X}_{10}/\widetilde{X}_{00} = b/a & \gamma = a/b \\ v_1 = X_{01}/X_{00} = \widetilde{X}_{01}/\widetilde{X}_{00} = c/a & w_1 = c/b \\ v_2 = X_{00} = \widetilde{X}_{00} = a^3 & w_2 = b^3 \end{array}$$

which gives transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2.$$

These are precisely what we started with!

Remark 5.1. In the resolved $\widehat{\mathcal{M}}$ geometry, what we have is a $\mathbb{P}^1$ inside a $\mathbb{P}^2$ (or any other del Pezzo surface). If you have a $\mathbb{P}^2$ inside a Calabi–Yau and blow it down, you get $\mathbb{C}^3/\mathbb{Z}^3$ as the singular point.

5.2 The case $\widehat{A}$

The resolved geometry $\widehat{\mathcal{M}}$. From the Intriligator–Wecht superpotential

$$W(x,y) = \frac{1}{2}x^2,$$

we compute the resolved geometry $\widehat{\mathcal{M}}$ in terms of transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1.$$

To find the $\mathbb{P}^1$s, we substitute $w_1(\gamma) = x + \gamma y$ into the v_2 transition function

$$v_2(\beta) = \beta^{-3}(w_2 + y) + \beta^{-2}x.$$

If we choose

$$w_2(\gamma) = -y,$$

in the β chart the section is

$$v_1(\beta) = \beta x + y, \quad v_2(\beta) = \beta^{-2}x.$$

This is only holomorphic $x = 0$. Since y is free, we have a one-parameter family of $\mathbb{P}^1$s located at

$$w_1(\gamma) = \gamma y, w_2(\gamma) = -y, \quad v_1(\beta) = y, v_2(\beta) = 0.$$

This is exactly what we expect from computing critical points of the superpotential

$$\delta W = x\delta x = 0.$$

Global holomorphic functions. The transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1$$

are quasi-homogeneous if we assign the weights

$$\begin{array}{c|c|c|c|c|c} \beta & v_1 & v_2 & \gamma & w_1 & w_2 \\ 1 & d+1 & d-2 & -1 & d & d+1. \end{array}$$

Notice the freedom in choosing d: there is a one-dimensional lattice of possible weight assignments. The global holomorphic functions will necessarily be quasi-homogeneous in these weights. We find global holomorphic functions:

$$\begin{array}{c|l} d-2 & y_1 = v_2 = \gamma^3 w_2 + \gamma^2 w_1 \\ d-1 & y_2 = \beta v_2 = \gamma^2 w_2 + \gamma w_1 \\ d & y_3 = \beta^2 v_2 = \gamma w_2 + w_1 \\ d+1 & y_4 = \beta^3 v_2 - v_1 = w_2 \end{array}$$

The singular geometry $\mathcal{M}_0$. The functions y_i satisfy the single degree $2d-2$ relation

$$y_2^2 - y_1 y_3 = 0,$$

with y_4 free. In other words, our singular geometry $\mathcal{M}_0$ is a curve of A_1 singularities, parametrized by y_4.

The blow-up. We now verify that we have identified the right singular space $\mathcal{M}_0$ by inverting the blow-down. In the β and γ charts we find

$$\begin{array}{l|l} \beta = y_2/y_1 & \gamma = y_1/y_2 \\ v_1 = (y_2y_3 - y_1y_4)/y_1 & w_1 = (y_2y_3 - y_1y_4)/y_2 \\ v_2 = y_1 & w_2 = y_4. \end{array}$$

This gives transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1}w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1,$$

as expected.

5.3 The case $\widehat{D}$

The resolved geometry $\widehat{\mathcal{M}}$. From the Intriligator–Wecht superpotential

$$W(x,y) = x^2 y,$$

we compute the resolved geometry $\widehat{\mathcal{M}}$ in terms of transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma w_1^2.$$

To find the $\mathbb{P}^1$s, we substitute $w_1(\gamma) = x + \gamma y$ into the v_2 transition function

$$v_2(\beta) = \beta^{-3}(w_2 + y^2) + \beta^{-2}(2xy) + \beta^{-1}x^2.$$

If we choose

$$w_2(\gamma) = -y^2,$$

in the β chart the section is

$$v_1(\beta) = \beta x + y, \quad v_2(\beta) = \beta^{-2}(2xy) + \beta^{-1}x^2.$$

