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Causal properties of AdS-isometry groups I: causal actions and limit sets

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Abstract

We study the causality relation in the 3-dimensional anti-de Sitter space AdS and its conformal boundary Ein_2. To any closed achronal subset Λ in Ein_2 we associate the invisible domain $E(\Lambda)$ from Λ in AdS. We show that if Γ is a torsion-free discrete group of isometries of AdS preserving Λ and is non-elementary (for example, not abelian) then the action of Γ on $E(\Lambda)$ is free, properly discontinuous and strongly causal. If Λ is a topological circle then the quotient space $M_\Lambda(\Gamma) = \Gamma\backslash E(\Lambda)$ is a maximal globally hyperbolic AdS-spacetime admitting a Cauchy surface S such that the induced metric on S is complete. In a forthcoming paper [7] we study the case where Γ is elementary and use the results of the present paper to define a large family of AdS-spacetimes including all the previously known examples of BTZ multi-black holes.

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1 Introduction

The anti-de Sitter space AdS is the complete Lorentzian manifold with constant sectional curvature -1 (Section 3.1). This is the lorentzian version of the hyperbolic space $\mathbb{H}^n$. Here we only consider the 3-dimensional case. In this introduction, Isom(AdS) denotes the group of isometries of AdS preserving the orientation and the time-orientation (see Definition 2.2).

We intend to reach two goals:

- For every discrete group Γ of isometries on AdS, study Γ-invariant domains of AdS on which Γ acts properly discontinuously,
- Provide a general geometrical framework including the notion of BTZ black-hole and multi-black hole.

BTZ black-holes were defined in [2, 3] and studied in many papers, serving as toy models in the attempt to put together the black-hole notion (which arises from general relativity) and quantum physics. We will discuss these spacetimes with detail in the second part of this work.

The present paper is devoted to the first problem. We essentially mimic the classical study of groups of isometry of the hyperbolic space. Let's recall few basic facts of this well-known theory: let Γ be a discrete group of isometry of $\mathbb{H}^n$. The action of Γ on the entire $\mathbb{H}^n$ is properly discontinuous and if Γ is torsion-free, this action is free. Moreover, the action of Γ on the boundary at infinity $\partial\mathbb{H}^n \approx \mathbb{S}^{n-1}$ admits a unique minimal invariant closed subset $\Lambda(\Gamma)$ and the action of Γ on $\partial\mathbb{H}^n \setminus \Lambda(\Gamma)$ is properly discontinuous.

These features do not apply directly in the AdS context, mainly because the action of Isom(AdS) on AdS is not proper. In this paper, we adopt the following point of view: when dealing with this kind of questions it is pertinent to take into account related *causality notions,* which, in the Riemannian context, remain hidden since automatically fulfilled. Moreover, this causal aspect lies in the very foundation of the notion of BTZ black hole.

1.1 Causality notions

A lorentzian manifold — in short, a spacetime — is causal if no point can be joined to itself by a non-trivial causal curve, i.e., a C^1 immersed curve for which the tangent vectors have non-positive norm for the ambient lorentzian metric. Observe that this notion remains meaningful for any pseudo-Riemannian metric. In the Riemannian case, this property is always true since all tangent vectors have positive norms.

This notion extends to isometry groups in the following way: a group Γ of isometries on a pseudo-Riemannian manifold Ω is causal if for every x in M and every γ in Γ there is no causal nontrivial curve joining x to γx. In the Riemannian context any action is causal.

In Section 2, we collect general definitions and facts about causality notions in general spacetimes.

1.2 Einstein universe

Actually, we spend a great amount of time to the detailed discussion of the subtle causality notion in AdS-spacetimes: see Section 5.3. One important aspect is that in AdS the causality relation is trivial: every pair of points in AdS is causally related! But this triviality vanishes in the universal covering of AdS and we can define *achronal* subsets (see next Section) in AdS as the projections in AdS of achronal subsets of the universal covering (see Section 5).

It appears extremely useful for the causal study of AdS to use the conformal embedding of the AdS spacetime in the universal conformally flat Lorentzian manifold: the *Einstein universe* Ein_3 (Section 4). The causality notion persists in the conformal framework and the understanding of the causality relation in AdS follows easily from the study in Ein_3 (Section 5.1). This ingredient is particularly useful for the study of spacelike surfaces (see Section 7) in AdS-spacetimes: compare our proof of Proposition 7.4 with the similar statement (Lemma 7) in the pioneering paper [30].

Einstein universe plays another important role: the 2-dimensional Einstein universe Ein_2 is the natural conformal boundary of AdS (Remark 4.7). This feature is completely similar to the fact that the natural conformal boundary of $\mathbb{H}^3$ is the universal conformally flat Riemannian surface; i.e., the round sphere $\mathbb{S}^2$ (observe also that $\mathbb{H}^3$ admits a conformal embedding in $\mathbb{S}^3$). The causality notion extends to the conformal completion $\mathrm{AdS} \cup \partial\mathrm{AdS}$ (Section 5.4) simply by restriction of the causality relation in Ein_3.

1.3 Invisible domains

A subset Λ of the conformal boundary Ein_2 is *achronal* if pairs of points in Λ are not causally related. The *invisible domain from* Λ, denoted $E(\Lambda)$, is the set of points in AdS which are not causally related to any point in

Λ (Section 8.6). When Λ is *generic,* i.e., not *pure lightlike* (which is a particularly exceptional case, see Definitions 5.7, 5.11), then $E(\Lambda)$ is a non-empty convex open domain containing Λ in its closure which is *geodesically convex,* i.e., any timelike geodesic segment in AdS joining two points in U is contained in $E(\Lambda)$.

We study extensively this notion in Section 8 in the "non-elementary" case, i.e., when Λ is not contained in one or two lightlike geodesics (Section 8.7). The elementary case will be studied in [7]. We also consider in detail the case where Λ is a topological circle in Ein_2: $E(\Lambda)$ is then *globally hyperbolic,* with *regular cosmological time* (see Definitions 2.9, 2.23, 2.24 and Propositions 8.21, 8.23, 8.15).

The main issue of Section 8 is Proposition 8.50: in the general case, invisible domains decompose in (non-disjoint) two globally hyperbolic domains, called globally hyperbolic cores, and *closed ends.* This notion will be more detailed in the next paper when we will consider BTZ black-holes. We just mention here that they are simply intersections between the Klein model $\mathbb{ADS}$ and tetraedra in the projective space $\mathbb{R}P^3$. They are also simple pieces of elementary invisible domains.

1.4 Proper and causal actions

Let Γ be a discrete group of isometries of AdS. We say that Γ is *admissible* if it preserves a generic non-elementary achronal subset of Ein_2. This definition extends to the elementary cases, but requires then a more detailed discussion that we postpone to [7]. Actually, it is easy to see that Λ is necessarily non-elementary if Γ is not abelian.

However, one of the main result of the present paper is Theorem 10.1 that we reproduce here:

Theorem 1.1. *Let Λ be a non-elementary generic achronal subset preserved by a discrete group $\Gamma \subset Isom(AdS)$. Then the action of Γ on $E(\Lambda)$ is properly discontinuous.*

Theorem 10.1 states also another result: the quotient spacetime $M_\Lambda(\Gamma)$ is *strongly causal* (see Section 2.3).

Remark 1.2. The theorem above is still true when Γ is a non-cyclic abelian group: we then obtain as quotient spaces the (AdS)-*Torus universes* described in [29, 18]. They correspond, through the AdS-rescaling [10, 11] to the flat Torus Universes described in [6, 18, 9].

Furthermore, we have a description of admissible groups (Theorem 10.7). In the non-abelian case, the formulation is as follows: Isom(AdS) is isomorphic to the quotient of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ by a subgroup of order 2 (see Section 3.3). Under this identification every admissible group is the projection of the image of a representation $\rho = (\rho_L, \rho_R) : \Gamma \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ where $\rho_{L,R}$ are fuchsian (i.e., faithfull and discrete) representations into $\mathrm{SL}(2,\mathbb{R})$ one semi-conjugate to the other, i.e., defining the same bounded Euler cohomology class in $H^2_b(\Gamma, \mathbb{Z})$ (see [24, 25], Remark 10.10).

1.5 Limit sets

We still assume that Γ is admissible and non-abelian. Then, there is an unique minimal closed generic achronal Γ-invariant subset $\Lambda(\Gamma)$ which is contained in every closed achronal Γ-invariant subset (Theorem 10.13, Corollary 10.14). Hence, following the classical terminology used for isometry groups of $\mathbb{H}^n$, it is natural to call $\Lambda(\Gamma)$ the *limit set* of Γ.

This analogy with the Riemannian case can be pursued further: consider the Klein model of $\mathbb{H}^n$ as a (convex) ellipsoid in $\mathbb{R}P^n$. Let Λ be the limit set of a discrete group of isometries Γ (a Kleinian group). For any pair (x, y) of points in Λ let E_{xy} be the unique connected component of $\mathbb{R}P^n \setminus (T_x \cup T_y)$ containing $\mathbb{H}^n$ where T_x, T_y are the projective hyperplanes tangent to $\mathbb{H}^n$ at x, y. Then the intersection of all the E_{xy} with (x, y) describing $\Lambda \times \Lambda$ is a convex domain $E(\Lambda)$ on which Γ acts properly: this statement is the true Riemannian counterpart of Theorem 1.1.

The complement in $\mathbb{R}P^n$ of the closure of $\mathbb{H}^n$ admits a natural $\mathrm{SO}(1, n)$-invariant lorentzian metric: this is the (Klein model of) the de Sitter space dS^n. Then the action of Γ on $\mathrm{dS}^n \cap E(\Lambda)$ is causal. When Γ is torsion-free, the quotient space is strongly causal — even more, it is globally hyperbolic. This is a particular case of the content of Scannell thesis [31] where maximal globally hyperbolic dS-spacetimes are classified. For more details in this direction we refer to the survey [5].

Remark 1.3. However, there is an important difference between the de Sitter case and the anti de Sitter one: whereas Scannell proved that the quotient spaces described above are all globally hyperbolic, the quotient space $M_\Lambda(\Gamma) = \Gamma \backslash E(\Lambda)$ is *not* globally hyperbolic, except if Λ is a topological circle.

1.6 Maximal globally hyperbolic AdS-spacetimes

We mention here another result of the present paper (Corollary 11.2): *the quotient spacetimes $M_\Lambda(\Gamma)$, where Λ is an achronal topological circle, admits*

Cauchy surfaces such that the ambient AdS-metric restricts as a Cauchy-complete Riemannian metric (for the definition of Cauchy surfaces, see Section 2.4). This theorem is also proved in [11] Proposition 6.4.19. Our proof relies on an general construction interesting by itself and which presumably works for the the higher dimensional case, associating to any embedded spacelike surfaces in AdS a surface embedded in $\mathbb{H}^2 \times \mathbb{H}^2$ (Sections 3.6, 7.3). This result completes nicely the classification of maximal globally hyperbolic AdS-spacetimes admitting Cauchy-complete Cauchy surfaces (Proposition 11.1, see also the Proposition 6.5.7 in [11]). Thus the AdS-rescaling defined in [10, 11] establishes a natural bijection between these maximal globally hyperbolic AdS-spacetimes and non-complete maximal Cauchy-complete globally hyperbolic *flat* spacetimes which have been classified in [4].

The spacetimes $M_\Lambda(\Gamma)$ defined here, even if Λ is not a topological sphere, are always strongly causal. In particular, they are never compact. This is an important difference between our work with usual studies of discrete subgroups of Isom(AdS) where a special focus is put on the cocompact case. In our framework, the spacetimes enjoying a "compact" character are the globally hyperbolic AdS-spacetimes admitting a closed Cauchy surface.

1.7 Admissible groups

The main drawback of our approach is that many discrete subgroups are not admissible, i.e., do not preserve generic achronal subsets of Ein_2 (for example the lattices of Isom(AdS)). In Section 10.6, we give a characterization of (non-abelian) admissible groups: they are the subgroups of $\mathrm{SO}_0(2,2)$ preserving some proper convex domain of $\mathbb{R}P^3$ which, in the terminology of [12], are *positively proximal.* In some way, the present paper provides a geometrical illustration of some cases considered in [12]. Actually, we need to be more precise. The claim above is not exactly correct: it is true that admissible groups are positively proximal, but positively proximal subgroups are not always admissible. But our claim is not so far to be correct: the Klein model $\overline{\mathbb{ADS}}$ in $\mathbb{R}P^3$ is one connected component of the complement of a quadric. The other connected component is another copy $\overline{\mathbb{ADS}}'$ of AdS (see Remark 3.1, 5.23). Observe that $\mathrm{SO}_0(2,2)$ is also the isometry group of $\overline{\mathbb{ADS}}$. Then the correct statement is Proposition 10.22: any positively proximal subgroup of $\mathrm{SO}_0(2,2)$ is either admissible, or admissible as considered as a group of isometry of $\overline{\mathbb{ADS}}'$.

1.8 Higher dimension

Many results in the present paper extend in higher dimensions, particularly if we restrict ourselves to strongly irreducible subgroups. But the number of "elementary" cases increases with the dimension and a systematic treatment requires a non-elementary case-by-case study of these "elementary" cases. We prefer to postpone such a study to another circumstance. Dropping here the elementary case is not conceivable since it corresponds to our second main goal: the systematic description of BTZ multi-black holes, in particular, of single BTZ black holes.

2 General notions

A *spacetime* M is a manifold equipped with a lorentzian metric — actually we will soon restrict to the constant curvature case. In our convention, a lorentzian metric has signature $(-,+,\dots,+)$; an orthonormal frame is a frame $(e_1, e_2, \dots, e_n)$ where e_1 has norm -1, every e_i $(i \geq 2)$ has norm $+1$ and every scalar product $\langle e_i \mid e_j \rangle$ $(i \neq j)$ is 0. A tangent vector is *spacelike* if its norm is positive; *timelike* if its norm is negative; *lightlike* if its norm is 0. We also define *causal* vectors as tangent vectors which are timelike or lightlike. An immersed surface S is spacelike if all vectors tangent to S are spacelike; it is non-timelike if tangent vectors are all spacelike or lightlike.

A causal (resp. timelike) curve is an immersion $c : I \subset \mathbb{R} \to M$ such that for every t in I the derivative $c'(t)$ is causal (resp. timelike). This notion extends naturally to non-differentiable curves (see [8]). Such a curve is *extendible* if there is another causal curve $\hat{c} : J \to M$ and a homeomorphism $\varphi : I \to K \subset J$ such that $K \neq J$ and c coincide with $\hat{c} \circ \varphi$. The causal curve c is *inextendible* if it is not extendible.

2.1 Time oriention

We always assume that the lorentzian manifold is oriented. On spacetimes we have another orientability notion:

Definition 2.1. A spacetime M is time-orientable (or chronologically orientable) if it admits a a continuous field of timelike vectors.

Definition 2.2. A time-orientation on M is an equivalence class of continuous timelike vector fields, for the following equivalence relation: two timelike vector fields X, Y are equivalent if for any x in M the scalar product $\langle X(x) \mid Y(x) \rangle$ is negative.

M is time oriented when a time-orientation on M has been selected.

It is easy to show that any spacetime admits a continuous field of timelike lines. Hence, every spacetime is doubly covered by a time-orientable lorentzian manifold. We will always assume that M is time-oriented. Once selected the time-orientation X, the set of causal tangent vectors splits into the union of two bundles of convex cones: the cone of future-oriented vectors $\{v \in T_x M \mid \langle v \mid X(p) \rangle < 0\}$ and the cone of past-oriented vectors $\{v \in T_x M \mid \langle v \mid X(p) \rangle > 0\}$.

Time-orientation provides naturally an orientation on every causal curve: a causal curve is either future-oriented or past-oriented.

2.2 Causality notions

Two points in M are causally related if there exists a causal curve joining them; they are *strictly* causally related if moreover this joining curve can be chosen timelike.

More generally: let E a subset of M and U an open neighborhood of E in M. A subset E of M is said *achronal* in U if there is no timelike curve contained in U joining two points of the subset. It is *acausal,* or *strictly achronal* in U if there is no causal curve contained in U joining two points of E. We say simply that E is (strictly) achronal if it is (strictly) achronal in $U = M$. Finally, we say that E is locally (strictly) achronal if every point x in E admits a neighborhood U in M such that $E \cap U$ is (strictly) achronal in U.

Remark 2.3. Spacelike hypersurfaces are locally acausal; non-timelike hypersurfaces are locally achronal.

2.3 Past, future

Definition 2.4. The future of a subset A of M is the set of final points of future oriented timelike curves not reduced to one point and starting from a point of A. The causal future of A is the set of final points of future oriented causal curves, maybe reduced to one point and starting from a point of S (hence A itself belongs to its causal future). The (causal) past of A is the (causal) future of A when the time-orientation of M is reversed.

Definition 2.5. Let x, y be two points in M with y in the future of x. The common past-future region $U(x, y)$ is the intersection between the past of y and the future of x.

The domains $U(x, y)$ form the basis for some topology on M, the so-called *Alexandrov topology* (see [8]). Observe that every $U(x, y)$ is open for the manifold topology. The converse in general is false:

Definition 2.6. If the Alexandrov topology coincide with the manifold topology, M is strongly causal.

Remark 2.7. If M is strongly causal, every open domain $U \subset M$ equipped with the restriction of the ambient lorentzian metric is strongly causal.

Proposition 2.8 (Proposition 3.11 of [8])**.** *The lorentzian manifold M is strongly causal if and only if it satisfies the following property: for every point x in M every neighborhood of x contains an open neighborhood U (for the usual manifold topology) which is* causally convex, *i.e., such that any causal curve in M joining two points in U is actually contained in U.*

2.4 Global hyperbolicity

Definition 2.9. M is globally hyperbolic if:

- it is strongly causal,
- for any x, y in M the intersection between the causal future of x and the causal past of y is compact or empty.

From now we assume that M is strongly causal.

The notion of global hyperbolicity is closely related to the notion of Cauchy surfaces: let S be a spacelike surface embedded in M.

Definition 2.10. The past development $P(S)$ (resp. the future development $F(S)$) is the set of points x in M such that every inextendible causal path containing x meets S in its future (resp. in its past). The Cauchy development $\mathcal{C}(S)$ is the union $P(S) \cup F(S)$.

Definition 2.11. If $\mathcal{C}(S)$ is the entire M, S is a Cauchy surface.

Theorem 2.12 ([23])**.** *A strongly causal lorentzian manifold M is globally hyperbolic if and only if it admits a Cauchy surface.*

Theorem 2.13 ([23], Proposition 6.6.8 of [28])**.** *If M is globally hyperbolic and S a Cauchy surface of M, there is a diffeomorphism $f : M \to S \times \mathbb{R}$ such that every $f^{-1}(S \times \{*\})$ is a Cauchy surface in M.*

Remark 2.14. There has been some imprecision in the litterature concerning the proof the smoothness of the splitting of globally hyperbolic spacetimes. See [13–15] for a survey on this question and a complete proof of the smoothness of the splitting $M \approx S \times \mathbb{R}$.

Remark 2.15. All the notions discussed in this section above only depends on the conformal class of the metric. Hence they are well defined in every conformally lorentzian manifolds, in particular, in Ein_n (see Section 4).

Proposition 2.16. *Let M be a globally hyperbolic spacetime and let Γ be a group of isometries of M acting freely and properly discontinuously on M and preserving the chronological orientation. Assume that Γ preserves a Cauchy surface S in M. Then, the quotient spacetime $\Gamma\backslash M$ is globally hyperbolic and the projection of S in this quotient is a Cauchy surface.*

Sketch of proof. Let M' be the quotient $\Gamma\backslash M$ and S' be the projection of S in M'. Since Γ preserves the chronological orientation, the future (resp. past) of S' in M' is the projection of the future (resp. past) of S in M. Every inextendible causal curve in M' lifts in M as a inextendible causal curve in M. The proposition follows.

2.5 Maximal globally hyperbolic spacetimes

In this Section, we assume that M has constant curvature[1].

Definition 2.17. An isometric embedding $f : M \to N$ is a Cauchy embedding if the image by f of any Cauchy surface in M is a Cauchy surface of N.

Definition 2.18. A globally hyperbolic spacetime M is maximal if every Cauchy embedding $f : M \to N$ in a spacetime with constant curvature is surjective.

Theorem 2.19 (see Choquet–Bruhat–Geroch [19])**.** *Let M be a globally hyperbolic spacetime with constant curvature. Then, there is a Cauchy embedding f: $M \to N$ in a maximal globally hyperbolic spacetime N with constant curvature. Moreover, this maximal globally hyperbolic extension is unique up to right composition by an isometry.*

2.6 Lorentzian distance

Let M be a time-oriented spacetime.

Definition 2.20. The length-time $L(c)$ of a causal curve $c : I \to M$ is the integral over I of the square root of $-\langle c'(t) \mid c'(t)\rangle$.

Observe that it is well defined, since causal curves are always Lipschitz.

[1]We could actually only suppose that (M, g) is a solution of the Einstein equation $\mathrm{Ricci}_g - \frac{R}{2} g = \Lambda g$.

Definition 2.21. The lorentzian distance $d_{lor}(x, y)$ between two points x, y in M is $\mathrm{Sup}\{L(c)/c \in C(x, y)\}$ where $C(x, y)$ is the set of causal curves with extremities x, y. By convention, if x, y are not causally related, $d_{lor}(x, y) = 0$.

Theorem 2.22 (Corollary 4.7 and Theorem 6.1 of [8]). *If M is globally hyperbolic, then $d : M \times M \to [0, +\infty]$ is continuous and admits only finite values. Moreover, if y is in the causal future of x, then there exist a geodesic c with extremities x, y such that $L(c) = d(x, y)$.*

2.7 Cosmological time

In any spacetime, we can define the *cosmological time* (see [1]):

Definition 2.23. For any x in M, the cosmological time $\tau(x)$ is the Sup $\{L(c)/c \in \mathcal{R}^-(x)\}$, where $\mathcal{R}^-(x)$ is the set of past-oriented causal curves starting at x,

This function could have in general a bad behavior: for example, in Minkowski space, the cosmological time is everywhere infinite.

Definition 2.24. M has regular cosmological time if:

– M has finite existence time, i.e., $\tau(x) < \infty$ for every x in M,
– for every past-oriented inextendible curve $c : [0, +\infty[\to M$, $\lim_{t\to\infty} \tau(c(t)) = 0$.

Theorem 1.2 in [1] expresses many nice properties of spacetimes with regular cosmological time function. We need only the following statement:

Theorem 2.25. *If M has regular cosmological time, then the cosmological time is Lipschitz regular and M is globally hyperbolic.*

Obviously:

Lemma 2.26. *On any spacetime M, every level set of the cosmological time is preserved by the isometry group of M.*

Proposition 2.27. *If M has regular cosmological time and if Γ is a group of isometries acting freely and properly discontinuously on M, then the quotient space $\Gamma\backslash M$ has regular cosmological time.*

Sketch of proof. It follows from the fact that inextendible causal curves in the quotient are projections of inextendible causal curves in M.

3 Anti-de Sitter and Einstein spaces

3.1 Anti-de Sitter space

Let E denote the vector space $\mathbb{R}^4$ equipped with the quadratic form $Q = -u^2 - v^2 + x_1^2 + x_2^2$ (we also denote $E = \mathbb{R}^{2,2}$). The associated bilinear form is denoted by $Q(.,.)$ or $\langle . \mid . \rangle$ as well. The 3-dimensional anti-de Sitter space AdS_3 is the set $\{Q = -1\}$, equipped with the lorentzian metric obtained by restriction of Q. We will most of the time drop the index 3, since here we mainly consider the 3-dimensional case. This space has constant negative curvature. Its isometry group is naturally $O(2,2)$, acting freely transitively on the bundle of orthonormal frames of AdS. Let $\mathrm{SO}_0(2,2)$ the neutral component of $O(2,2)$ (the so-called orthochronal component). Fix an orientation of $\mathrm{SO}(2) \subset \mathrm{SO}_0(2,2)$ (the subgroup preserving the x_1, x_2-coordinates): it defines a time-orientation on AdS. The neutral component $\mathrm{SO}_0(2,2)$ is precisely the group of isometries of AdS preserving the orientation and the time orientation.

3.2 Klein models

Let $S(E)$ be the half-projectivization of E, i.e., the space of rays. It is the double covering of the projective space $P(E) \approx \mathbb{R}P^3$. We lift to $S(E)$ all the usual notions in $P(E)$: for example, a projective line in $S(E)$ is the radial projection of a 2-plane in E.

$S(E)$ is homeomorphic to the sphere $\mathbb{S}^3$. Let $\mathbb{ADS}$ be the radial projection of AdS in $S(E)$: we call it the Klein model of AdS. Let $\overline{\mathbb{ADS}}$ be the radial projection of AdS in $P(E)$. Observe that $\overline{\mathbb{ADS}}$ is still time-oriented.

The boundary of $\mathbb{ADS}$ (resp. $\overline{\mathbb{ADS}}$) in $S(E)$ (resp. $P(E)$) is the quadric $\mathcal{Q}$ (resp. $\overline{\mathcal{Q}}$), projection of the zero set of Q. We call it the *Klein boundary* of AdS.

The main interesting feature of the Klein model is that geodesics there are connected components of intersections with projective lines. More generally, the totally geodesic subspaces in $\mathbb{ADS}$ are the traces in $\mathbb{ADS}$ of projective subspaces. Projective lines avoiding $\mathcal{Q}$ are timelike geodesics, projective lines intersecting transversely $\mathcal{Q}$ (resp. tangent to $\mathcal{Q}$) induce spacelike (resp. lightlike) geodesics.

Remark 3.1. The complement in $S(E)$ of the closure of $\mathbb{ADS}$ can be considered as another copy of anti-de Sitter space: it is the projection of $\{Q = +1\}$;

this locus, equipped with the restriction of $-Q$, is isometric to AdS. Hence, $S(E)$ is the union of two copies of AdS and of their common boundary.

3.3 The SL$(2,\mathbb{R})$-model

Consider the linear space gl$(2,\mathbb{R})$ of 2 by two matrices, equipped with the quadratic form $-\det$. It is obviously isometric to (E,Q). Hence, AdS is canonically identified with the group SL$(2,\mathbb{R})$ of 2 by two matrices with determinant 1. The actions of SL$(2,\mathbb{R})$ on itself by left and right translations are both isometric actions: we obtain a morphism $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\to\mathrm{SO}_0(2,2)$. A dimension argument proves easily that this morphism is surjective. Its kernel is the pair $\{(id,id),(-id,-id)\}$.

The Klein model $\overline{\mathbb{ADS}}$ is canonically identified with PSL$(2,\mathbb{R})$. The group of orientation and time orientation preserving isometries is $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$, acting by left and right translations.

This Lie group structure on $\overline{\mathbb{ADS}}$ provides a natural parallelism on the tangent bundle: if $\mathcal{G}$ denotes the Lie algebra of $G=\mathrm{PSL}(2,\mathbb{R})$, i.e., the algebra of 2 by two matrices with zero trace, the differential of left translations identify $G\times\mathcal{G}$ with $TG=T\overline{\mathbb{ADS}}$. Then, the AdS-norm of a pair (g,Y) is simply $-det(Y)$.

3.4 The universal anti-de Sitter space

The anti-de Sitter space AdS is homeomorphic to $\mathbb{R}^2\times\mathbb{S}^1$: in particular, it is not simply connected. Let $p:\widetilde{\mathrm{AdS}}\to\mathrm{AdS}$ be the universal covering. The composition of p with the radial projection $\mathrm{AdS}\to\overline{\mathbb{ADS}}$ is denoted by $\bar{p}$: this is a universal covering of $\overline{\mathbb{ADS}}$.

Let δ be a generator of the Galois group of $\bar{p}$, i.e., the group of covering automorphisms: then, δ^2 generates the Galois group of p.

3.5 Affine domains

Definition 3.2. Let x be an element of AdS. The affine domain $A(x)$ is the subset of AdS formed by elements y satisfying $\langle x \mid y\rangle < 0$.

The restrictions of the radial projections of AdS over $\mathbb{ADS}$ or $\overline{\mathbb{ADS}}$ to affine domains are injective. The images of these projections are also called affine domains.

For any point x of $\mathbb{ADS}$, let x^* be the projection in $\mathbb{ADS}$ of the Q-orthogonal hyperplane in E of the direction defined by x: we call it the (totally geodesic) hypersurface dual to x. Observe that x^* has two connected components. Every connected component is a spacelike totally geodesic disc in $\mathbb{ADS}$, isometric to (the Klein model of) the hyperbolic disc $\mathbb{H}^2$. The boundary of x^* in $S(E)$ is the set of tangency between $\mathcal{Q}$ and lightlike geodesics containing $\mathbb{ADS}$. Moreover, x^* is orthogonal to every timelike geodesic containing x. All these geodesics also contain $-x$.

Denote also by $A(x)$ the projection in $\mathbb{ADS}$ of the affine domain $A(y)$ in AdS, where x is the projection of y in $\mathbb{ADS}$. Observe that $A(x)$ is the connected component of $\mathbb{ADS} \setminus x^*$ containing x. It is also the intersection between $\mathbb{ADS}$ and the affine patch $V(x) = S(E) \cap S(\{y/\langle y \mid x\rangle < 0\})$. $V(x)$ admits a natural affine structure and is affinely isomorphic to $\mathbb{R}^3$.

Definition 3.3. Let $\tilde{x}$ be an element of $\widetilde{\mathrm{AdS}}$. The affine domain $A(\tilde{x})$ is the connected component of $p^{-1}(A(p(\tilde{x})))$ containing $\tilde{x}$.

Affine domains are simple blocks quite easy to visualize, from which $\widetilde{\mathrm{AdS}}$ can be nicely figured out:

– every affine domain is naturally identified with the interior in $\mathbb{R}^3$ of the one-sheet hyperboloid: $\{(x, y, z)/x^2 + y^2 < 1 + z^2\}$.
– for any $\tilde{x}$ in $\widetilde{\mathrm{AdS}}$, let A_i be the affine domain $A(\delta^i \tilde{x})$ (recall that δ generates the Galois group of $\bar{p}$). Then, the affine domains A_i are disjoint 2 by 2, $\widetilde{\mathrm{AdS}}$ is the union of the closures $\overline{A}_i$ and two such closures $\overline{A}_i$, $\overline{A}_j$ are disjoint, except if $j = i \pm 1$ (keeping away the trivial case $i = j$), in which case $\overline{A}_j \cap \overline{A}_i$ is a totally geodesic surface isometric to a connected component of $p(\tilde{x})^*$.

In other words, the universal anti-de Sitter space can be obtained by adding up a bi-infinite sequence of affine domains, every affine domain being attached to the next one along a copy of the hyperbolic plane.

3.6 The projectivized timelike tangent bundle

We will also consider the *projectivized timelike tangent bundle:* it is the bundle $PT_{-1}\mathrm{AdS}$ over AdS admitting as fibers over a point x of AdS the set of timelike rays in $T_x\mathrm{AdS}$. It can also be defined as the subset $\mathcal{T}$ of $\mathrm{AdS} \times \mathrm{AdS}$ formed by pairs (x, y) such that $\langle x \mid y\rangle = 0$. Indeed, for such a pair, the tangent direction at $t = 0$ of the curve $t \mapsto \cos tx + \sin ty$ for $t \geq 0$ defines a timelike ray in $T_x\mathrm{AdS}$ and every timelike ray can be obtained in this manner in an unique way.

It will be useful to consider $\mathcal{T}$ as a quadric in $E \times E$. For convenience, we write explicitly the definition:

Definition 3.4. The projectivized timelike tangent bundle $\mathcal{T}$ is the set of pairs (x, y) in $E \times E$ such that:

- $\mid x \mid = \mid y \mid = -1$,
- $\langle x \mid y \rangle = 0$.

Using the canonical parallelism of the vector space $E \times E$, we can identify the tangent space to $\mathcal{T}$ over (x, y) with the vector space of vectors $(u, v) \in E \times E$ such that:

- $\langle x \mid u \rangle = \langle y \mid v \rangle = 0$,
- $\langle x \mid v \rangle + \langle u \mid y \rangle = 0$.

Define $\mid (u, v) \mid^2$ as the sum $\frac{1}{4}(Q(u) + Q(v))$. It endows $\mathcal{T}$ with a pseudo-Riemannian metric.

The diagonal action of $O(2,2)$ on $E \times E$ preserves $\mathcal{T}$ and the restriction of this action on $\mathcal{T}$ is isometric for the pseudo-Riemannian metric we have just defined. We claim that this metric is lorentzian. Indeed, by transitivity of the $O(2,2)$-action, it suffices to check at the special point $(x, y) = ((1,0,0,0),(0,1,0,0))$. Tangent vectors at this point correspond to pairs (u, v), with $u = (0, \alpha, \eta, \nu)$ and $v = (-\alpha, 0, \eta', \nu')$. The pseudo-Riemannian norm is therefore $-\frac{1}{2}\alpha^2 + \frac{1}{4}(\eta^2 + \nu^2 + \eta'^2 + \nu'^2)$. The claim follows.

Observe that the identification of $\mathcal{T}$ with PT_{-1}AdS defined above is $O(2,2)$-equivariant, where the $O(2,2)$-action on PT_{-1}AdS to be considered is the action induced by the differential of its isometric action on AdS.

The space $\mathcal{T}$ admits two connected components: one of them, called $\mathcal{T}^+$, corresponds to future-oriented timelike tangent vectors to AdS and the other, called $\mathcal{T}^-$, corresponds to past-oriented timelike tangent vectors.

4 Conformal embedding of AdS in the Einstein Universe

Sometimes (for example, when the causality notion is involved, see next Section), it is worth considering the natural embedding of AdS in the so-called *Einstein Universe* (see [21]).

Let Q_n be the quadratic form $-u^2 - v^2 + x_1^2 + \cdots + x_n^2$ on $\mathbb{R}^{2,n}$ (we only consider here the cases $n = 2$ or $n = 3$). Let $\mathcal{Q}_n$ be the projection of

$\{Q_n = 0\}$ is the sphere $S(\mathbb{R}^{2,n})$ of half-directions. As we have seen above, $\mathcal{Q}_2$ can be naturally thought as the (Klein) boundary of AdS.

The n-dimensional Einstein universe, denoted by Ein_n, is the quadric $\mathcal{Q}_n$, equipped with a conformally lorentzian structure as follows: let $\pi : \mathbb{R}^{2,n} \setminus \{0\} \to S(\mathbb{R}^{2,n})$ be the radial projection. For any open domain U in $\mathcal{Q}_n$ and any section $\sigma : U \to \mathbb{R}^{2,n}$, we can define the norm $Q_\sigma(v)$ of any tangent vector v as the Q_n norm of $d\sigma(v)$. We obtain by this procedure a lorentzian metric on U. This lorentzian metric depends on the selected section σ, but if $\sigma' = f\sigma$ is another section, then $Q_{\sigma'} = f^2 Q_\sigma$. Hence, the *conformal class* of Q_σ does not depend on σ. Moreover, the choice of a representant (of class C^r) of this conformal class is equivalent to the choice of a section (of class C^r) of π over U. Eventually, the group of conformal transformations of Ein_n is $O(2, n)$.

As a first application of this remark, we obtain that Ein_n is conformally isometric to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ equipped with the metric $ds^2 - dt^2$, where ds^2 is the usual metric on the unit sphere $\mathbb{S}^{n-1}$, and dt^2 the usual metric on $\mathbb{S}^1 \approx \mathbb{R}/2\pi\mathbb{Z}$. This lorentzian metric appears when we select the global section σ with image contained in the sphere $u^2 + v^2 + x_1^2 + \cdots + x_n^2 = 2$ of $\mathbb{R}^{2,n}$. We will denote by $p : \widehat{\mathrm{Ein}}_n \approx \mathbb{S}^{n-1} \times \mathbb{R} \to \mathrm{Ein}_n$ the cyclic covering (it is the universal covering when $n \geq 3$) (it is coherent with the convention in Section 3.4 in view of the natural embedding $\widehat{\mathrm{AdS}} \subset \widehat{\mathrm{Ein}}_3$, see Remark 4.8 below). Observe that Ein_n is time-orientable.