This is only holomorphic if $x = 0$. Since y is free, we have a one-parameter family of $\mathbb{P}^1$s located at

$$w_1(\gamma) = \gamma y, w_2(\gamma) = -y^2, \quad v_1(\beta) = y, v_2(\beta) = 0.$$

This is exactly what we expect from computing critical points of the superpotential

$$\delta W = 2xy\; \delta x + x^2 \delta y = 0.$$

Global holomorphic functions. The transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma w_1^2$$

are quasi-homogeneous if we assign the weights

$$\begin{array}{c|c|c|c|c|c} \beta & v_1 & v_2 & \gamma & w_1 & w_2 \\ 1 & d+1 & 2d-1 & -1 & d & 2d+2 \end{array}.$$

Notice the freedom in choosing d: there is a one-dimensional lattice of possible weight assignments. The global holomorphic functions will necessarily be quasi-homogeneous in these weights. We find global holomorphic functions:

$$\begin{array}{c|l} 2d-1 & y_1 = v_2 = \gamma^3 w_2 + \gamma w_1^2 \\ 2d & y_2 = \beta v_2 = \gamma^2 w_2 + w_1^2 \\ 3d & y_3 = v_1 v_2 = \gamma^2 w_1 w_2 + w_1^3 \\ 2d+2 & y_4 = \beta^3 v_2 - v_1^2 = w_2 \end{array}$$

The singular geometry $\mathcal{M}_0$. The functions y_i satisfy the single-degree $6d$ relation

$$y_3^2 - y_2^3 + y_1^2 y_4 = 0.$$

The blow up. We now verify that we have identified the right singular space $\mathcal{M}_0$ by inverting the blow-down. In the β and γ charts we find

$$\begin{array}{r|l} \beta = y_2/y_1 & \quad \gamma = y_1/y_2 \\ v_1 = y_3/y_1 & w_1 = y_3/y_2 \\ v_2 = y_1 \quad\;\; & w_2 = y_4, \end{array}$$

with transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma w_1^2.$$

5.4 The case $\widehat{E}$

The resolved geometry $\widehat{\mathcal{M}}$. From the Intriligator–Wecht superpotential

$$W(x,y) = \frac{1}{3}x^3,$$

we compute the resolved geometry $\widehat{\mathcal{M}}$ in terms of transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^2.$$

To find the $\mathbb{P}^1$s, we substitute $w_1(\gamma) = x + \gamma y$ into the v_2 transition function

$$v_2(\beta) = \beta^{-3}(w_2 + 2xy + \gamma y^2) + \beta^{-2} x^2.$$

If we choose

$$w_2(\gamma) = -2xy - \gamma y^2,$$

in the β chart the section is

$$v_1(\beta) = \beta x + y, \quad v_2(\beta) = \beta^{-2} x^2.$$

This is only holomorphic if $x = 0$. Since y is free, we have a one-parameter family of $\mathbb{P}^1$s located at

$$w_1(\gamma) = \gamma y, w_2(\gamma) = -\gamma y^2, \quad v_1(\beta) = y, v_2(\beta) = 0.$$

This is exactly what we expect from computing critical points of the superpotential

$$\delta W = x^2 \, \delta x = 0.$$

Global holomorphic functions. The transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^2$$

are quasi-homogeneous if we assign the weights

$$\begin{array}{c|c|c|c|c|c} \beta & v_1 & v_2 & \gamma & w_1 & w_2 \\ 1 & d+1 & 2d-2 & -1 & d & 2d+1 \end{array}.$$

Notice the freedom in choosing d: there is a one-dimensional lattice of possible weight assignments. The global holomorphic functions will necessarily be quasi-homogeneous in these weights. We find global holomorphic functions:

$$\begin{array}{c|l} 2d-2 & y_1 = v_2 \\ 2d-1 & y_2 = \beta v_2 \\ 2d & y_3 = \beta^2 v_2 \\ 3d-1 & y_4 = v_1 v_2 \\ 3d & y_5 = \beta v_1 v_2 \\ 4d & y_6 = v_1^2 v_2 \\ 4d+1 & y_7 = \beta(\beta^4 v_2 - v_1^2) v_2 = \beta v_3 v_2 \\ 5d+1 & y_8 = v_1(\beta^4 v_2 - v_1^2) v_2 = v_1 v_3 v_2 \\ 6d+2 & y_9 = (\beta^4 v_2 - v_1^2)^2 v_2 = v_3^2 v_2 \end{array}$$