Throughout this paper, we denote by d the spherical distance on $\mathbb{S}^{n-1}$. Keeping in mind the identification $\widehat{\mathrm{Ein}}_n \approx \mathbb{S}^{n-1} \times \mathbb{R}$, timelike curves in $\widehat{\mathrm{Ein}}_n$ correspond to curves $t \mapsto (\varphi(t), t)$ where t describes some segment $I \subset \mathbb{R}$ and $\varphi : I \to \mathbb{S}^{n-1}$ is a contracting map (i.e., the spherical distance $d(\varphi(t), \varphi(t'))$ is strictly less than $\mid t - t' \mid$). When φ is just 1-Lipschitz, the curve $t \mapsto (\varphi(t), t)$ is only causal.

It is well known that the notion of lightlike geodesic is still meaningful in the conformally lorentzian context, but they are not naturally parametrized. More precisely, if we forget their parametrizations, lightlike geodesics does not depend on the lorentzian metric in a given conformal class. Under the identification $\widehat{\mathrm{Ein}}_n \approx \mathbb{S}^{n-1} \times \mathbb{R}$, inextendible lightlike geodesics are curves $t \mapsto (\varphi(t), t)$ where $\varphi : \mathbb{R} \to \mathbb{S}^{n-1}$ is a geodesic on the sphere.

Theorem 4.1 ([22]). *The Einstein space $\widehat{Ein}_n$ is universal in the category of locally conformally flat lorentzian spaces; i.e., every simply connected lorentzian manifold of dimension n which is conformally flat can be conformally immersed in $\widehat{Ein}_n$.*

We will not give a proof of this theorem here, but will exhibit the natural embedding of anti-de Sitter space AdS in Ein_3: let v be any spacelike vector in $\mathbb{R}^{2,3}$; and $v^\perp$ its Q_3-orthogonal hyperplane in $\mathbb{R}^{2,3}$. Let $\mathcal{A}(v)$ be the projection in $S(\mathbb{R}^{2,3})$ of the intersection between $\{Q_3 = 0\}$ and one connected component of the complementary part of $v^\perp$ and let $\partial\mathcal{A}(v)$ be the projection of $\{Q_3 = 0\} \cap v^\perp$. The notations $\mathcal{A}$ and $\partial\mathcal{A}$ will be reserved to the special case $v = v_0 = (0, \dots, 0, 1)$. There is a natural section σ over $\mathcal{A}(v)$: take $\sigma(x)$ such that the Q_3-scalar product between $\sigma(x)$ and v is equal to ± 1. A straightforward computation shows that $(\mathcal{A}(v), Q_\sigma)$ is isometric to AdS.

Definition 4.2. An anti-de Sitter domain in Ein_3 is an open domain $\mathcal{A}(v)$ for some spacelike vector v in $\mathbb{R}^{2,3}$.

Remark 4.3. The notation is a little misleading since every v defines actually *two* domains, since $\mathbb{R}^{2,3} \setminus v^\perp$ has two connected components. We can withdraw this undeterminancy by defining more precisely $\mathcal{A}(v)$ as the radial projection of $\{x / \langle x|v\rangle = -1\}$. Anyway, both connected components are conformal copies of AdS, glued along their common conformal boundary $\approx \mathrm{Ein}_2$. This decomposition is *not* the decomposition discussed in the Remark 3.1: indeed, $\mathrm{Ein}_3 \approx \mathbb{S}^2 \times \mathbb{S}^1$ is not homeomorphic to $S(E) \approx \mathbb{S}^3$! See also Remark 5.23.

Remark 4.4. Similar constructions can be performed even when v is not spacelike. When v is timelike, we obtain an open domain $\mathcal{S}(v)$ conformally isometric to de 3-dimensional de Sitter space. When v is lightlike, this procedure provides a conformal idendification, called *stereographic projection,* between every connected component of the complement of a lightcone with the 3-dimensional Minkowski space.

Remark 4.5. In order to get a satisfying understanding of the geometry involved, it is useful to "scan" these domains in $\mathbb{S}^2 \times \mathbb{S}^1$: the standard anti-de Sitter domain $\mathcal{A}$ is the domain in $\mathbb{D}^2 \times \mathbb{S}^1$ is $\mathbb{S}^2 \times \mathbb{S}^1$, where $\mathbb{D}^2$ is the upper hemisphere $\{x_2 > 0\}$. A typical de Sitter domain is $\mathbb{S}^2 \times]0, \pi[$. A typical affine domain (Section 3.5) of AdS conformally embedded in Ein_3 is $\mathbb{D}^2 \times]0, \pi[$: this is the intersection between an anti-de Sitter domain and some de Sitter domain.

Finally, the conformal embedding of AdS in Ein_3 lifts to a conformal embedding of $\widetilde{\mathrm{AdS}}$ in $\widehat{\mathrm{Ein}}_3$: it follows that $\widetilde{\mathrm{AdS}}$ is conformally equivalent to $\mathbb{D}^2 \times \mathbb{R}$ equipped with the metric $ds^2 - dt^2$, where ds^2 is the restriction to the hemisphere $\mathbb{D}^2$ of the spherical metric.

More precisely, if O is the "North pole" of $\mathbb{D}^2$ (i.e., the unique point in $\mathbb{D}^2$ which is at d-distance $\pi/2$ to every element of $\partial\mathbb{D}^2$), then $\widetilde{\mathrm{AdS}}$ is isometric to $\mathbb{D}^2 \times \mathbb{R}$ equipped with the metric:

$$\frac{1}{\cos^2(d(p,O))}(ds_0^2 - dt^2)$$

Remark 4.6. Sometimes, we will consider the projection of Ein_n in $P(\mathbb{R}^{2,n}) \approx \mathbb{R}P^{n+1}$: we denote it by $\overline{\mathrm{Ein}}_n$. It is still time-orientable. Observe that de Sitter domains in Ein_n projects injectively in $\overline{\mathrm{Ein}}_n$ as complementary parts of spacelike hyperplanes. It follows that de Sitter domains in Ein_n correspond to intersections of $\overline{\mathrm{Ein}}_n$ with affine patches of $\mathbb{R}P^{n+1}$ with spacelike boundaries.

Remark 4.7. When v is spacelike, $\partial\mathcal{A}(v)$ is a copy of Ein_2: we call it a *Einstein flat subspace.* We can also see $\partial\mathcal{A}(v)$ as the *conformal boundary of* $\mathcal{A}(v)$. The closure $\overline{\mathcal{A}}(v) = \mathcal{A}(v) \cup \partial\mathcal{A}(v)$ is canonically identified with the closure $\mathbb{ADS} \cup \mathrm{Ein}_2$ of $\mathbb{ADS}$ in $S(E)$, but this identification is not analytic, even if its restrictions to respectively $\mathcal{A}(v)$, $\partial\mathcal{A}(v)$ are individually analytic.

Remark 4.8. The projection of $\widehat{\mathrm{Ein}}_n$ over $\overline{\mathrm{Ein}}_n$ is a cyclic covering. Denote by δ a generator of the group of covering transformations. It is coherent with the convention in Section 3.4 since the restriction of this covering transformation to the image of the natural embedding $\widetilde{\mathrm{AdS}} \subset \widehat{\mathrm{Ein}}_3$ is indeed the generator of the Galois group of $\bar{p}$.

We say that two points in $\widehat{\mathrm{Ein}}_n$ are *opposite* if one of them is the image under δ of the other. It is easy to give another equivalent definition: for every element x of $\widehat{\mathrm{Ein}}_n$, the lightlike geodesics containing x admits many other common intersection points, that are precisely the iterates of x under δ. Therefore, two points x, y are opposite if every lightlike geodesic containing one of them contains the other and if the lightlike segments joining x to y are all disjoint.

Remark 4.9. The Einstein space $\widehat{\mathrm{Ein}}_n$ admits of course many different parametrizations by $\mathbb{S}^{n-1} \times \mathbb{R}$ for which the conformal lorentzian structure is represented by $ds^2 - dt^2$. Anyway, in all these parametrizations, pair of opposite points always have coordinates of the form (x, θ), $(-x, \theta + \pi)$.

Remark 4.10. According to Section 3.3, $\overline{\mathrm{Ein}}_2$ is naturally identified with the projection in $P(\mathrm{gl}(2,\mathbb{R}))$ of non-zero non-invertible matrices. For any such matrice A, let $K(A)$ be the kernel of A and $I(A)$ the image of A. Then, $A \mapsto ([I(A)], [K(A)])$ identifies $\overline{\mathrm{Ein}}_2$ with $\mathbb{R}P^1 \times \mathbb{R}P^1$. Denote by $\mathbb{R}P^1_L$ (resp. $\mathbb{R}P^1_R$) the leaf space of the *left* (resp. *right*) foliation $\overline{\mathcal{G}}_L$ (resp. $\overline{\mathcal{G}}_R$), i.e., the foliation of $\mathbb{R}P^1 \times \mathbb{R}P^1$ with leaves $\{*\} \times \mathbb{R}P^1$ (resp. $\mathbb{R}P^1 \times \{*\}$). An usual point of view is to consider every point of $\overline{\mathrm{Ein}}_2$ as the intersection between a leaf of the left foliation and a leaf of the right foliation. In this spirit, we can write $\overline{\mathrm{Ein}}_2 = \mathbb{R}P^1_L \times \mathbb{R}P^1_R$.

The extension of the isometric action of $G \times G$ (with $G = \mathrm{PSL}(2, \mathbb{R})$) to $\overline{\mathrm{Ein}_2}$ corresponds to the diagonal action of $G \times G$ on $\mathbb{R}P^1 \times \mathbb{R}P^1$, where the action of G on $\mathbb{R}P^1$ is the usual projective action. Let G_L (resp. G_R) be the group of left (resp. right) translations of G on itself: the group of conformal isometries of $\overline{\mathcal{Q}}$ is then the product $G_L \times G_R$. Observe that the leaves of the left and right foliations are lightlike geodesics.

Ein_2 can be considered in a similar way: it is bifoliated by two transverse foliations $\mathcal{G}_L$, $\mathcal{G}_R$. Every leaf of the left or right foliation is canonically the double covering $\widehat{\mathbb{R}P}^1$ of $\mathbb{R}P^1$. Every leaf of $\mathcal{G}_L$ intersects every leaf of $\mathcal{G}_R$ at *two* points, one opposite to the other. Finally, the leaf space of the left (resp. right) foliation is canonically identified to $\mathbb{R}P^1_L$ (resp. $\mathbb{R}P^1_R$).

This description lifts to $\widehat{\mathrm{Ein}}_2$: it admits a pair of transverse foliations $\widehat{\mathcal{G}}_L$, $\widehat{\mathcal{G}}_R$. The respective leaf spaces are still projective lines $\mathbb{R}P^1_L$, $\mathbb{R}P^1_R$. The leaves themselves are universal coverings of the projective line. Finally, the intersection between a leaf of the right foliation and a leaf of the left foliation is a δ-orbit.

5 Causality relation

In this section, we discuss the notion of causality in AdS and Ein_n. A fundamental observation is that the causality relation in AdS and Ein_n is trivial: for any pair (x, y) in AdS or Ein_n, there is a timelike curve joining x and y! Actually, this notion is interesting only in the universal covering $\widetilde{\mathrm{AdS}} \approx \mathbb{D}^2 \times \mathbb{R}$ and in the cyclic covering $\widehat{\mathrm{Ein}}_n \approx \mathbb{S}^{n-1} \times \mathbb{R}$.

The main purpose of this Section is to show that, even if the achronality notion is not *stricto sensu* well defined in AdS or in Ein_n, projections in these spaces of achronal subspaces of $\widetilde{\mathrm{AdS}}$ or of $\widehat{\mathrm{Ein}}_n$ are nicely described.

5.1 Achronality in $\widehat{\mathbf{Ein}}_n$

We leave as an exercise to the reader the following lemma (we just stress out that any causal curve can intersect every $\mathbb{S}^{n-1} \times \{*\}$ in at most one point):

Lemma 5.1. *Two points $(x_1, \tilde{\theta}_1)$ and $(x_2, \tilde{\theta}_2)$ are causally related in $\widehat{Ein}_n \approx \mathbb{S}^{n-1} \times \mathbb{R}$ if and only if the distance in $\mathbb{S}^{n-1}$ between x_1 and x_2 is less or equal to $|\theta_2 - \theta_1|$. These points are strictly causally related if the distance between x_1 and x_2 is less than $|\theta_2 - \theta_1|$. In particular, they are necessarily causally related if $|\theta_2 - \theta_1|$ is greater than π.*

Corollary 5.2. *Achronal subsets of $\widehat{Ein}_n$ are graphs of 1-Lipschitz functions $f : E \to \mathbb{R}$, where E is a subset of $\mathbb{S}^{n-1}$. Such a subset is strictly achronal if and only if f is contracting.*

Corollary 5.3. *The closure of an achronal subset is achronal.*

Corollary 5.4. *$\widehat{Ein}_n$ is strongly causal.*

Corollary 5.5. *Every $\mathbb{S}^{n-1} \times \{*\}$ is a Cauchy hypersurface in $\widehat{Ein}_n$. In particular, $\widehat{Ein}_n$ is globally hyperbolic.*

Lemma 5.6. *Every closed achronal subset Λ of $\widehat{Ein}_n$ is contained in a de Sitter domain, except if it is contained in the past lightcone and future lightcone of two opposite elements of itself (see Remark 4.8).*

Proof. Let (x^+, θ^+), (x^-, θ^-) be elements of Λ where the θ-coordinate attains, respectively, its maximum and mimimum value: if $\theta^+ - \theta^-$ is strictly less than π, then the lemma is proved, since Λ is contained in some de Sitter domain of the form $\mathbb{S}^{n-1}\times] - \theta^- - \epsilon, \theta^+ + \epsilon[$, with 2ϵ less than $\pi - \theta^+ + \theta^-$. On the other hand, $\theta^+ - \theta^-$ is less than π since π is the diameter of the hemisphere. Hence, we have only to deal with the case $\theta^+ - \theta^- = \pi$. In this case, the distance between x^+ and x^- on the sphere has to be precisely π and (x^+, θ^+), (x^-, θ^-) are opposite points (in the meaning of Definition 4.8). Moreover, for any (x, θ) in Λ, x lies on a minimizing geodesic of $\mathbb{S}^{n-1}$ between x^+ and x^-. It follows that θ must be equal to $\theta^+ - d(x, x^+) = \theta^- + d(x, x^-)$. In other words, Λ is contained in the past lightcone of (x^+, θ^+) and the future lightcone of (x^-, θ^-). □

The particular case appearing in Lemma 5.6 deserves a particular appellation.

Definition 5.7. A subset of $\widehat{\mathrm{Ein}}_n$ is pure lightlike if it is contained in the past lightcone and future lightcone of two opposite elements of itself. If not, it is generic.

We point out the obvious fact that a strictly achronal set is generic.

Remark 5.8. The proof of Lemma 5.6 actually shows that an achronal subset is pure lightlike as soon as it contains two opposite points.

5.2 Achronality in $\mathbf{Ein}_n$

Definition 5.9. An achronal (resp. strictly achronal) subset of Ein_n is the projection of an achronal (resp. strictly) achronal subset of $\widehat{\mathrm{Ein}}_n$.

Our main purpose here is to provide an effective criterion recognizing achronal subsets of Ein_n.

Observe that the scalar product $\langle [x] \mid [y] \rangle$ of two elements of the sphere $S(\mathbb{R}^{2,n})$ of half-rays is not well defined but has a well defined sign.

Proposition 5.10. *A subset Λ of Ein_n is (strictly) achronal if and only if it is contained in the closure of a de Sitter domain and for every pair $([x],[y])$ of elements of Λ the scalar product $\langle [x] \mid [y] \rangle$ is non-positive (resp. negative).*

Proof. We detail the proof only in the achronal case; the strictly achronal case being similar.

Assume first that Λ is achronal, i.e., is the projection of an achronal subset $\widehat{\Lambda}$ of $\widehat{\mathrm{Ein}}_n$. If $\widehat{\Lambda}$ is pure lightlike the conclusion holds quite easily. Assume now that $\widehat{\Lambda}$ is generic. According to Lemma 5.6, Λ has to be contained in some de Sitter domain $\mathcal{S}(v_0)$ (with v_0 a timelike vector of $\mathbb{R}^{2,n}$). In this de Sitter domain it appears clearly that if $\langle [x] \mid [y] \rangle$ is positive then the segment in the affine domain $\mathcal{S}(v_0)$ with extremities $[x]$, $[y]$ is timelike (check in obvious cases and use the transitivity of the action of the stabilizer of v_0 in $\mathrm{SO}(2,n)$ on the sets of timelike, lightlike and spacelike lines in $\mathcal{S}(v_0)$). The first implication of the proposition then follows.

For the reverse implication: assume that Λ is contained in the closure of a de Sitter domain and that for every pair $([x],[y])$ of elements of Λ the scalar product $\langle [x] \mid [y] \rangle$ is non-positive. Consider any connected component $\widetilde{\mathcal{S}}(v_0)$ of the preimage of $\mathcal{S}(v_0)$ and let $\widehat{\Lambda}$ be the preimage of Λ in the closure $\overline{\mathcal{S}}(v_0)$ of $\widetilde{\mathcal{S}}(v_0)$: the projection of $\widehat{\Lambda}$ in $S(\mathbb{R}^{2,n})$ is Λ. Using the conformal identification $\overline{\mathcal{S}}(v_0) \approx \mathbb{S}^{n-1} \times [-\pi/2, +\pi/2]$ it is quite straightforward to check that $\widehat{\Lambda}$ is achronal (hint: de Sitter domains in $\widehat{\mathrm{Ein}}_n$ are causally convex). □

The way to recognize projections of pure lightlike subsets of $\widehat{\mathrm{Ein}}_n$ is obvious:

Definition 5.11. A closed subset Λ of Ein_n is pure lightlike if:

- it contains two opposite points $[x_0]$, $-[x_0]$,
- for every element $[x]$ of Λ the scalar product $\langle [x] \mid [x_0] \rangle$ is zero,
- Λ is contained in the closure of a de Sitter domain.

Non-pure lightlike closed subsets of Ein_n are generic.

Corollary 5.12. *A generic subset of Ein_n is (strictly) achronal if and only if it is contained in some de Sitter domain and (strictly) achronal in every de Sitter domain containing it.*

We have a well-defined notion of convex hull in $S(E)$.

Corollary 5.13. *A generic subset of Ein_n is achronal if and only if its convex hull in $S(E)$ is contained in the closure in $S(E)$ of the Klein model $\mathbb{ADS}$.*

Proof. Immediate corollary of Proposition 5.10. □

Remember that *extreme points* of a closed convex C set are points which do not belong to segments $]x, y[$ with x, y in C:

Lemma 5.14. *An achronal subset of Ein_n is strictly achronal precisely when the intersection points between its convex hull C and Ein_2 are all extreme points of C.*

Proof. A non-extreme point of the convex hull belonging to Ein_n is the projection of a sum $\sum_{i=1,..,k} t_i u_i$ where every u_i projects to an element of Λ, $0 < t_i < 1$, $k \geq 2$ and $[u_i] \neq [u_j]$ if $i \neq j$. Then, $\langle \sum t_i u_i | \sum t_i u_i \rangle = \sum t_i t_j \langle u_i | u_j \rangle$ can be zero only if all products $\langle u_i | u_j \rangle$ are null, which precisely means that every u_i is causally related to every u_j. □

Remark 5.15. Let y, y' be two non-causally related points in $\widehat{\mathrm{Ein}}_n$. Let p, p' be two points in $\mathbb{R}^{2,n}$ such that $p(y) = [p]$, $p(y') = [p']$. Since y and y' are not causally related, according to Proposition 5.10, the quantity $\langle p | p' \rangle$ is negative: we can select p, p' such that this quantity is actually -2. Then, there is a basis of $\mathbb{R}^{2,n}$ for which the quadratic form Q_n still admits the expression $-u^2 - v^2 + x_1^2 + x_2^2 + \cdots + x_n^2$ and for which the coordinates of p and p' are, respectively $(1, 0, 1, 0, 0, \ldots, 0)$ and $(1, 0, -1, 0, 0, \ldots, 0)$. For this choice of coordinates, when we select the section $\sigma : \mathrm{Ein}_n \to \mathbb{R}^{2,n}$ taking value in the sphere $u^2 + v^2 + x_1^2 + \cdots + x_n^2 = 2$, we obtain an identification $\mathrm{Ein}_n \approx \mathbb{S}^{n-1} \times \mathbb{R}$ where the conformal structure is still represented by $ds^2 - dt^2$ but now with the additional requirement that p, p' have coordinates $(x, 0)$, $(-x, 0)$.

5.3 Achronality in AdS

Keeping in mind the identification of $\widetilde{\mathrm{AdS}}$ with $\mathbb{D}^2 \times \mathbb{R} \subset \mathbb{S}^2 \times \mathbb{R} \approx \widehat{\mathrm{Ein}}_3$:

Lemma 5.16. *Two points in $\widetilde{AdS}$ are (strictly) causally related if and only if they are (strictly) causally related in $\widehat{Ein}_3$.*

Proof. Let $\tilde{x}$, $\tilde{y}$ be two points in $\widetilde{\mathrm{AdS}}$. Clearly, if they are (strictly) causally related in $\widetilde{\mathrm{AdS}}$, they are (strictly) causally related in $\widehat{\mathrm{Ein}}_3$. The inverse follows directly from Lemma 5.1. □

According to Remark 2.7 and Corollary 5.4:

Lemma 5.17. *$\widetilde{AdS}$ is strongly causal.*

Corollaries 5.2 and 5.3 imply:

Lemma 5.18. *Achronal subsets of $\widetilde{AdS}$ are graphs of 1-Lipschitz functions $f : E \to \mathbb{R}$, where Λ_0 is a subset of $\mathbb{D}^2$. Such a subset is strictly achronal if and only if f is contracting.*

Lemma 5.19. *The intersection between the closure of an achronal subset of $\widetilde{AdS}$ and $\partial\widetilde{AdS}$ is an achronal subset of $\widehat{Ein}_2$.*

Define generic subsets of $\widetilde{\mathrm{AdS}}$ as subsets for which the intersection between their closure and $\partial\widetilde{\mathrm{AdS}}$ is a generic in $\widehat{\mathrm{Ein}}_2$. Lemma 5.6 now becomes:

Lemma 5.20. *Generic achronal subsets of $\widetilde{AdS}$ are contained in affine domains.*

Define (strictly) achronal domains of AdS as projections of (strictly) achronal domains of $\widetilde{\mathrm{AdS}}$. Proposition 5.12 now becomes:

Proposition 5.21. *A generic subset of AdS is (strictly) achronal if and only if it is contained in some affine domain and (strictly) achronal in every affine domain containing it.*

5.4 Causality relation between AdS and ∂AdS

Since Ein_2 is the boundary of AdS in Ein_3, we have the notion that points in AdS can be causally related to points in $\mathrm{Ein}_2 \approx \partial\mathrm{AdS}$. This notion can be easily understood by considering the identification between AdS and $\mathbb{D}^2 \times \mathbb{S}^1$: the boundary of AdS is then $\partial\mathbb{D}^2 \times \mathbb{S}^1 \approx \mathrm{Ein}_2$. Then, (x, θ) in $\mathbb{D}^2 \times \mathbb{S}^1$ is causally related to (y, θ') in $\partial\mathbb{D}^2 \times \mathbb{S}^1$ if and only if $d(x, y) \leq |\theta - \theta'|$.

The conformal completion $\mathcal{A} \cup \partial\mathcal{A}$ of AdS is naturally identified with the Klein completion $\mathbb{ADS} \cup \mathrm{Ein}_2$ (see Remark 4.7). It is also useful to understand the causality relation between points of AdS and points in ∂AdS, but when the last one is considered as the Klein boundary, not the conformal boundary:

Lemma 5.22. *Let x be an element of $\mathbb{ADS}$ and let y be an element of the Klein boundary Ein_2. Let $I =]x, y[$ be the shortest segment between x and y (observe that x and y are never opposite in $S(E)$). Then, x and y are causally related if and only if I is contained in $\mathbb{ADS}$. There are strictly causally related if and only if the projective line d containing I is not lightlike, i.e., is transverse to Ein_2. If d is tangent to Ein_2, then I is a lightlike geodesic: considered as elements of Ein_3, x and y are joined by a lightlike geodesic.*

Remark 5.23. In Remark 4.3, we have observed that Ein_3 is obtained by glueing conformally along their boundaries two copies of AdS. In Remark 3.1, we have seen that $S(E)$ can also be considered as the union of two copies of AdS, glued along their common Klein boundary. But there is a main difference here: $\mathbb{ADS}$ is the projection of $\{Q = -1\}$ equipped with the restriction of Q, whereas the complement in $S(E)$ of its closure is the projection of $\{Q = 1\}$ equipped with the restriction of $-Q$. Hence, the identification between the boundaries of these copies of AdS does not preserve the causality notion: it sends causal curves to achronal topological circles!

6 Dualities

Let E^* be the dual of E. The quadratic form Q defines a map $\flat : E \to E^*$ by $x^\flat(y) = Q(x, y)$. The image under $\flat$ of Q is a quadratic form Q^* on E^*. Let $S(E)$ and $S(E^*)$ be the associated half-projective spaces. The map $\flat$ induces a polarity $S(E) \to S(E^*)$ that we still denote by $\flat$. We denote by $\sharp$ the inverse map of $\flat$.

We have denoted by Ein_2 the projection of the null cone of Q in $S(E)$; the nullcone of Q^* is a dual copy Ein_2^* of Ein_2. Elements of Ein_2^* can be interpreted as lightcones in Ein_2; more precisely, $x^\flat$ is the lightcone emitted from x in $\mathbb{ADS}$.

Observe that $\flat$ and $\sharp$ respect the causality notion. In particular:

Lemma 6.1. *The image $\Lambda^\flat$ of a (strictly) achronal subset Λ of Ein_n by $\flat$ is (strictly) achronal.*

We will use another notion of duality, more traditional, completely independant from the notion discussed above: the duality of convex subsets of $S(E)$. We recall basic facts (cf. [17]):

A convex cone J of E is a convex subset stable by positive homotheties. It is *proper* if it is non-empty and its closure $\bar{J}$ does not contain a complete affine line. A convex subset C of $S(E)$ is the projection of a convex cone $J(C)$ of E. It is called proper if $J(C)$ is proper.

For any convex cone J, we define its dual by $J^* = \{\alpha \in E^* / \forall x \in \bar{J} \setminus \{0\} \ \ \alpha(x) < 0\}$. This provides a construction of dual convex $C^* \subset S(E^*)$ for any convex subset of $S(E)$ (which could be empty!).

Proposition 6.2. *A convex subset C has empty interior if and only if its dual C^* is not proper. If C is open and proper, then the same is true for C^*, and $C^{**} = C$.*

Recall that a support hyperplane to an open convex subset C is a projective hyperplane meeting the closure of C but not C itself.

Proposition 6.3. *Let C be a proper open convex subset of $S(E)$. The support hyperplanes of C are the projections in $S(E)$ of the boundary points of C^*. More precisely, if $[Ker\alpha]$ is a support hyperplane of C at $[x] \in \partial C$, then $[Ker x^*]$ is a support hyperplane of C^* at $[\alpha] \in \partial C^*$.*

7 Spacelike and non-timelike surfaces

The notion of non-timelike hypersurfaces in $\mathrm{Ein}_n \approx \mathbb{S}^{n-1} \times \mathbb{S}^1$ or $\mathrm{AdS} \approx \mathbb{D}^2 \times \mathbb{R}$ can be easily extended to the non-smooth case: define it as closed subsets which are locally the graphs of 1-Lipschitz maps from $\mathbb{S}^{n-1}$ into $\mathbb{S}^1$ or from $\mathbb{D}^2$ into $\mathbb{S}^1$. If moreover the Lipschitz maps are contracting, i.e., are functions f such that the equality $|f(x) - f(y)| = d(x,y)$ implies $x = y$, then the non-timelike surface is said *spacelike*. The same notions apply in the coverings $\widehat{\mathrm{Ein}}_n$ and $\widetilde{\mathrm{AdS}}$.

Since it is Lipschitz, f is differentiable almost everywhere. For any C^1-curve $c : [0,a] \to \mathbb{D}^2$ in the domain of definition of f, let $l(c)$ be the integral over $[0,a]$ of the square root of the AdS-norm of $(c'(t), D_{c(t)}f(c'(t)))$. Define then the distance between $(x, f(x))$ and $(y, f(y))$ as the infimum of the $l(c)$. This procedure endows the spacelike surface S with a distance. Of course, this construction applies more generally to spacelike hypersurfaces in any lorentzian space. For more details, see [8].

Achronal (resp. strictly achronal) hypersurfaces are nontimelike (resp. spacelike) hypersurfaces, but the converse is not true. The main goal of this Section is to discuss under which additionnal hypothesis a nontimelike hypersurface is achronal.

7.1 The redshift phenomenom

Let M be a lorentzian manifold. For any timelike tangent vector v at a point x of M, the orthogonal projection in T_xM on the orthogonal hyperplane $v^\perp$ increases the length of spacelike vectors: this fact is at the origin of the so-called "redshift phenomenom". It implies the well-known "twins paradox". There is another powerfull consequence, already observed in [30] (see also [26, 27]).

Assume the existence of one-parameter group Φ^t of isometries of M such that the orbits of Φ^t are the fibers of a fibration π: $M \to Q$. The base space Q can be equipped with a Riemannian metric as follows: for any point x of Q and any tangent vector v of Q at x, define the norm of v as the norm in M of any vector orthogonal to the fiber $\pi^{-1}(x)$ and projecting on v by the differential $d\pi$. This is well defined since the Φ^t are isometries.

Recall that we equipped spacelike surfaces with a distance function:

Lemma 7.1. *Let $\varphi : S \to M$ be an isometric immersion of a Riemannian manifold such that $\varphi(S)$ is an immersed locally acausal hypersurface. The composition $\pi \circ f : S \to Q$ is distance increasing.*

Proof. When φ is C^1, the lemma follows from the observation above. The general case is a limit case. Details are left to the reader. □

When M is the anti-de Sitter space AdS, we can take as one-parameter subgroup the subgroup SO(2) of $\mathrm{SO}_0(2,2)$ acting in E on the (u, v) coordinates and fixing the coordinates x_1, x_2 (recall that we actually used this subgroup to define the time-orientation of AdS). The quotient space of this timelike action equipped as above with a Riemannian metric is isometric to the hyperbolic space: therefore, spacelike hypersurfaces in anti-de Sitter space correspond to distance increasing maps into $\mathbb{H}^2$.

Proposition 7.2 (Lemma 6 in [30])**.** *Let S be a complete Riemannian surface and let $\varphi : S \to AdS$ be an isometric immersion. Then, φ is an embedding and $\varphi(S)$ is the graph of some contracting map $\mathbb{H}^2 \to \mathbb{S}^1$.*

Every timelike geodesic in AdS is the fiber of some fibration π as above. It follows that under the hypothesis of Proposition 7.2, $\varphi(S)$ meets every timelike geodesic in one and only one point.

7.2 Proper non-timelike surfaces

The conformal coordinates enable to extend Proposition 7.2 to the non-timelike case. By *proper* non-timelike surface φ: $S \to$ AdS, we mean an immersion such that the immersion φ is proper, i.e., that the preimage of a compact domain is compact.

Proposition 7.3. *Let φ: $S \to AdS$ be a proper non-timelike surface without boundary. Then, φ is an embedding and $\varphi(S)$ is the graph of some 1-Lipschitz map $\mathbb{D}^2 \to \mathbb{S}^1$.*

Proof. The projection $\pi : \mathrm{Ein}_3 \to \mathbb{S}^2$ induces a projection $\pi_a : \mathcal{A} \to \mathbb{D}^2$ (cf. Remark 4.5). We claim that $\pi_a \circ \varphi$ is covering map. We will justify it by proving that it has the path lifting property. For this purpose, it is enough to prove it for paths in the open hemisphere $\mathbb{D}^2$ which are segments of geodesics of $\mathbb{D}^2 \subset \mathbb{S}^2$: let $[a, b]$ be such a geodesic segment and such that $a = \pi(\varphi(x))$ for some element x of S. Since $\pi \circ \varphi$ is open, there is a section σ of $\pi \circ \varphi$ defined over a subinterval $[a, c[\subset [a, b]$. The point is that $p \circ \varphi \circ \sigma : [a, c[\subset [a, b] \to \mathbb{S}^1$ is then a 1-Lipschitz map (where p is the projection on the second factor). Therefore, it can be continuously extended over c. The properness of φ then implies that σ can be extended over c. Hence, $c = b$ and σ can be extended over all $[a, b]$: $\pi \circ \varphi$ has the path lifting property.

We thus know that $\pi \circ \varphi$ is a covering map over $\mathbb{D}^2$, hence a homeomorphism. It follows that $\varphi(S)$ is the graph of a 1-Lipschitz map $\mathbb{D}^2 \to \mathbb{S}^1$ and that φ is an embedding. □

From now, we will assume that S fits *inside* AdS, i.e., that $\varphi : S \to \mathrm{AdS}$ is an inclusion map. The additional advantage of our point of view is that the proof of Lemma 7 of [30], which was a delicate matter in this paper, now appears as completely obvious:

Proposition 7.4. *A proper non-timelike hypersurface S in AdS extends continuously in $\mathcal{A} \cup \partial\mathcal{A} \subset Ein_3$ as a closed topological disk, whose boundary ∂S is a topological non-timelike circle in $\partial\mathcal{A} \approx Ein_2$.*

Proof. Any 1-Lipschitz map from $\mathbb{D}^2$ into $\mathbb{S}^1$ extends continuously as a 1-Lipschitz map on the closure of $\mathbb{D}^2$ in $\mathbb{S}^2$. □

Consider now the universal covering $\widetilde{\mathrm{AdS}}$. Select any connected component $\tilde{S}$ of the preimage of S in $\widetilde{\mathrm{AdS}}$ by the covering map. Clearly, $\tilde{S}$ is the graph of a 1-Lipschitz maps from $\mathbb{D}^2$ into $\mathbb{R}$. Moreover,

Corollary 7.5. *A proper non-timelike surface in $\widetilde{AdS}$ meets every timelike geodesic in one and only one point.*

Corollary 7.6. *Proper non-timelike surfaces in AdS are achronal subsets of AdS.*

Observe also:

Lemma 7.7. *Let S be a proper spacelike hypersurface in $\widetilde{S}$. Then, the closure $\overline{S}$ of S in $\widetilde{AdS} \cup \partial\widetilde{AdS}$ is not necessarily strictly achronal, but if two points in this closure are causally related, then they both belong to $\partial\widetilde{AdS}$. In particular, no point of S is causally related to a point of $\overline{S} \cap \partial\widetilde{AdS}$.*

Sketch of proof. The closure $\overline{S}$ is the graph of a 1-Lipschitz function $f\colon \overline{\mathbb{D}}^2 \to \mathbb{R}$ such that the restriction of f to $\mathbb{D}^2$ is contracting. If a point $(x, f(x))$ of S is causally related to a point $(y, f(y))$ of $\overline{S} \cap \partial\widetilde{\mathrm{AdS}}$, and if $|f(y) - f(x)| = d(x, y)$, then for any z on the geodesic $[x, y]$ in $\overline{\mathbb{D}}^2$ the equality $|f(z) - f(x)| = d(x, z)$ must hold. This is a contradiction since S is assumed acausal. □

Pure lightlike non-timelike surfaces in $\mathbb{ADS} \approx \mathrm{AdS}$ are easy to describe:

Corollary 7.8. *Pure lightlike surfaces in $\mathbb{ADS}$ are connected components of intersections between $\mathbb{ADS}$ and projections in $S(E)$ of lightlike hyperplanes of E.*

7.3 Embeddings in $\mathbb{H}^2 \times \mathbb{H}^2$

Recall the projectivized timelike tangent bundle $\mathcal{T}$ (cf. Definition 3.4); more precisely, the connected component $\mathcal{T}^+$ corresponding to future oriented timelike vectors.