where we have defined

$$v_3 = \beta^4 v_2 - v_1^2 = \gamma^{-1} w_2.$$

The singular geometry $\mathcal{M}_0$. The functions y_i satisfy a total of 20 distinct relations, most of which are obvious. To simplify things, consider the monomial mapping

$$\beta^i v_1^j v_3^k v_2 \longmapsto a^{2-i-j-k} b^i c^j f^k.$$

Our functions now become

$$\begin{array}{c|l} 2d-2 & y_1 = a^2 \\ 2d-1 & y_2 = ab \\ 2d & y_3 = b^2 \\ 3d-1 & y_4 = ac \\ 3d & y_5 = bc \\ 4d & y_6 = c^2 \\ 4d+1 & y_7 = bf \\ 5d+1 & y_8 = cf \\ 6d+2 & y_9 = f^2. \end{array}$$

Note that

$$\begin{aligned} \beta &= y_2/y_1 = b/a \\ v_1 &= y_4/y_1 = c/a \\ v_2 &= y_1 = a^2 \\ v_3 &= y_7/y_2 = f/a, \end{aligned}$$

so the relation defining v_3 becomes

$$af = b^4 - c^2.$$

This means we can add the function af to our list, together with the relation:

$$y_{10} = af = b^4 - c^2.$$

Now the functions $y_1, \ldots, y_{10}$ are exactly the 10 monomials of degree 2 in four variables, together with the previous relation. The ring of global holomorphic functions is thus

$$(\mathbb{C}[a,b,c,f]/\mathbb{Z}_2)/(af - b^4 + c^2),$$

where the $\mathbb{Z}_2$ acts diagonally as -1. In other words, we have a hypersurface in a $\mathbb{Z}_2$ quotient space:

$$(b^2 + c)(b^2 - c) = af \quad \text{in} \quad \mathbb{C}^4/\mathbb{Z}_2.$$

We can immediately see from this equation that a small resolution, where we blow up an ideal of the form

$$b^2 + c = a = 0,$$

would not work, since the $\mathbb{Z}_2$ action interchanges $b^2 + c$ and $b^2 - c$. We will need to do a big blow up of the origin instead.

The blow up. We now verify that we have identified the right singular space $\mathcal{M}_0$ by inverting the blow-down. In the β and γ charts we find

$$\begin{array}{r|l} \beta = b/a & \gamma = a/b \\ v_1 = c/a & w_1 = c/b \\ v_2 = a^2 & w_2 = f/b \\ v_3 = f/a & w_3 = b^2. \end{array}$$

We will perform the big blow-up of the origin, with corresponding $\mathbb{P}^3$ co-ordinates:

$$\begin{array}{ccccccc} a & = & b & = & c & = & f = 0. \\ \alpha & & \delta & & \rho & & \nu \end{array}$$

Note that all eight co-ordinates switch sign under the $\mathbb{Z}_2$ action. The blow-up has four co-ordinate charts:

$$
\begin{array}{c|c|c|c}
\alpha = 1 & \delta = 1 & \rho = 1 & \nu = 1 \\
 & & & \\
\begin{aligned} a &= a \\ b &= \delta_1 a \\ c &= \rho_1 a \\ f &= \nu_1 a \end{aligned} &
\begin{aligned} a &= \alpha_2 b \\ b &= b \\ c &= \rho_2 b \\ f &= \nu_2 b \end{aligned} &
\begin{aligned} a &= \alpha_3 c \\ b &= \delta_3 c \\ c &= c \\ f &= \nu_3 c \end{aligned} &
\begin{aligned} a &= \alpha_4 f \\ b &= \delta_4 f \\ c &= \rho_4 f \\ f &= f \end{aligned} \\
 & & & \\
\nu_1 = \delta_1^4 a^2 - \rho_1^2 & b^2 = \alpha_2 \nu_2 + \rho_2^2 & \alpha_3 \nu_3 = \delta_3^4 c^2 - 1 & \alpha_4 = \delta_4^4 f^2 - \rho_4^2 \\
 & & & \\
(a^2, \delta_1, \rho_1) & (\alpha_2, \rho_2, \nu_2) & (\alpha_3, \delta_3, c^2, \nu_3) & (\delta_4, \rho_4, f^2).
\end{array}
$$