Definition 7.9. The Gauss flow is the flow G^t on $\mathcal{T}^+$ defined by:

$$G^t(x, y) = (x\cos t + y\sin t, -x\sin t + y\cos t)$$

This flow commutes with the $O_0(2, 2)$-action. Moreover, it is easy to check that G^t is isometric. The Killing vector field Z generating G^t is easy to describe (see Section 3.6 for the convention on tangent vectors to $\mathcal{T}^+$):

$$Z(x, y) = (y, -x)$$

Let Q_G be the orbit space of G^t and let $\pi_G : \mathcal{T}^+ \to Q_G$ be the quotient map. We equip Q_G with a Riemannian metric as discussed in Section 7.1: the norm of a tangent vector ζ to Q_G is the norm of any tangent w to $\mathcal{T}^+$ orthogonal to Z and such that $d\pi_G(w) = \zeta$.

Remark 7.10. Let's be more precise: let $w = (u, v)$ be a tangent vector at $(x, y) \in \mathcal{T}^+$. The norm of $\zeta = d\pi_G(w)$ is the norm of $w + \lambda Z$, where λ is the unique real number such that $w + \lambda Z$ is orthogonal to Z. A straightforward computation shows $\lambda = \langle y \mid u\rangle = -\langle x \mid v\rangle$. Hence, the norm of ζ is $\frac{1}{4}(\mid u \mid^2 + \mid v \mid^2) + \frac{1}{2}\langle x \mid v\rangle^2$.

Proposition 7.11. *The Riemannian orbit space Q_G is homothetic to the Riemannian product $\mathbb{H}^2 \times \mathbb{H}^2$ of two copies of the hyperbolic plane.*

Proof. Identify AdS with $G = \mathrm{SL}(2,\mathbb{R})$ and consider the upper-half plane model of $\mathbb{H}^2$. Let i denote the point $\sqrt{-1}$ in $\mathbb{H}^2$. Let x_0 be the identity matrice and y_0 be the matrice representing the rotation by angle $\pi/2$ around i. Observe that under the indentifications above, (x_0, y_0) belongs to $\mathcal{T}^+ \subset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Define a $G \times G$-equivariant map $F : \mathcal{T}^+ \to \mathbb{H}^2 \times \mathbb{H}^2$ as follows: if $(x, y) = (g_L x_0 g_R^{-1}, g_L y_0 g_R^{-1})$, then $F(x, y) = (g_L i, g_R i)$. Observe that it is well-defined: indeed, if (g_L, g_R) fixes (x_0, y_0), then $g_L = g_R$ commutes with y_0: $g_L = g_R$ preserves i in $\mathbb{H}^2$. Moreover, the preimage of (i, i) is precisely the G^t-orbit of (x_0, y_0): it follows that F induces a homeomorphism between Q_G and $\mathbb{H}^2 \times \mathbb{H}^2$. The only remaining point to check is that F is a homothety. Since it is equivariant, we just have to consider the differential of f at (x_0, y_0). The computation can be performed as follows: let A be an element of the Lie algebra $\mathcal{G}$:

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

Using Lemma 7.10, we obtain that the norm in $\mathcal{T}^+$ of the tangent vector to x_0 of the curve $t \mapsto (\exp(tA)x_0, \exp(tA)y_0)$ is $\frac{\alpha^2}{2} + \frac{1}{8}(\beta+\gamma)^2$, whereas the norm in $\mathbb{H}^2 \times \mathbb{H}^2$ of the tangent vector at (i, i) of the image curve $t \mapsto (\exp(tA)i, i)$ is $(\beta+\gamma)^2 + 4\alpha^2$. It follows that the restriction of F to every left G_L-orbit is an isometry on the image $\mathbb{H}^2 \times \{*\}$ with the metric divided by $2\sqrt{2}$.

A similar calculus holds for curves $t \mapsto (x_0 \exp(-tA), y_0 \exp(-tA))$, proving that the restriction of F to every G_R-orbit is an homothety of factor $8^{-1/2}$ on $\{*\} \times \mathbb{H}^2$. The proposition follows from the fact that in $\mathcal{T}^+$, G_L-orbits are orthogonal to G_R-orbits and that every $\mathbb{H}^2 \times \{*\}$ are orthogonal to every $\{*\} \times \mathbb{H}^2$. □

Consider a C^1 embedded spacelike surface $S \subset$ AdS. For any x in S, let $n(x)$ be the unique future-oriented unit timelike vector normal to S at x: $(x, n(x))$ is an element of $PT^+_{-1}\mathrm{AdS} \approx \mathcal{T}^+$. In other words, the embedding of S in AdS lifts to an embedding $n \colon S \to \mathcal{T}^+$.

Definition 7.12. The Gauss map of S is $n \colon S \to \mathcal{T}^+$.

Lemma 7.13. *The image of the Gauss map is a topological spacelike surface in $\mathcal{T}^+$. Moreover, the restriction of π_G to the image of the Gauss map endowed with the induced metric is an isometry.*

Proof. Consider first the case where f is C^2. Then, the Gauss map is C^1. Let (u,v) be a tangent vector to the image of n. By definition of the Gauss map, the tangent vector u to S satisfies: $\langle y \mid u\rangle = 0$. Hence, $\langle x \mid v\rangle = -\langle y \mid u\rangle = 0$: u and v both belong to the spacelike 2-plane $x^\perp \cap y^\perp$: the sum of their norms is positive. These identities mean also that (u,v) is orthogonal to the Killing vector field $Z(x,y)$. It follows that the restriction of π_G to the image is an isometry. The C^1-case is a limit case: any C^1 spacelike surface can be C^1-approximated by a C^2 spacelike surface. It follows that even in the C^1-case, the image of the Gauss map is locally achronal. Observe that a locally achronal surface which is not locally acausal must contain a lightlike geodesic segment: we can apply the argument above, leading to a contradiction. It follows that the image of the Gauss map is a topological spacelike surface.

The distance between two points in the image of the Gauss map is the infimum of the length of Lipschitz curves joining the two points. Let $c : t \mapsto (x(t), y(t))$ be such a Lipschitz curve. It is differentiable almost everywhere, with derivative $(x'(t), y'(t))$. The tangent vector $x'(t)$, where it is defined, is orthogonal to $y(t)$. The derivation of the identity $\langle x(t) \mid y(t)\rangle = 0$ then implies that $\langle x(t) \mid y'(t)\rangle = 0$. It follows that $(x'(t), y'(t))$ is orthogonal to $Z(x(t), y(t))$ almost everywhere. Hence the length of c is equal to the length of its projection in Q_G. The restriction of π_G to the image of the Gauss map is thus an isometry. □

Remark 7.14. The Gauss map $n : S \to \mathcal{T}^+$ in general is not isometric! Actually, the metric along the image of n involves the second fundamental form of S. Observe that n is isometric if and only if S is totally geodesic and that n is conformal if and only if S is totally umbilic.

Remark 7.15. Our choice of terminology is justified by the following observation: if S_t is the image of S under the Gauss flow in the usual meaning, then $n(S_t)$ is the image of $n(S)$ by the Gauss flow G^t we have defined above.

This observation extends to a much less regular situation: the case where S is maybe non C^1, but *convex.* This notion is meaningful, due to the local real projective structure of AdS $\approx \mathbb{ADS}$:

Definition 7.16. An embedded topological spacelike surface S in AdS is future-convex if any point x in S admits a geodesically convex neighborhood U in *AdS* such that for any y in $S \cap U$, the geodesic segment $[x,y]$ is contained in the causal future of $S \cap U$ relatively to U.

Remark 7.17. Observe that geodesic segments in U are restrictions of projective lines of $\mathbb{ADS}$. With the local description of spacelike subsets as graphs of functions, it follows that S is future-convex if and only if it is convex in $\mathbb{ADS} \subset S(E)$ in the usual meaning and that it is "curved in the future direction".

Definition 7.18. Let S be a future-convex spacelike surface in AdS. The Gauss graph of S is the set of pairs $(x, y) \in \mathcal{T}^+$ such that:

– x belongs to S,
– there is a neighborhood U of x in AdS such that for every for any x' in $U \cap S$ the scalar product $\langle x' \mid y \rangle$ is nonpositive. In other words, the connected component P of y^* containing x is a support hyperplane of S, such that $S \cap U$ is contained in the causal future of $P \cap U$.

Remark 7.19. When S is C^1, the Gauss graph is the graph of the Gauss map as defined in Definition 7.12.

In the sequel, we assume the reader acquainted with the familiar notion of convex surfaces in $P(E)$.

Proposition 7.20. *The Gauss graph of a future-convex spacelike surface is a locally embedded topological spacelike surface in $\mathcal{T}^+$.*

Sketch of proof. Let (x_0, y_0) be an element of the Gauss graph $\mathcal{N}(S)$ of a future-convex spacelike surface S. There is a neighborhood U of x_0 such that $S \cap U$ is contained in the boundary of a proper compact convex subset C of $S(E)$. Then, $\mathcal{N}(S)$ is contained in the set $\mathcal{C}$ of pairs (x, y), where x belongs to ∂C, $y^\flat$ belongs to the boundary of the dual convex $C^* \subset S(E^*)$ and $y^\flat$ is a support hyperplane to C at x. The set $\mathcal{C}$ is notoriously a topological surface and it should be clear to the reader that $\mathcal{N}(S)$ contains a neighborhood of (x_0, y_0) in $\mathcal{C}$. It follows that $\mathcal{N}(S)$ is a topological surface near (x_0, y_0).

The local achronality of $\mathcal{N}(S)$ follows from the local achronality of S and the fact that if y is a point near y_0 in the future of y_0, then $\langle x_0 \mid y \rangle$ is positive, which is a contradiction since $\langle y \mid x \rangle$ should be nonpositive for every x in S. □

Hence, the Gauss graph admits a natural distance (recall the definition in the beginning of this Section). Once more, we consider the general case as a "limit case"of the regular one, leaving to the reader the proof of the following lemma:

Lemma 7.21. *The restriction of π_G to $\mathcal{N}(S)$ is an isometry.*

The distance in $\mathcal{N}(S)$ is evaluated by the computation of the "length" of Lipschitz curves. We will also need the following fact:

Lemma 7.22. *Let $c : [0, a] \to \mathcal{N}(S)$ be a Lipschitz curve. It is differentiable almost everywhere. Any tangent vector $c'(t) = (u, v)$ satisfies:*

$$\langle u \mid v \rangle \geq 0$$

Sketch of proof. Once more, we only consider the regular case: we assume that S is C^2 and that the curve $(x(t), y(t))$ has tangent vectors $(u(t), v(t)) = (x'(t), y'(t))$. The derivation of $\langle y \mid x' \rangle = 0$ implies $\langle x' \mid y' \rangle = -\langle y \mid x'' \rangle$. Since S is future oriented, the second derivative $x''(t)$ must point towards the future of S, hence, in the future of the support hyperplane at x contained in y^*. The negativity of $\langle y \mid x'' \rangle$ follows.

The general case is similar, once observed that convex surfaces are C^2 almost everywhere. □

8 Cauchy developments

In this Section, we study Cauchy development in AdS of achronal subsets of AdS. It turns out that Cauchy developments can be defined as *invisible domains* from achronal subsets of ∂AdS. We start with the most familiar notion of Cauchy development of spacelike surfaces (domain of dependance in [30]) and then extend to the most general context.

8.1 Cauchy developments of spacelike surfaces

We consider in this section a proper spacelike hypersurface $\tilde{S}$ in $\widetilde{\mathrm{AdS}}$. Actually, all the results apply if $\tilde{S}$ is more generally any strictly achronal surface.

According to Proposition 7.4, the boundary $\partial\tilde{S}$ of $\tilde{S}$ in $\partial\widetilde{\mathrm{AdS}} \approx \widetilde{\mathrm{Ein}}_2$ is an achronal topological circle. We denote by S, ∂S the projections in AdS, Ein_2.

Lemma 8.1. *The past development $P(\tilde{S})$ is the set of points x in $\widetilde{AdS}$ such that every lightlike geodesic containing x meets $\tilde{S}$ in its future.*

Of course, the analogous property for $F(\tilde{S})$ is true.

Proof. Assume that every lightlike geodesic containing x meets $\tilde{S}$ in its future. According to corollary 7.5 every timelike geodesic containing x intersect $\tilde{S}$ in one and only one point. The set of timelike geodesics containing x coincide with the space of future oriented timelike elements of $T_x\widetilde{\mathrm{AdS}}$, i.e., a copy of $\mathbb{H}^2$. In particular, it is connected, and the boundary of this space is the space of lightlike geodesics containing x. It follows that every timelike geodesic containing x meets $\tilde{S}$ in the future of x. The union of the non-spacelike geodesic segments $[x, y]$ with y in $\tilde{S}$ is a topological disk B'. Any properly embedded causal path starting from x cannot escape from B' through L: it must therefore intersect $B' \subset \tilde{S}$. □

We pursue our investigation in the Klein model $\mathbb{ADS}$. According to Lemma 5.20, since it is generic, $\tilde{S}$ is contained in some affine domain. Therefore, it projects injectively in AdS as a strictly achronal surface S, contained in some affine domain.

Definition 8.2. We define $T(S)$ as the set of points x in $\mathbb{ADS}$ such that the affine domain $A(x)$ contains S.

Lemma 8.3. *$T(S)$ is a neighborhood of S.*

Proof. By definition, $T(S)$ is the set of elements x of $\mathbb{ADS}$ such that $\langle x|y\rangle$ is negative for every y in S. In other words, in the terminology of Section 6, $T(S)$ is the intersection between $\mathbb{ADS}$ and the image under $\sharp$ of the dual of the convex hull $\mathrm{Conv}(S)$ of S in $S(E)$.

Consider an element x_0 of S. It is the projection of some vector v_0 in E. Then, S is the projection of some subset $S(v_0)$ in P_0 (see Section 5.2). According to Proposition 5.10, since S is achronal, for every x, y in $S(v_0)$, the scalar product $\langle x \mid y\rangle$ is negative. It follows that $T(S)$ contains S.

Moreover, an element x of $S(v_0)$ does not project to a point in the interior of $T(S)$ if and only if there is a sequence of points x_n in $S(v_0)$ such that $\langle x \mid x_n\rangle$ tends to 0. Up to some subsequence, the projections in x_n in $\mathbb{ADS}$ converge to some element $\bar{x}$ of $\mathbb{ADS} \cup \partial\mathbb{ADS}$. Then, $\bar{x}$ would be a point in the closure of S in $\mathrm{AdS} \cup \partial\mathrm{AdS}$ causally related to the point x in S. It contradicts Lemma 7.7. □

Let $T_0(S)$ be the interior of $T(S)$. It contains S. Select any element x_0 in $T_0(S)$. Then, S is contained in the affine domain $A_0 = A(x_0)$. Actually, the fact that x_0 belongs to the interior of $T(S)$ means that S is contained in a compact domain of the affine patch $V_0 = V(x_0)$. Hence, the closure $\overline{S}$ in $\mathrm{AdS} \cup \partial\mathrm{AdS}$ is a closed topological disc in $V(x_0)$, with boundary ∂S contained in the one-sheet hyperboloid $\mathrm{Ein}_2 \cap V_0$.

Proposition 8.4. *The restriction of p to the Cauchy development $\mathcal{C}(\tilde{S})$ is injective, with image $T_0(S)$.*

Proof. First observe that $T_0(S)$ is contained in every affine domain $A = A(x)$, with x in S. Let $\widetilde{A}$ be the affine domain in $\widetilde{\mathrm{AdS}}$ containing $\tilde{S}$ such that $p(\widetilde{A}) = A$.

Let $\tilde{x}$ be an element of $P(\tilde{S})$. Consider a conformal parametrization of $\widetilde{\mathrm{AdS}}$ by $\mathbb{D}^2 \times \mathbb{R}$ such that $\tilde{x}$ has coordinates $(x_0, 0)$, where x_0 is the north pole, i.e., is the unique point of $\mathbb{D}^2$ at distance $\pi/2$ of $\partial\mathbb{D}^2$. In these coordinates, $\tilde{S}$ is the graph of some contracting function $f : \mathbb{D}^2 \to \mathbb{R}$. Since $\tilde{x}$ is in the past of $\tilde{S}$, we have $f(x_0) > 0$.

Every future oriented lightlike geodesic starting from $\tilde{x}$ intersect $\tilde{S}$: it follows that there is an open topological disc B in $\mathbb{D}^2$ containing x_0 and such that $f(x) = d(x_0, x)$ for every x in ∂B. Since f is contracting and since any point in $\overline{\mathbb{D}}^2$ is at distance strictly less than $\pi/2$ of some point in ∂B, it follows that the extension $\bar{f}$ of f over $\overline{\mathbb{D}}^2$ takes value in $]-\pi/2, \pi/2[$. Since $\overline{\mathbb{D}}^2$ is compact, $\bar{f}$ takes actually value in some closed intervall $[-\pi/2+\epsilon, \pi/2-\epsilon]$. It follows that $p(\tilde{x})$ belongs to $T_0(S)$. In other words, the image $p(P(\tilde{S}))$ is contained in $T_0(S)$. Hence, since $P(\tilde{S})$ is connected and since A contains $T_0(S)$, $P(\tilde{S})$ is contained in $\widetilde{A}$.

Applying a similar argument to $F(\tilde{S})$, we obtain that $C(\tilde{S})$ is contained in $\widetilde{A}$ and that $p(C(\tilde{S})) \subset T_0(S)$.

Assume now that $\tilde{x}$ is an element of $\widetilde{A}$ such that $p(\tilde{x})$ belongs to $T_0(S)$ and select once more a conformal parametrization of $\widetilde{\mathrm{AdS}}$ by $\mathbb{D}^2 \times \mathbb{R}$ such that $\tilde{x}$ has coordinates $(x_0, 0)$, where x_0 is the north pole. The affine domain associated to $\tilde{x}$ is then the open domain $\{(y, \theta)/ \mid \theta \mid < \pi/2\}$. Hence, since $\tilde{S}$ has to be contained in this affine domain, the map $f : \mathbb{D}^2 \to \mathbb{R}$ admitting $\tilde{S}$ as graph takes value in $]-\pi/2, \pi/2[$. More precisely, since $p(\tilde{x})$ belongs to the interior of $T(S)$, f takes value in some interval $[-\pi/2+\epsilon, \pi/2-\epsilon]$.

If $f(x_0) = 0$, $\tilde{x}$ belongs to $\tilde{S} \subset C(\tilde{S})$. Assume $f(\tilde{x}_0) > 0$, i.e., assume that $\tilde{x}$ is in the past of $\tilde{S}$. Define $g(y) = f(y) - d(y, x_0)$. This function is continuous, positive on x_0 and negative near $\partial\mathbb{D}^2$. Hence, any geodesic ray in $\mathbb{D}^2$ starting from x_0 admits some point where g is 0. It means that any future oriented lightlike geodesic starting form $\tilde{x}$ intersect $\tilde{S}$: $\tilde{x}$ belongs to $P(\tilde{S})$.

A similar argument proves that in the remaining case, i.e., when $\tilde{x}$ belongs to the future of $\tilde{S}$, then it belongs to $F(\tilde{S})$. The proposition follows. □

Corollary 8.5. *$T_0(S)$ is globally hyperbolic, with Cauchy surface S.*

Corollary 8.6. *The Cauchy development $\mathcal{C}(\tilde{S})$ is contained in a de Sitter domain.*

8.2 The Cauchy development as the invisible domain from ∂S

Definition 8.7. For any point x of Ein_2, let T_x be the projective hypersurface in $S(E)$ containing x and tangent to Ein_2. For any pair of points x, y of Ein_2, let E_{xy} be the open half-space in $S(E)$ bounded by T_x and T_y and containing the segment $]x, y[$ contained in $\mathbb{ADS}$.

Definition 8.8. Let $E(\partial S)$ be the intersection of all E_{xy} when (x, y) describes $\partial S \times \partial S$ minus the diagonal.

Remark 8.9. The presentation above is suitable for getting some geometrical vision of $E(\partial S)$. It is more relevant for the proofs below to consider the following equivalent definition: recall that ∂S is the projection in $\mathbb{ADS}$ of a compact subset $\partial S(v_0)$ in $\{Q = 0\} \cap P(v_0)$, where $P(v_0)$ is some affine hyperplane in E. Then, $E(\partial S)$ is the projection of the set of elements v of E satisfying $\langle v \mid x\rangle < 0$ for every x in $\partial S(v_0)$. In other words: *$E(\partial S)$ is the dual of the convex hull in $S(E^*)$ of $\partial S^\flat$.* We write: $E(\partial S) = \mathrm{Conv}(\partial S^\flat)^*$. Alternatively: $E(\partial S) = (\mathrm{Conv}(\partial S)^*)^\sharp$. Observe that the compactness of $\partial S(v_0)$ implies that $E(\partial S)$ is open.

Remark 8.10. Using the conformal model $\widetilde{\mathrm{AdS}} \approx \mathbb{D}^2 \times \mathbb{R}$, we obtain another equivalent definition: let $f : \partial\mathbb{D}^2 \to \mathbb{R}$ be the 1-Lipschitz map admitting as graph the topological circle $\partial\tilde{S}$. Define $f_\pm : \mathbb{D}^2 \to \mathbb{R}$ by:

- $f_-(x)$ is the supremum of $f(y) - d(x, y)$ when y describes $\mathbb{S}^1$,
- $f_+(x)$ is the infimum of $f(y) + d(x, y)$ when y describes $\mathbb{S}^1$.

Then, it is easy to check that $f_\pm$ are both 1-Lipschitz extensions of f. Moreover, it follows easily from the arguments used in the proof of Proposition 8.4 that $E(\partial S) \cap \mathbb{ADS}$ is the projection of the domain $E(\partial\tilde{S}) = \{(x, \theta) \in \mathbb{D}^2 \times \mathbb{R}/f_-(x) < \theta < f_+(x)\}$. The advantage of our Definition 8.8 is to explicit the *convex* character of $E(\partial S)$.

Remark 8.11. The definition in Remark 8.10 can be interpreted in the following way (recall Section 5.4): *$E(\partial S)$ is the set of points of AdS which are not causally related to any element of ∂S.* Hence, it is appropriate to consider $E(\partial S)$ as the *invisible domain from ∂S.*

Remark 8.12. The extensions $f_\pm$ above can be defined in any metric space X for any 1-Lipschitz map f defined over a closed subset $Y \subset X$: they are still 1-Lipschitz and any extension F of f satisfy $f_- \leq g \leq f_+$. In general, f_- and f_+ can coincide on some closed subset of $X \setminus Y$: the points belonging to some minimizing geodesic joining two points x, y of Y such that $|f(x) - f(y)| = d(x, y)$. In the case, where X is the closed unit disc in the euclidean plane and Y its boundary, the set $f_+ = f_-$ is a lamination.

In the particular case, we consider here, where X is a hemisphere and Y its boundary, Y is totally geodesic and extremities in Y of minimizing geodesics are opposite points of the boundary sphere. Moreover, minimizing geodesics joining two given points form a foliation of the hemisphere. We recover easily from this that when f correspond to the generic topological sphere ∂S, then the associated $E(\partial S) \cap \mathbb{ADS}$ is open, i.e., $f_- < f_+$.

The definition in Remark 8.10 implies easily:

Lemma 8.13. *The convex $E(\partial S)$ is contained in $\mathbb{ADS}$. The intersection between its closure and Ein_2 is ∂S.*

Proof. Indeed, since $f_- = f_+$ over $\partial\mathbb{D}^2$, the intersection between $\partial\mathbb{D}^2 \times \mathbb{R}$ and the closure $\{(x, \theta) \in \mathbb{D}^2 \times \mathbb{R} / f_-(x) \leq \theta \leq f_+(x)\}$ of $E(\partial\tilde{S})$ in $\mathbb{D}^2 \times \mathbb{R}$ is simply the graph of f. The lemma follows easily, since $E(\partial S)$ is convex. □

In the same spirit:

Lemma 8.14. *Any proper nontimelike topological hypersurface contained in $\mathbb{ADS}$ and containing x, y in its closure is necessarily contained in the closure of E_{xy}.*

$E(\partial S)$ is thus a convex subset of $S(E)$ containing S: in particular, it is not empty! Actually, using the definition in Remark 8.9 and since ∂S is in the closure of S, it is straightforward to show the inclusion $T_0(S) \subset E(\partial S)$. The inverse inclusion is true:

Proposition 8.15. *The invisible domain $E(\partial S)$ is equal to the Cauchy development $T_0(S)$.*

Proof. Let x be a point in $\partial T_0(S) \cap E(\partial S)$. Select as usual a conformal parametrization of the affine domain $A(x)$ by $\mathbb{D}^2\times]-\pi/2, +\pi/2[$, such that x has coordinates $(x_0, 0)$, where x_0 is the North pole. The surface S is the graph of a 1-Lipschitz function $f : \mathbb{D}^2 \to \mathbb{R}$. Since x belongs to $E(\partial S)$, the restriction of f to $\partial\mathbb{D}^2$ takes value in $]-\pi/2, +\pi/2[$. On the other hand, since x belongs to $\partial T_0(S)$, the map f takes value in $[-\pi/2, +\pi/2]$, but there is some element y of $\overline{\mathbb{D}}^2$ such that $f(y) = \pm\pi/2$; let's say, $+\pi/2$. Observe that y cannot belong to $\partial\mathbb{D}^2$. Hence, $f(x_0) \geq \pi/2 - d(x_0, y) > 0$: x is in the past of S. Consider the function $g : z \mapsto f(z) - d(x_0, z)$: it is negative on $\partial\mathbb{D}^2$ and $g(x_0)$ is positive: it follows, as in the proof of Proposition 8.4 that x belongs to the past development $P(S)$. Similarly, if $f(y) = -\pi/2$, we infer that x belongs to the future development of S. This is a contradiction since $T_0(S)$ is the Cauchy development of S. □

According to Corollary 8.6, $T_0(S) = E(\partial S)$ is contained in a de Sitter domain. We can now say more:

Lemma 8.16. *If ∂S is not a round circle, then the closure of $E(\partial S)$ is contained in a de Sitter domain.*

Proof. If ∂S is not a round circle, then the interior of $\mathrm{Conv}(\partial S)$ is not empty. Thus, the same is true for $\mathrm{Conv}(\partial S^\flat)$. Points in the interior of $\mathrm{Conv}(\partial S^\flat)$ correspond to an open set of flat spheres avoiding $E(\partial S)$. □

8.3 Support hyperplanes

Lemma 8.17. *The boundary of $Conv(\partial S)$ (resp. $E(\partial S)$) in $S(E)$ is the set of points dual to support hyperplanes to the closure of $E(\partial S)$ (resp. $Conv(\partial S)$) in $S(E)$.*

Proof. Corollary of remark 8.9, proposition 6.3, and lemma 8.13. □

We can be slightly more precise. When ∂S is not contained in a round circle, the complement of ∂S in the boundary $\partial\mathrm{Conv}(\partial S)$ is contained in $\mathbb{ADS}$ and admits two connected components.

Definition 8.18. The future (respectively past) convex boundary $\partial^+ C(\partial S)$ (respectively $\partial^- C(\partial S)$) is the connected component of $\partial\mathrm{Conv}(\partial S)\setminus\partial S$ such that the interior of $\mathrm{Conv}(\partial S)$ is contained in the past (resp. future) of $\partial^+ C(\partial S)$ (resp. $\partial^- C(\partial S)$).

Observe that in the flat case, $\partial^+ C(\partial S) = \partial^- C(\partial S)$

Similarly, the complement of ∂S in $\partial E(\partial S)$ has two connected components:

Definition 8.19. The future (respectively past) boundary $\partial^+ E(\partial S)$ (resp. $\partial^- E(\partial S)$) is the connected component of $\partial E(\partial S)\setminus\partial S$ such that the interior of $E(\partial S)$ is contained in the past (resp. future) of $\partial^+ E(\partial S)$ (resp. $\partial^- E(\partial S)$).

Then:

Proposition 8.20. *$\partial^+ C(\partial S)$ is the set of points dual to spacelike support hyperplanes to $E(\partial S)$ at a point of $\partial^- E(\partial S)$.*

8.4 Cosmological time

Proposition 8.21. *$E(\partial S)$ has regular cosmological time.*

Proof. According to Remark 8.10:

$$E(\partial\tilde{S}) = \{(x, \theta) \in \mathbb{D}^2 \times \mathbb{R} / f_-(x) < \theta < f_+(x)\}$$

Since it is contained in a de Sitter domain, we can assume that $f_\pm$ take value in $[-\pi/2, \pi/2]$. We denote by O the north pole of $\mathbb{D}^2$.

Let $p_0 = (x_0, \theta_0)$ be a point in $E(\partial\tilde{S})$. By the very definition of f_+, for x on the boundary $\partial\mathbb{D}^2$ we have $\theta_0 - d(x, x_0) < f(x)$. Hence, the map $x \to \theta_0 - d(x, x_0) - f_-(x)$ is negative on $\partial\mathbb{D}^2$. By compactness of $\partial\mathbb{D}^2$, there is a compact subset K of $\mathbb{D}^2$ such that each x satisfying $f_-(x) < \theta_0 - d(x, x_0)$ belongs to K. But the past $I^-(p_0)$ of p_0 in $E(\partial\tilde{S})$ is the set of points (x, θ) with $f_-(x) < \theta < \theta_0 - d(x, x_0)$: it follows that $I^-(p_0)$ is contained in $K \times [-\pi/2, \pi/2]$. In particular, the conformal factor $(1/\cos^2(d(O, x)))$ of the AdS metric versus $ds_0^2 - dt^2$ (cf. Remark 4.5) is uniformly bounded on $I^-(p_0)$ by a factor μ^2. It follows easily that the time length of causal curves contained in $I^-(p_0)$ is uniformly bounded by $\mu\pi$. Hence, the cosmological time τ on $E(\partial\tilde{S})$ has finite existence time.

Consider now an inextendible past oriented causal curve c in $E(\partial\tilde{S})$ starting from p_0. Forgetting the parametrization, such a causal curve can always be expressed as the graph of a 1-Lipschitz function $x : [\theta_0, \theta_1[\to \mathbb{D}^2$ (with $\theta_1 < \theta_0$) such that $x(\theta_0) = x_0$ and:

$$f_-(x(\theta)) < \theta < f_+(x(\theta))$$

Since it is 1-Lipschitz, $\theta \to x(\theta)$ admits a limit at θ_1, that we denote by $x_1 = x(\theta_1)$. Observe that x_1 must belong to the compact K defined above: in particular, x_1 belongs to $\mathbb{D}^2$. Moreover, since c is inextendible, $p_1 = (x_1, \theta_1)$ belongs to the boundary of $E(\partial\tilde{S})$: $f_-(x_1) = \theta_1$.

Assume that τ does not converge to 0 along c. Then, there is some $\epsilon > 0$, and a sequence of θ_n converging to θ_1 such that each $\tau((x(\theta_n), \theta_n))$ is bigger than ϵ. In other words, there is a past-oriented causal curve c_n starting from $(x(\theta_n), \theta_n)$ contained in $E(\partial\tilde{S})$ and with time length equal to ϵ. All these curves are contained in the past of p_0. We have just proved that this past is contained in a region $K \times [-\pi/2, \pi/2]$ where K is compact, and where the conformal factor between the AdS metric and the Ein metric is between 1 and μ^2. Hence the curves c_n, considered as causal curves in the Einstein universe $\mathbb{S}^2 \times \mathbb{R}$ with the metric $ds_0^2 - dt^2$ have time length between ϵ and $\mu\epsilon$. Since the Einstein Universe is globally hyperbolic, it follows that the curves c_n converges in the Hausdorff topology to a past-oriented causal curve $\bar{c}$ with time length (for the Einstein metric) strictly positive and contained in $K \times [-\pi/2, \pi/2]$. Obviously, the starting point of

$\bar{c}$ must be p_1. Hence, the final part of $\bar{c}$ is in the (strict) past of p_1. The same must be true for the c_n, for sufficiently big n. In particular, the past of p_1 contains points of $E(\partial\tilde{S})$. This is a contradiction, since the past of p_1 is the domain (x, θ) with $\theta < \theta_1 - d(x, x_1)$, and every element (x, θ) of $E(\partial\tilde{S})$ satisfy $\theta > f_-(p) \geq f_-(x_1) - d(x, x_1) = \theta_1 - d(x, x_1)$. □

Remark 8.22. The cosmological time is Lipschitz continuous, but it is not C^1 in general. It can be proved that τ is $C^{1,1}$ on $\{\tau < \pi/2\}$, i.e., the past in $E(\partial S)$ of $\partial^+ C(\partial S)$. See [10, 11].

8.5 Generic achronal circles in Ein_2

Most considerations above apply when ∂S is any achronal topological circle in Ein_2. Anyway:

Proposition 8.23. *Every generic achronal topological circle of Ein_2 is the boundary of a smooth spacelike hypersurface of AdS.*

Proof. A generic achronal topological circle correspond to the graph Λ of some 1-Lipschitz map $f : \mathbb{S}^1 \to \mathbb{R}$. We can define the open set $E(\Lambda)$ as in Remarks 8.9 and 8.10 (the two definitions still coincide). Following the second definition, it is the open set contained between the graphs of two 1-Lipschiz maps f_-, f_+. The proposition is proved as soon as we prove the existence of the smooth contracting map $g : \mathbb{D}^2 \to \mathbb{R}$ with $f_- < g < f_+$. Indeed, such a g will necessarily coincide with $f_+ = f_-$ on $\partial\mathbb{D}^2$.

After adding some positive contant, we can assume $f_+ > 0$. Define then, for every integer n, $g_n(x) = \mathrm{Sup}(f_-(x), (1 - \frac{1}{n})f_+(x) - \frac{1}{n})$. Then, the sum $g(x) = \sum(g_n(x)/2^n)$ provides a contracting map g with all the required property, except smoothness. This map can be approximated with a smooth one, still contracting [20].

There is another proof: $E(\Lambda)$ is strongly causal since $\widehat{\mathrm{Ein}_3}$ is strongly causal. It follows from the definition that if x, y are points in $E(\Lambda)$, the intersection between the future (in $\widehat{\mathrm{Ein}_3}$) of x and the past (in $\widehat{\mathrm{Ein}_3}$) of y is contained in $E(\Lambda)$: it is compact and coincide with the intersection between the future in $E(\Lambda)$ of x and the past in $E(\Lambda)$ of y. Hence, according to Theorem 2.12, $E(\Lambda)$ is globally hyperbolic. In particular, it admits a *smooth* Cauchy surface S (see Remark 2.14). Any inextendible causal curve intersect S; in particular, this is true for the curves $\{x\}\times]f_-(x), f_+(x)[$. It follows that S is the graph of a smooth contracting map $\mathbb{D}^2 \to \mathbb{R}$ which extends on $\partial\mathbb{D}^2$ as f.

We propose now a third and last proof, more adapted to the equivariant case to be considered later (see Section 10.2): prove as in Proposition 8.21 that $E(\Lambda)$ has regular cosmological time and then apply Theorems 2.12, 2.25. □

8.6 Invisible domains from achronal subsets of $\widehat{\mathrm{Ein}}_2$

In this Section $\widetilde{\Lambda}$ is a generic closed achronal subset of $\widehat{\mathrm{Ein}}_2$. We assume $\mathrm{card}(\widetilde{\Lambda}) \geq 2$. Then $\widetilde{\Lambda}$ is the graph of a 1-Lipschitz map $f_0 : \Lambda_0 \to \mathbb{R}$, where Λ_0 is a closed subset of $\mathbb{S}^1 = \partial\mathbb{D}^2$. We can define as before the *invisible domain from* $\widetilde{\Lambda}$ *in* $\widehat{AdS}$ that we denote by $E(\widetilde{\Lambda})$: it is the set $\{(x,\theta) \in \mathbb{D}^2 \times \mathbb{R} / f_-(x) < \theta < f_+(x)\}$ where

- $f_-(x) = \mathrm{Sup}_{y\in\Lambda_0}\{f(y) - d(x,y)\}$,
- $f_+(x) = \mathrm{Inf}_{y\in\Lambda_0}\{f(y) + d(x,y)\}$.