Remark 5.2. The functions $\alpha_i, \delta_i, \rho_i$, and ν_i are all invariant under the $\mathbb{Z}_2$ action, since they are all ratios of functions which change sign:

$$\delta_1 = \delta/\alpha, \quad \rho_2 = \rho/\delta, \ldots \text{ etc.}$$

Because a, b, c and f all change sign under the $\mathbb{Z}_2$ action, we must take their invariant counterparts a^2, b^2, c^2 and f^2 when we list the final co-ordinates for each chart. In the $\delta = 1$ chart, we solve for b^2 instead of b, because b is not an invariant function. The blow up is non-singular. In the $\alpha = 1$, $\delta = 1$ and $\nu = 1$ charts we see this because we are left with three co-ordinates and no relations, so these charts are all isomorphic to $\mathbb{C}^3$. In the $\rho = 1$ chart, we have a hypersurface in $\mathbb{C}^4$ defined by the non-singular equation

$$\alpha_3 \nu_3 = \delta_3^4 c^2 - 1.$$

Transition functions. Between the first two charts $\alpha = 1$ and $\delta = 1$, we have transition functions

$$
\begin{aligned}
\delta_1 &= \delta/\alpha = \alpha_2^{-1} \\
\rho_1 &= \rho/\alpha = (\delta/\alpha)(\rho/\delta) = \alpha_2^{-1} \rho_2 \\
a^2 &= \alpha_2^2 b^2 = \alpha_2^2 (\alpha_2 \nu_2 + \rho_2^2) \\
&= \alpha_2^3 \nu_2 + \alpha_2^2 \rho_2^2.
\end{aligned}
$$

Notice that

$$
\begin{array}{l|l}
\delta_1 = b/a = \beta & \alpha_2 = a/b = \gamma \\
\rho_1 = c/a = v_1 & \rho_2 = c/b = w_1 \\
a^2 = a^2 = v_2 & \nu_2 = f/b = w_2,
\end{array}
$$

and so our transition functions are really

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma^2 w_1^2.$$

These are exactly the $\widehat{E}$ transition functions we started with!

5.5 Comparison with ADE cases

We have seen that the singular geometries corresponding to Intriligator and Wecht's "hat" cases are given by

$\widehat{O}$	$W(x,y) = 0$	$\mathbb{C}^3/\mathbb{Z}_3$
$\widehat{A}$	$W(x,y) = \frac{1}{2}y^2$	$\mathbb{C}[X,Y,Z,T]/(XY - Z^2) \cong \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ geometry has curve of A_1 singularities
$\widehat{D}$	$W(x,y) = xy^2$	$y_1^2 - y_2^3 + y_3^2 y_4 = 0$ recall geometry for D_{k+2}: $y_1^2 + y_2^3 + y_3^2 y_4 + y_4^k y_2 = 0$
$\widehat{E}$	$W(x,y) = \frac{1}{3}y^3$	$(\mathbb{C}[a,b,u,v]/\mathbb{Z}_2)/(b^4 - u^2 - av)$ this is a hypersurface in $\mathbb{C}^4/\mathbb{Z}_2$.

Note that in both the $\widehat{A}$ and $\widehat{D}$ cases, the resulting equations can be obtained from the A_k and D_{k+2} equations by dropping the k-dependent terms. In other words, we are tempted to think of $\widehat{A}$ and $\widehat{D}$ as the $k \to \infty$ limit. Perhaps in trying to come up with an analogous statement for $\widehat{E}$ we can learn something about the "missing" E_6 and E_8 cases. In particular, it will be interesting to understand the role of these spaces in a geometric model for RG flow.

6 Conclusions

We end by posing a series of questions for the future which are beyond the scope of this work.

From the Intriligator–Wecht classification, we still need to understand the E_6 and E_8 cases. Using our algorithm we have found many global holomorphic functions, but not enough to give us the blow-down [7]. There are also questions which arise from the extra "hat" cases. What is the interpretation of the $\widehat{O}$, $\widehat{A}$, $\widehat{D}$ and $\widehat{E}$ geometries from the string theory perspective? The $\mathbb{P}^1$s are no longer isolated; do they correspond to D-branes wrapping families of $\mathbb{P}^1$s? Moreover, what is the role of higher order terms in the superpotential?