Of course $f_\pm$ are actually defined on the closure $\overline{\mathbb{D}}^2 \times \mathbb{R}$. Define $\Omega(\widetilde{\Lambda}) = \{(x,t) \in \partial\mathbb{D}^2 \times \mathbb{R}/f_-(x) < t < f_+(x)\}$. It is the *invisible domain* from $\widetilde{\Lambda}$ in $\widehat{\mathrm{Ein}}_2$. Observe that $\Omega(\widetilde{\Lambda})$ is an open subset of $\widehat{\mathrm{Ein}}_2$.

Moreover, if we consider $\widetilde{\Lambda}$ as a closed achronal subset of $\widehat{\mathrm{Ein}}_3$ containing $\widehat{\mathrm{Ein}}_2$ as the boundary of an anti-de Sitter domain, then $E(\widetilde{\Lambda})$ is the intersection between the anti-de Sitter domain and the invisible domain from $\widetilde{\Lambda}$ in $\widehat{\mathrm{Ein}}_3$.

Let $\widetilde{\Lambda}^\pm$ be the graphs of the restriction of $f_\pm$ to $\partial\mathbb{D}^2$. These graphs are achronal topological circles in $\widehat{\mathrm{Ein}}_2$ containing $\widetilde{\Lambda}$.

If we change the parametrization $\widetilde{\mathrm{AdS}} \approx \mathbb{D}^2 \times \mathbb{R}$, we change of course the closed subset $\Lambda'_0 \subset \partial\mathbb{D}^2$, but there is a diffeomorphism from $\partial\mathbb{D}^2$ into itself mapping Λ_0 on Λ'_0. In particular, if the first coordinates of x, y for the first parametrization are extremities of a connected component of $\partial\mathbb{D}^2 \setminus \Lambda_0$, then their projections in Λ'_0 are also extremities of a connected component of $\partial\mathbb{D}^2 \setminus \Lambda_0$.

Definition 8.24. A gap pair is a pair (x,y) of points $\widetilde{\Lambda}$ corresponding to points (x_0,y_0) in Λ_0 which are extremities of a connected component of $\partial\mathbb{D}^2 \setminus \Lambda_0$. An ordered gap pair is the data of a gap pair with a connected component of $\partial\mathbb{D}^2 \setminus \Lambda_0$ with extremities x_0, y_0. A gap pair is achronal if x, y are not causally related in $\widehat{\mathrm{Ein}}_2$. An ordered gap pair (x,y,I) is lightlike if x, y are extremities of a lightlike segment in $\widehat{\mathrm{Ein}}_2$ projecting in $\partial\mathbb{D}^2$ on I.

An ordered gap pair (x, y, I) is extreme if x, y are extremities of a lightlike segment in $\widehat{\mathrm{Ein}_2}$ projecting in $\partial\mathbb{D}^2$ on a segment disjoint from I.

Remark 8.25. It is quite clear that for any lightlike gap pair (x, y), $\widetilde{\Lambda}$ and $\widetilde{\Lambda} \cup [x, y]$ define the same invisible domain. Hence, by adding all these lightlike segments and if the initial $\widetilde{\Lambda}$ did contains at least one pair of non-causally related points, we can reduce the study to the case where all gap pairs are achronal or extreme. Moreover, if $\widetilde{\Lambda}$ contains at least three points, every gap pair defines an ordered gap pair. Finally, extreme gap pairs occurs only in the case where $\widetilde{\Lambda}$ is contained in a lightlike segment.

Definition 8.26. A gap segment is a segment $]x, y[$ where (x, y) is a gap pair.

Gap segments associated to achronal gap pairs are contained in $\mathbb{ADS}$. Observe that gap segments are contained in $\partial\mathrm{Conv}(\Lambda)$.

According to Lemma 5.6, the projection of $\widetilde{\Lambda}$ in $\mathbb{ADS}$ is injective and the image Λ is a compact subset in Ein_2. Moreover, since Λ is contained in a de Sitter domain, we can select some element x_0 of AdS such that Λ is contained in the affine patch $V(x_0)$.

Lemma 8.27. *If $\widetilde{\Lambda}$ contains at least two non-causally related points, then $E(\widetilde{\Lambda})$ is contained in a de Sitter domain.*

Proof. $\widetilde{\Lambda}$ contains two elements which are not causally related, i.e., with coordinates $(x, 0)$, $(-x, 0)$ (Remark 5.15). Then, for any (y, θ) in $E(\widetilde{\Lambda})$ we have

$$\mathrm{Sup}\{-d(x, y), -d(-x, y)\} < \theta < \inf\{d(x, y), d(-x, y)\}$$

Hence, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and the lemma follows. □

Thanks to this lemma, we can project everything in AdS. Remark 8.10 remains true: define $C(\Lambda) = (\mathrm{Conv}(\Lambda)^*)^\sharp = \mathrm{Conv}(\Lambda^\flat)^*$. The projection of $E(\widetilde{\Lambda})$ in this affine patch is the intersection $E(\Lambda)$ between $\mathbb{ADS}$ and all the half-spaces E_{xy} where x, y describes $\Lambda \times \Lambda$ minus the diagonal. It follows that $E(\Lambda)$ is equal to $C(\Lambda) \cap \mathbb{ADS}$. $\Omega(\Lambda)$ projects in Ein_2 to $\Omega(\Lambda) = \mathrm{Ein}_2 \cap C(\Lambda)$.

8.7 Elementary cases

In this section, we assume that $\widetilde{\Lambda}$ is a generic achronal subset, containing at least two points and without lightlike gap pair (see Remark 8.25).

Definition 8.28. The common future of $\widetilde{\Lambda}$ is the set of points of $\widehat{\mathrm{Ein}}_2$ containing the entire $\widetilde{\Lambda}$ in their past lightcones. We denote it by $X^+(\widetilde{\Lambda})$. Similarly, the common past $X^-(\widetilde{\Lambda})$ is the set of points in $\widehat{\mathrm{Ein}}_2$ containing $\widetilde{\Lambda}$ in their future lightcones.

Definition 8.29. When $X^+(\widetilde{\Lambda})$ or $X^-(\widetilde{\Lambda})$ is not empty, $\widetilde{\Lambda}$ is elementary. If not, $\widetilde{\Lambda}$ is said non-elementary.

Lemma 8.30. *The elementary case admits three subcases:*

- the conical case: *it is the case where $X^+(\widetilde{\Lambda})$ or $X^-(\widetilde{\Lambda})$ is reduced to one point x_0 and $X(\widetilde{\Lambda}) = \{x_0\}$. If $X^+(\widetilde{\Lambda}) = \{x_0\}$: let L_1, L_2 be the two past oriented closed lightlike segments in $\widehat{Ein}_2$ with extremities x_0, x_1, where x_1 is the point opposite to x_0 in the past. Then, $\widetilde{\Lambda}$ is the union of two lightlike segments I_1, I_2, not reduced to single points, contained, respectively, in L_1, L_2.*

  The case $X^-(\widetilde{\Lambda}) = \{x_0\}$ admits a similar description: inverse the time orientation.
- the splitting case: *it is the case where $X(\widetilde{\Lambda})$ is a pair of non-causally related points. $\widetilde{\Lambda}$ is then a pair of non-causally related points.*
- the extreme case: *it is the case where $X^+(\widetilde{\Lambda})$ and $X^-(\widetilde{\Lambda})$ are lightlike rays, contained in a lightlike geodesic Δ. $\widetilde{\Lambda}$ is then a lightlike segment inside Δ.*

Proof. Reversing the time orientation if necessary, we can assume that $X^+(\widetilde{\Lambda})$ contains a point x_0. Let $-x_0$ be the point opposite to x_0 in the past and L_1, L_2 the two lightlike segments with extremities x_0, $-x_0$. Then, $\widetilde{\Lambda}$ is contained in the pure lightlike circle $L_1 \cup L_2$. Since we assume that $\widetilde{\Lambda}$ is generic, it is not the union $L_1 \cup L_2$. Since we assume that there is no lightlike gap pair, the intersections $\widetilde{\Lambda}_i = \widetilde{\Lambda} \cap L_i$ $(i = 1, 2)$ are connected, i.e., intervals.

Assume that $\widetilde{\Lambda}$ contains at least two non-causally related points. Then, $\widetilde{\Lambda}_1$ and $\widetilde{\Lambda}_2$ have non-empty interior. If $X(\widetilde{\Lambda}) = \{x_0\}$, we are in the conical case. If $X^+(\widetilde{\Lambda})$ contains another point x_0', we can define similarly two lightlike segments L_1', L_2' and $\widetilde{\Lambda}_i$ must be contained in $L_i \cap L_i'$. Since these intersections are not empty, x_0 and x_0' are not causally related, and these intersections are both reduced to one point. The lemma follows in this case: we are in the splitting case.

The last case to consider is the case where all points in $\widetilde{\Lambda}$ are causally related one to the other. $\widetilde{\Lambda}$ is then equal to $\widetilde{\Lambda}_1$ or $\widetilde{\Lambda}_2$. We are in the extreme case. □

8.8 Description of the splitting case

In [7], we will describe $E(\widetilde{\Lambda})$ for every elementary $\widetilde{\Lambda}$. For the present paper, we just need to understand the splitting case $\widetilde{\Lambda} = \{x, y\}$, where x, y are two non-causally related points in $\widehat{\mathrm{Ein}}_2$. Then, $\{x, y\}$ is a gap pair, and there are two associated ordered gap pairs, that we denote, respectively, by (x, y) and (y, x). $\widetilde{\Lambda}^+$ is an union $\mathcal{T}^+_{xy} \cup \mathcal{T}^+_{yx}$ of two nontimelike segments with extremities x, y, that we call *upper tents*. Such an upper tent is the union of two lightlike segments, one starting from x, the other from y, and stopping at their first intersection point, that we call the *upper corner.*

Similarly, $\widetilde{\Lambda}^-$ is an union $\mathcal{T}^-_{xy} \cup \mathcal{T}^-_{yx}$ of two *lower tents* admitting a similar description, but where the lightlike segments starting from x, y are now past oriented (see figure 1) and sharing a common extremity: the *lower corner.*

The invisible domain $\Omega(\widetilde{\Lambda})$ from $\widetilde{\Lambda}$ in $\widehat{\mathrm{Ein}}_2$ is the union of two diamond-shape regions $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$. The boundary of $\widetilde{\Delta}_1$ is the union $\mathcal{T}^+_{xy} \cup \mathcal{T}^-_{xy}$ and the boundary of $\widetilde{\Delta}_2$ is $\mathcal{T}^+_{yx} \cup \mathcal{T}^-_{yx}$. We project all the picture in some affine region $V \approx \mathbb{R}^3$ of $S(E)$ such that:

- $V \cap \mathbb{ADS}$ is the interior of the hyperboloid: $\{x^2 + y^2 < 1 + z^2\}$,
- $\Lambda = \{(1, 0, 0), (-1, 0, 0)\}$.

Then, $E(\Lambda)$ is region $\{-1 < x < 1\} \cap \mathbb{ADS}$. One of the diamond-shape region $\widetilde{\Delta}_i$ projects to $\Delta_1 = \{-1 < x < 1, y > 0, x^2 + y^2 = 1 + z^2\}$, the other projects to $\Delta_2 = \{-1 < x < 1, y < 0, x^2 + y^2 = 1 + z^2\}$. The past of Δ_1 in $E(\Lambda)$ is $P_1 = \{(x, y, z) \in E(\Lambda)/z < y\}$. and the future of Δ_1 in $E(\Lambda)$ is

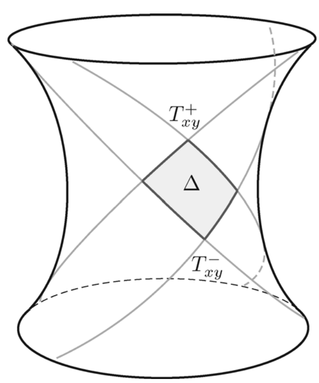

Figure 1: Upper and lower tents

$F_1 = \{(x,y,z) \in E(\Lambda)/z > -y\}$. We have of course a similar description for the future F_2 and the past P_2 of Δ_2 in $E(\Lambda)$. Observe:

- the intersections $F_1 \cap F_2$ and $P_1 \cap P_2$ are disjoint. They are tetraedra in $S(E)$: $F_1 \cap F_2$ is the interior of the convex hull of Λ^+, and $P_1 \cap P_2$ is the interior of the convex hull of Λ^-.
- the intersection $F_1 \cap P_1$ (resp. $F_2 \cap P_2$) is the intersection between $\mathbb{ADS}$ and the interior of a tetraedron in $S(E)$: the convex hull of Δ_1 (resp. Δ_2).

Definition 8.31. $E^+(\Lambda) = F_1 \cap F_2$ is the future globally hyperbolic convex core; $E^-(\Lambda) = P_1 \cap P_2$ is the past globally hyperbolic convex core.

This terminology is justified by the following (easy) fact: $F_1 \cap F_2$ (resp. $P_1 \cap P_2$) is the invisible domain $E(\Lambda^+)$ (resp. $E(\Lambda^-)$). Hence, they are indeed globally hyperbolic.

The intersection between the closure of $E(\Lambda)$ in $S(E)$ and the boundary Ein_2 of $\mathbb{ADS}$ is the union of the closures of the diamond-shape regions. Hence, $\Delta_{1,2}$ can be thought as the conformal boundaries at infinity of $E(\Lambda)$. Starting from any point in $E(\Lambda)$, to Δ_i we have to enter in $F_i \cap P_i$, hence we can adopt the following definition (figure 2):

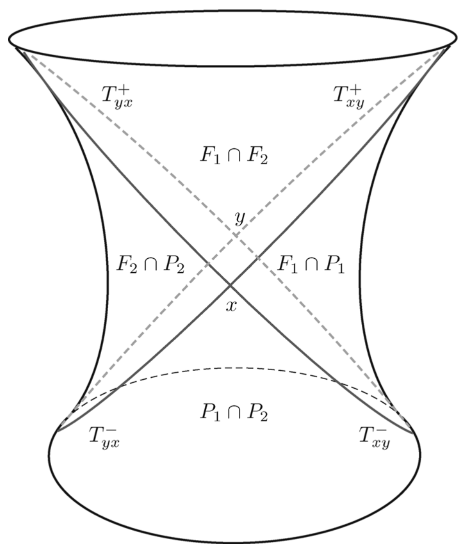

Figure 2: The splitting case. The domain $E(\Lambda)$ is between the hyperplanes $x^\perp$ and $y^\perp$. These hyperplanes, tangent to the hyperboloid, are not drawn, except their intersections with the hyperboloid, which are the upper and lower tents $\mathcal{T}^\pm_{xy}$, $\mathcal{T}^\pm_{yx}$.

Definition 8.32. $F_1 \cap P_1$ is an end of $E(\Lambda)$.

Finally:

Definition 8.33. The future horizon is the past boundary of $F_1 \cap F_2$; the past horizon is the future boundary of $P_1 \cap P_2$.

Proposition 8.34. *$E(\Lambda)$ is the disjoint union of the future and past globally hyperbolic cores $E^{\pm}(\Lambda)$, of the two ends and of the past and future horizons.*

Remark 8.35. In the conventions of [2, 3, 16], the globally hyperbolic convex cores $F_1 \cap F_2$ and $P_1 \cap P_2$ are *regions of type II,* also called *intermediate regions.* The ends $F_1 \cap P_1$ and $F_2 \cap P_2$ are *outer regions,* or *regions of type I.*

8.9 The non-elementary case

From now, we assume that $\widetilde{\Lambda}$ is generic and non-elementary. Then, it contains at least two non-causally related points: every $\widetilde{\Lambda}^{\pm}$ is generic. Moreover, every gap pair defines uniquely an ordered gap pair, that we can assume to be achronal (Remark 8.25).

Definition 8.36. $\widetilde{\Lambda}$ is proper if it is nonelementary and not contained in a flat sphere.

Proposition 8.37. *The closure of $E(\Lambda)$ in $S(E)$ is contained in an affine patch if and only if Λ is proper.*

Remark 8.38. We have chosen the terminology so that Λ is proper if and only if the convex $E(\Lambda)$ is proper in the meaning of Section 6.

Proof of Proposition 8.37. $E(\Lambda)$ is the intersection between $(\mathrm{Conv}(\Lambda)^*)^\sharp$ and $\mathbb{ADS}$. It follows from Proposition 6.2 that its closure is contained in an affine patch, except if $\mathrm{Conv}(\Lambda)$ has empty interior, i.e., is contained in a projective hyperplane P. But then P must be spacelike since it contains the non-elementary set Λ. Hence, $P \cap \mathrm{Ein}_2$ is a flat sphere containing Λ.

Inversely, if Λ is contained in a flat sphere $S(v_0^\perp \cap \{Q = 0\})$, then v_0 and $-v_0$ belong to the closure of $E(\Lambda)$. □

Remark 8.39. Observe that if Λ is non-elementary but non-proper, the flat sphere containing it is unique.

8.10 The decomposition in ends and globally hyperbolic cores

We still assume that $\widetilde{\Lambda}$ is non-elementary. It is easy to see that $\widetilde{\Lambda}^+$ (resp. $\widetilde{\Lambda}^-$) is obtained from $\widetilde{\Lambda}$ by adding for any lightlike gap pair (x, y) the lightlike segment $[x, y]$ and for any achronal gap pair (x, y) the upper (resp. lower) tent $\mathcal{T}^+_{xy}$ (resp. $\mathcal{T}^-_{xy}$) (figure 3).

The connected components of $\Omega(\Lambda)$ are precisely the diamonds Δ_{xy}. The convex hull in $S(E)$ of Δ_{xy} is a tetraedron (see Section 8.7), the intersection of this tetraedron with $\mathbb{ADS}$ has been described above.