Do we have a geometric model for RG flow? Proposition 1.1 suggests that the geometry might encode something about the RG fixed points of the corresponding matrix models or gauge theories. Can Proposition 1.1 be extended? Finding more general co-ordinate changes which can show how the rest of the terms in the superpotential are affected when a bundle-changing co-ordinate is "integrated out" is a necessary step in developing this kind of geometric picture.

Furthermore, Intriligator and Wecht have a chart of all possible flows between the RG fixed points. We can make a similar chart based on our geometric framework. Do they match? Finally, what is the role of fundamentals? Our entire analysis involves only adjoint fields, which correspond geometrically to parameters of the $\mathbb{P}^1$ deformation space. Intriligator and Wecht only find the ADE classification for superpotentials involving two adjoint fields, but their paper also analyzes many cases with fundamentals. Is it possible to have a geometric interpretation for these fields?

As far as Ferrari's construction is concerned, there are many open ends to be explored. Can we generalize Ferrari's framework to include perturbation terms for both v_1 and v_2 transition functions? Can we generalize for cases where the geometry is specified by more than two charts? This would enable more flexibility in identifying superpotentials in a "bottom–up" approach. On the other hand, the techniques developed in [3] in principle allow computation of the superpotential in general. In cases where the superpotential cannot be easily identified in the transition functions, perhaps this approach should be used instead.

Moreover, in all of our new cases there is still work to be done to complete the remaining steps in Ferrari's program. For example, what is the solution to the matrix model corresponding to the length 3 singularity? What can we learn about the matrix models corresponding to the "hat" cases? Although the singularities are no longer isolated, is it still possible to compute resolvents from the geometry? If Ferrari's conjecture about the Calabi–Yau geometry encoding the solution to the matrix model is correct, we should now be able to solve the matrix models corresponding to the length 3 and "hat" cases. If solutions are already known (or can be computed using traditional matrix model techniques), these examples will provide new tests to the conjecture.

Appendix

We reformulate our problem of finding global holomorphic functions as an ideal membership problem. We begin by illustrating the reformulation in an example. Consider the resolved geometry for the E_7 Intriligator–Wecht potential:

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-1} w_1, \quad v_2 = \gamma^3 w_2 + \gamma w_1^3 + \gamma^{-1} w_1^2.$$

We can think of this as describing a variety in $\mathbb{C}^6$, defined by the following ideal $I \subset \mathbb{C}[\gamma, w_1, w_2, \beta, v_1, v_2]$:

$$I = \langle \beta\gamma - 1, v_1 - \beta w_1, v_2 - \gamma^3 w_2 - \gamma w_1^3 - \beta w_1^2 \rangle.$$

In order to blow down the exceptional $\mathbb{P}^1$, we must find functions which are holomorphic in each coordinate chart, and will therefore be constant on the $\mathbb{P}^1$. Such global holomorphic functions correspond to elements of the ideal I that can be written in the form[17]

$$f - g \in I, \quad \text{where } f \in \mathbb{C}[\beta, v_1, v_2], \quad g \in \mathbb{C}[\gamma, w_1, w_2].$$

For each such element, the global holomorphic function is $f = g$.

In general, we begin with transition functions

$$\beta = \gamma^{-1}, \quad v_1 = \gamma^{-n} w_1, \quad v_2 = \gamma^{-m} w_2 + \partial_{w_1} E(\gamma, w_1),$$

and form the ideal

$$I = \langle \beta\gamma - 1, v_1 - \beta^n w_1, v_2 - \beta^m w_2 - \partial_{w_1} \widetilde{E}(\gamma, w_1) \rangle,$$

where $\widetilde{E}(\gamma, w_1)$ is obtained from $E(\gamma, w_1)$ by replacing all instances of γ^{-1} with β.

A.1 The algorithm

Consider a monomial $\beta^i v_1^j v_2^k \in \mathbb{C}[\beta, v_1, v_2]$. Using Groebner basis techniques, we can easily reduce this modulo the ideal

$$I = \langle \beta\gamma - 1, v_1 - \beta^n w_1, v_2 - \beta^m w_2 - \partial_{w_1} \widetilde{E}(\gamma, w_1) \rangle,$$

which is determined by our particular geometry. In general, we will find

$$\beta^i v_1^j v_2^k \overset{\text{mod } I}{\equiv} \text{pure}(\beta, v_1, v_2) + \text{pure}(\gamma, w_1, w_2) + \text{mixed},$$

where "pure" and "mixed" stand for pure and mixed terms[18] in the appropriate variables. We can then bring the pure(β, v_1, v_2) terms to the left-hand

[17] Note that in our example, none of the defining generators for I are of this form!

[18] We will refer to any monomial in $\mathbb{C}[\gamma, w_1, w_2, \beta, v_1, v_2]$ which does not belong to either $\mathbb{C}[\beta, v_1, v_2]$ or $\mathbb{C}[\gamma, w_1, w_2]$ as a mixed term.

side, "updating" our initial monomial to the polynomial

$$\beta^i v_1^j v_2^k - \mathrm{pure}(\beta, v_1, v_2) \overset{\mathrm{mod}\ I}{\equiv} \mathrm{pure}(\gamma, w_1, w_2) + \mathrm{mixed}.$$

Now the challenge is to find a linear combination f of such polynomials in $\mathbb{C}[\beta, v_1, v_2]$ such that the mixed terms cancel, and we are left with

$$f \overset{\mathrm{mod}\ I}{\equiv} g, \quad \text{where} \quad f \in \mathbb{C}[\beta, v_1, v_2], \;\; g \in \mathbb{C}[\gamma, w_1, w_2].$$

The central idea (as in the Euclidean division algorithm) is to put a term order on the mixed terms we are trying to cancel. In this way, we can make sure we are cancelling mixed terms in an efficient manner, and the cancellation procedure terminates. Because mixed terms (such as βw_1) correspond to "poles" in the γ coordinate chart (such as $\gamma^{-1} w_1$), we use the weighted degree term order

$$>\mathrm{TP} := \mathrm{wdeg}([1, 1, 1, -1, 0, 0], [\mathrm{b}, \mathrm{v}[1], \mathrm{v}[2], \mathrm{g}, \mathrm{w}[1], w[2]]):$$

which keeps track of the degree of the poles in γ.

Beginning with the superpotential, our algorithm thus consists of the following steps:

1. Compute transition functions following Ferrari's framework. This gives an ideal

$$I = \langle \beta\gamma - 1, v_1 - \beta^n w_1, v_2 - \beta^m w_2 - \partial_{w_1} \widetilde{E}(\gamma, w_1) \rangle \subset \mathbb{C}[\gamma, w_1, w_2, \beta, v_1, v_2].$$

2. Find a Groebner basis G for the ideal I, with respect to a term order T.
3. Generate a list L of monomials in β, v_1 and v_2 (up to some degree).
4. Reduce monomial $L[j]$ mod I, using G. What you have is

$$L[j] = \beta^i v_1^j v_2^k \overset{\mathrm{mod}\ I}{\equiv} \mathrm{pure}(\beta, v_1, v_2) + \mathrm{pure}(\gamma, w_1, w_2) + \mathrm{mixed},$$

where "pure" and "mixed" stand for pure and mixed terms in the appropriate variables. Bring the $\mathrm{pure}(\beta, v_1, v_2)$ terms over to the left-hand side to make a polynomial

$$\beta^i v_1^j v_2^k - \mathrm{pure}(\beta, v_1, v_2) \in \mathbb{C}[\beta, v_1, v_2].$$

(i) Record this polynomial in the array F as $F[j, 1]$.
(ii) Record the leading term (with respect to the term order TP) of the "mixed" part as $F[j, 2]$, and store the leading coefficient as $F[j, 3]$.

5. Reduction routine
   (i) Cycle through the list of previous polynomials $F[1..j-1,*]$ and cancel leading mixed terms as much as possible.
   (ii) The result is a new "updated" polynomial $F[j,1]$ which is *reduced* in the sense that its leading mixed term is as low as possible (with respect to the term order TP) due to cancellation with leading mixed terms from previous polynomials.
   (iii) Reduce the "updated" $F[j,1]$ modulo the ideal I to update $F[j,2]$ and $F[j,3]$.
   (iv) If the new leading mixed term $F[j,2]$ is 0, we have a global holomorphic function!
6. Determine which global holomorphic functions are "new," so that the final list is not redundant.
   (i) Check that the new global holomorphic function X_l is not in the ring $\mathbb{C}[X_1,\ldots,X_{l-1}]$ generated by the previous functions.
   (ii) To do this we find a Groebner basis for the ideal $\langle X_1,\ldots,X_l\rangle$ and compute partials to make sure we cannot solve for the new function in terms of the previous ones.
7. Find relations among the global holomorphic functions. These will determine the (singular) geometry of the blow down.

A.2 A shortcut

Of particular interest to us are the Intriligator–Wecht superpotentials [15]. As can be seen from Table A.1, each potential $W(x,y)$ has two possible expressions for $\partial_{w_1}E(\gamma,w_1)$, which corresponds to exchanging $x\leftrightarrow y$ in the matrix model potential.

In all of the Intriligator–Wecht cases, we can find weights for the variables β,v_1,v_2 and γ,w_1,w_2 such that the transition functions

$$\beta=\gamma^{-1},\quad v_1=\gamma^{-1}w_1,\quad v_2=\gamma^3w_2+\partial_{w_1}E(\gamma,w_1),$$

are quasi-homogeneous. For instance, in our previous example (the E_7 case), the transition functions

$$\beta=\gamma^{-1},\quad v_1=\gamma^{-1}w_1,\quad v_2=\gamma^3w_2+\gamma w_1^3+\gamma^{-1}w_1^2,$$

are quasi-homogeneous if we assign the weights

β	v_1	v_2	γ	w_1	w_2
1	3	5	-1	2	8

.

Table 5: Intriligator–Wecht superpotentials, and identification of corresponding resolved geometries.

Type	$W(x,y)$	$\partial_{w_1} E(\gamma, w_1)$	$\mathbb{P}^1 : (w_1, w_2), (v_1, v_2)$
$\widehat{O}$	0	0	$(x+\gamma y, 0), (\beta x + y, 0)$
$\widehat{A}$	$\frac{1}{2}y^2$	$w_1 \leftrightarrow \gamma^2 w_1$	$(x,0), (\beta x, x)$
$\widehat{D}$	xy^2	$w_1^2 \leftrightarrow \gamma w_1^2$	$(x,0), (\beta x, x^2)$
$\widehat{E}$	$\frac{1}{3}y^3$	$\gamma^{-1} w_1^2 \leftrightarrow \gamma^2 w_1^2$	$(x,0), (\beta x, \beta x^2)$
A_k	$\frac{1}{k+1}x^{k+1} + \frac{1}{2}y^2$	$\gamma^2 w_1^k + w_1$ $\leftrightarrow \gamma^{1-k} w_1^k + \gamma^2 w_1$	$(0,0), (0,0)$
D_{k+2}	$\frac{1}{k+1}x^{k+1} + xy^2$	$\gamma^2 w_1^k + w_1^2$ $\leftrightarrow \gamma^{1-k} w_1^k + \gamma w_1^2$	$(0,0), (0,0)$
E_6	$\frac{1}{3}y^3 + \frac{1}{4}x^4$	$\gamma^{-1} w_1^2 + \gamma^2 w_1^3$ $\leftrightarrow \gamma^2 w_1^2 + \gamma^{-2} w_1^3$	$(0,0), (0,0)$
E_7	$\frac{1}{3}y^3 + yx^3$	$\gamma^{-1} w_1^2 + \gamma w_1^3$ $\leftrightarrow \gamma^2 w_1^2 + \gamma^{-1} w_1^3$	$(0,0), (0,0)$
E_8	$\frac{1}{3}y^3 + \frac{1}{5}x^5$	$\gamma^{-1} w_1^2 + \gamma^2 w_1^4$ $\leftrightarrow \gamma^2 w_1^2 + \gamma^{-3} w_1^4$	$(0,0), (0,0)$

In particular, this means all elements of the ideal

$$I = \langle \beta\gamma - 1, v_1 - \beta w_1, v_2 - \gamma^3 w_2 - \gamma w_1^3 - \beta w_1^2 \rangle$$

are quasi-homogeneous in these weights, and all terms in the expression

$$\beta^i v_1^j v_2^k \stackrel{\text{mod } I}{\equiv} \text{pure}(\beta, v_1, v_2) + \text{pure}(\gamma, w_1, w_2) + \text{mixed},$$

will have the same weight.

This immediately tells us that only combinations of monomials *of the same weight* can be used to cancel mixed terms — i.e. the global holomorphic functions we build will themselves be quasi-homogeneous. This observation cuts computational time immensely, since it means that in the reduction routine we need only cycle through lists of polynomials of the same weight in order to reduce the order of the mixed terms. In particular, we can run the algorithm in parallel for different weights, restricting ourselves to lists of monomials in $\mathbb{C}[\beta, v_1, v_2]$ which are all in the same weighted degree.

For a detailed implementation (including actual Maple code) of the algorithm using this shortcut, see [7].

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