Definition 8.40. For any gap pair (x, y), the closed end $\overline{\mathcal{E}}_{xy}$ is the intersection between $\mathbb{ADS}$ and the convex hull in $S(E)$ of Δ_{xy}.

Lemma 8.41. *Every closed end $\overline{\mathcal{E}}_{xy}$ is contained in $E(\Lambda)$.*

Proof. Since $E(\Lambda)$ is convex and since Δ_{xy} is contained in the closure of $E(\Lambda)$, $\overline{\mathcal{E}}_{xy}$ is contained in the closure of $E(\Lambda)$. Let z^+ (resp. z^-) be the upper (resp. lower) corner of Δ_{xy}. Let $p = ax + by + cz^+ + dz^-$ $(a, b, c, d \geq 0)$ be a point in $\overline{\mathcal{E}}_{xy}$. Observe that the norm of p is $2ab\langle x \mid y\rangle + 2cd\langle z^- \mid z^+\rangle$. It has to be negative; since $\langle z^- \mid z^+\rangle$ is positive, we have $ab > 0$.

If p belongs to the boundary of $E(\Lambda)$ in $\mathbb{ADS}$, then there is an element z of Λ such that $\langle z \mid p\rangle = 0$. But the scalar products of z with x, y, z^+ and z^- are all nonnegative. Hence, $a\langle z \mid x\rangle$ and $b\langle z \mid y\rangle$ are both 0. Since a and b are positive, it follows that z is causally related to x *and* y. But $z^\pm$ are the

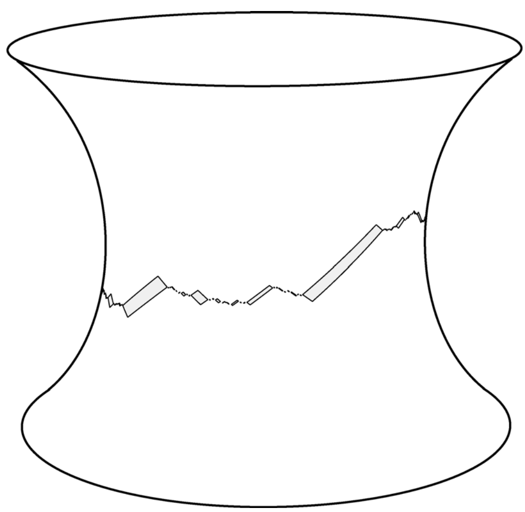

Figure 3: Filling the gaps.

only points in the affine patch in consideration which are causally related to x and y and they do not belong to Λ. We obtain a contradiction: $\overline{\mathcal{E}}_{xy}$ does not intersect the boundary of $E(\Lambda)$. The lemma follows. □

Definition 8.42. The end $\mathcal{E}_{xy}$ is the interior in $\mathbb{ADS}$ of $\overline{\mathcal{E}}_{xy}$.

Definition 8.43. The future globally hyperbolic core is $E(\Lambda^+)$, the past globally hyperbolic core is $E(\Lambda^-)$.

Definition 8.44. The past boundary in $\mathbb{ADS}$ of $E(\Lambda^+)$ is the past horizon. The future boundary in $\mathbb{ADS}$ of $E(\Lambda^-)$ is the future horizon.

Future and past horizons are achronal proper hypersurfaces.

Finally, another important feature is the convex hull $\mathrm{Conv}(\Lambda)$. We denote $\partial\mathrm{Conv}(\Lambda)$ its boundary in $S(E)$.

Definition 8.45. The edge part $\partial_{\mathrm{ed}}\mathrm{Conv}(\Lambda)$ of $\partial\mathrm{Conv}(\Lambda)$ is the union of all gap segments (see Definition 8.26).

Lemma 8.46. *The edge part is the set of points of $\partial\mathrm{Conv}(\Lambda) \cap \mathbb{ADS}$ admitting timelike support hyperplanes.*

Proof. Observe that every gap segment admits two lightlike hyperplanes: the dual planes to the middle points $z^{\pm}$ of the associated upper and lower tents. The dual plane to the middle of $[z^-, z^+]$ is then a timelike support hyperplane containing the gap segment.

Let's prove the reverse inclusion: let q be an element of $S(E)$ such that the dual plane $q^{\perp}$ is a timelike support hyperplane at a point $p = \sum_{i=1,\dots,k} a_i p_i$ of $\partial\mathrm{Conv}(\Lambda) \cap \mathbb{ADS}$. Since $q^{\perp}$ is timelike, $q^{\perp} \cap \mathrm{Ein}_2$ is a copy of the Klein model of the anti-de Sitter space of dimension 1. It follows that the integer k is less than 2, and since p belongs to $\mathbb{ADS}$, $k = 2$. Moreover, since $q^{\perp}$ is a support hyperplane, Λ is contained in one connected component of $\mathrm{Ein}_2 \setminus q^{\perp}$. It follows that (p_1, p_2) is a gap pair, and that p belongs to the gap segment $]p_1, p_2[$. □

In the proper case, i.e., when $\mathrm{Conv}(\Lambda)$ has non-empty interior, $\Lambda \cup \partial_{\mathrm{ed}}$ $\mathrm{Conv}(\Lambda)$ is a Jordan curve in the topological sphere $\partial\mathrm{Conv}(\Lambda)$ and $\partial\mathrm{Conv}$ $(\Lambda) \setminus (\Lambda \cup \partial_{\mathrm{ed}}\mathrm{Conv}(\Lambda))$ is the union of two (non-proper) topological discs. One of them — $C^+(\Lambda)$ — is in the future of the other, that we call $C^-(\Lambda)$.

In the non-proper case, i.e., when Λ is contained in the boundary of a totally geodesic embedding of $\mathbb{H}^2$ in $\mathbb{ADS}$, $\mathrm{Conv}(\Lambda)$ coincide with the convex hull of $\Lambda \subset \partial\mathbb{H}^2$ and the edge part $\partial_{\mathrm{ed}}\mathrm{Conv}(\Lambda)$ is the boundary of this convex hull in $\mathbb{H}^2$. We simply define $C^+(\Lambda) = C^-(\Lambda) = \mathrm{Conv}(\Lambda) \setminus (\partial_{\mathrm{ed}}\mathrm{Conv}(\Lambda) \cup \Lambda)$.

In both cases $C^\pm(\Lambda)$ are non-timelike topological surfaces (Lemma 8.46).

Lemma 8.47. *Every timelike geodesic intersecting $C^-(\Lambda)$ intersects $C^+(\Lambda)$.*

Proof. Let c be a timelike geodesic intersecting $C^-(\Lambda)$. Since $C^-(\Lambda)$ is non-timelike, c must enter in $\mathrm{Conv}(\Lambda)$. The only possible exit for c is then through $C^+(\Lambda)$. □

Corollary 8.48. *Let W be the projection of $C^-(\Lambda) \subset \mathbb{ADS} \approx \mathbb{D}^2 \times \mathbb{S}^1$ on the first factor $\mathbb{D}^2$. Then, it is also the projection of $C^+(\Lambda)$ on the first factor. Every connected component of ∂W is a curve c in $\mathbb{D}^2$ joining two elements x, y in $\Lambda_0 \subset \partial\mathbb{D}^2$ such that:*

- *$]x, y[$ is a gap of Λ_0,*
- *the curve c disconnect W from the gap segment $]x, y[\subset \partial\mathbb{D}^2$.*

Proof. The first statement follows immediatly from Lemma 8.47. Then, $C^\pm(\Lambda)$ are graphs of functions $g_\pm : W \to \mathbb{R}$. The proofs of the topological description of W, which follows from the concave–convex properties of $C^\pm(\Lambda)$, are left to the reader. □

Remark 8.49. Be careful! W is *not* in general the convex hull in $\mathbb{D}^2$ of Λ_0. Observe that $W \subset \mathbb{D}^2$ depends on the selected conformal parametrization $\mathbb{ADS} \approx \mathbb{D}^2 \times \mathbb{S}^1$.

Proposition 8.50. *$E(\Lambda)$ is the union of the past and future globally hyperbolic cores with the closed ends associated to gap pairs.*

Proof. One of the inclusion follows from Lemma 8.41 and the obvious inclusion $E(\Lambda^+) \cup E(\Lambda^-) \subset E(\Lambda)$.

For the reverse inclusion: first observe that if Λ is a topological circle, it has no ends and $E(\Lambda) = E(\Lambda^+) = E(\Lambda^-)$: there is nothing to prove. Hence, we assume that Λ admits at least one (achronal) gap. Recall that, in a suitable conformal domain $\approx \mathbb{D}^2 \times [-\pi/2, \pi/2]$, the invisibility domain $E(\Lambda)$ is the domain in $\mathbb{D}^2 \times [-\pi/2, \pi/2]$ between the graphs of $f_\pm$. Define:

- $F_-(x) = \mathrm{Sup}_{y \in \partial\mathbb{D}^2}\{f_+(y) - d(x, y)\}$,
- $F_+(x) = \mathrm{Inf}_{y \in \partial\mathbb{D}^2}\{f_-(y) + d(x, y)\}$.

Then, $E(\Lambda^+)$ is the domain between the graphs of F_-, f_+ and $E(\Lambda^-)$ is the domain between the graphs of f_-, F_+.

The discs $C^{\pm}(\Lambda)$ are the graphs of two functions $g_{\pm} : W \to]-\pi/2, \pi/2[$ described in Corollary 8.48.

Claim: for every x in W, $F^+(x) > F^-(x)$.

Since $\mathrm{Conv}(\Lambda) \subset \mathrm{Conv}(\Lambda^{\pm}) \subset E(\Lambda^{\pm})$, we have:

$$\forall x \in W, \quad f_-(x) \leq F_-(x) \leq g_-(x) \leq g_+(x) \leq F_+(x) \leq f_+(x)$$

In the proper case, we actually have $g_+(x) > g_-(x)$ for x in W: the claim follows. In the non-proper case, we can select the de Sitter domain so that $f(y) = 0$ for every y in Λ_0. Then, for every x in W, we have $g_+(x) = g_-(x) = 0$. It follows that if $F_+(x_0) = F_-(x_0)$ for some x_0 in W, then this common value is 0. By definition of $F_{\pm}$, 0 is the supremum of $f_+(y) - d(x_0, y)$ and this supremum is attained at some y_0. Then, $f_-(y_0) = -f_+(y_0) = -d(x_0, y_0)$. It means that $(y_0, d(x_0, y_0))$ is the upper corner of a upper tent associated to some gap (x, y) and $(y_0, -d(x_0, y_0))$ is the lower corner of the associated lower tent. It implies that $(x_0, 0)$ belongs to the gap segment $]x, y[$. Contradiction.

The claim is proved. Let now (x, θ) be a point in $E(\Lambda)$: $f_-(x) < \theta < f_+(x)$ holds.

If x belongs to W, then since $F_-(x) < F_+(x)$, we have either $f_-(x) < \theta < F_+(x)$, or $F_-(x) < \theta < f_+(x)$. In the former case, (x, θ) belongs to $E^-(\Lambda)$ and in the later case, (x, θ) belongs to $E^+(\Lambda)$.

According to Lemma 8.41, the same conclusion holds if x belongs to $\overline{W}$.

Assume now $x \in \mathbb{D}^2 \setminus \overline{W}$. W is a topological disc in $\mathbb{D}^2$ with boundary components the projection of gap segments. Hence x belongs to the connected component of $\mathbb{D}^2 \setminus l$ which does not contain W, where l is the projection of a gap segment $]y, z[$. Let z^+ be the upper corner of the upper tent $\mathcal{T}^+_{yz}$ and let z^- be the lower corner of the lower tent $\mathcal{T}^-_{yz}$. Since (x, θ) cannot be related to y or z we easily infer that it must belong to the intersection between the past of z^+ and the future of z^-, i.e., to the closed end $\overline{\mathcal{E}}_{yz}$. □

Remark 8.51. Furthermore, it follows quite easily from the proof above that the intersection $E^+(\Lambda) \cap E^-(\Lambda)$ is not empty: actually, it can be proved that, in the notation used in the proof of 8.50, $F_+ > F_-$ on W, $F_+ = F_-$ on ∂W and $F_+ < F_-$ on $\mathbb{D}^2 \setminus \overline{W}$.

Remark 8.52. There are other ways to characterize $E(\Lambda)$. For example, it is the union of every proper spacelike surfaces containing Λ in their natural extensions in $\partial\mathrm{AdS}$.

9 Synchronized isometries of AdS

We use the identification $\overline{\mathbb{ADS}} \approx G = \mathrm{PSL}(2,\mathbb{R})$ (cf. Section 3.3). Then, $\widetilde{\mathrm{AdS}}$ can be identified with the universal covering $\widetilde{G} = \widetilde{\mathrm{SL}}(2,\mathbb{R})$. Denote by $\bar{p} : \widetilde{G} \to G$ the covering map, and Z the kernel of $\bar{p}$: Z is cyclic, it is the center of $\widetilde{G}$. Let δ be a generator of Z: we select it in the future of the neutral element $\mathrm{i}d$.

$\widetilde{G} \times \widetilde{G}$ acts by left and right translations on $\widetilde{G}$. This action is not faithfull: the elements acting trivially are precisely the elements in $\mathcal{Z}$, the image of Z by the diagonal embedding. The isometry group $\widehat{\mathrm{SO}}_0(2,2)$ is then identified with $(\widetilde{G} \times \widetilde{G})_{/\mathcal{Z}}$.

Let $\mathcal{G}$ be the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ of G: the Lie algebra of $(\widetilde{G} \times \widetilde{G})_{/\mathcal{Z}}$ is $\mathcal{G} \times \mathcal{G}$. We assume the reader familiar with the notion of elliptic, parabolic, hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$. Observe that hyperbolic (resp. parabolic) elements of $\mathrm{PSL}(2,\mathbb{R})$ are the exponentials $\exp(A)$ of *hyperbolic* (resp. *parabolic, elliptic*) elements of $\mathcal{G} = \mathrm{sl}(2,\mathbb{R})$, i.e., such that $\det(A) < 0$ (resp. $\det(A) = 0$, $\det(A) > 0$).

Definition 9.1. An element of $\widetilde{G}$ is hyperbolic (resp. parabolic, elliptic) if it is the exponential of a hyperbolic (resp. parabolic, elliptic) element of $\mathcal{G}$.

Remark 9.2. There is another possible definition through the identification $\widetilde{G} \approx \widetilde{\mathrm{AdS}}$: hyperbolic (resp. parabolic) elements of $\mathcal{G}$ are spacelike (resp. lightlike) tangent vectors to $\widetilde{\mathrm{AdS}}$ at the neutral element $\mathrm{i}d$. Hyperbolic elements of $\widetilde{\mathrm{AdS}}$ are the elements which are not causally related to $\mathrm{i}d$ of $\widetilde{G}$. Parabolic elements are points in the lightcone of $\mathrm{i}d$. Hence, their union is the set of points in $\widetilde{\mathrm{AdS}}$ which are not strictly causally related to $\mathrm{i}d$. In particular, they belong to the affine domain associated to $\mathrm{i}d$.

Remark 9.3. Elements of $\widetilde{G}$ which are not hyperbolic, parabolic or elliptic have the form $\delta^k\gamma'$, where δ^k is a non-trivial element of the center of $\widetilde{G}$, and γ' a hyperbolic or parabolic element of $\widetilde{G}$.

Remark 9.4. Every element of $(\widetilde{G} \times \widetilde{G})_{/\mathcal{Z}}$ can be represented by a pair (γ_L, γ_R) such that:

- γ_L is the exponential of an element of $\mathcal{G}$,
- $\gamma_R = \gamma'_R\delta^k$, where γ'_R is the exponential of an element of $\mathcal{G}$, and δ^k an element of Z.

Definition 9.5. An element $\gamma = (\gamma_L, \gamma_R)$ of $\widetilde{G} \times \widetilde{G}$ is synchronized if, up to a permutation of left and right components, it has one of the following form:

- (hyperbolic translation): γ_L is trivial and γ_R is hyperbolic,
- (parabolic translation): γ_L is trivial and γ_R is parabolic,
- (hyperbolic–hyperbolic) γ_L and γ_R are both non-trivial and hyperbolic,
- (parabolic–hyperbolic) γ_L is parabolic and γ_R is hyperbolic,
- (parabolic–parabolic) γ_L and γ_R are both non-trivial and parabolic,
- (elliptic) γ_L and γ_R are elliptic elements conjugate in $\widetilde{G}$.

An element γ of $(\widetilde{G} \times \widetilde{G})_{/\mathcal{Z}}$ is synchronized if it is represented by a synchronized element of $\widetilde{G} \times \widetilde{G}$.

We will see that synchronized isometries are precisely those preserving some generic achronal subset. This statement essentially follows from the lemma:

Lemma 9.6. *An isometry γ is synchronized if and only if there is an affine domain U in $\widetilde{AdS}$ such that $\gamma^n(U) \cap U \neq \emptyset$ for every n in $\mathbb{Z}$.*

Proof. Assume that γ is synchronized. Consider first the case where γ_L or γ_R is a non-trivial elliptic element. Then, after conjugacy, we can assume $\gamma_L = \gamma_R$. Then, γ preserves the affine domain $A(\mathrm{id})$.

Consider now the case where γ_R and γ_R are not elliptic. After conjugacy in $\widetilde{G} \times \widetilde{G}$, we can assume that γ_L and γ_R are exponentials of matrices of the form:

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$$

In particular, $\gamma_L\gamma_R^{-1}$ is the exponential of a matrix X of the form above.

Consider the affine domain $A(\mathrm{id})$. Then, for every integer n, $(\gamma_R^n, \gamma_R^n)A(\mathrm{id}) = A(\mathrm{id})$, and $(\gamma_L^n\gamma_R^{-n}, \mathrm{id})A(\mathrm{id}) = A(\gamma_L^n\gamma_R^{-n}) = A(\exp(nX))$. Since $\exp(nX)$ belongs to $A(\mathrm{id})$, we obtain that $\gamma A(\mathrm{id}) \cap A(\mathrm{id}) \neq \emptyset$.

Assume now that γ is *not* synchronized, but that there exists a affine domain A such that all the $\gamma^n A$ $(n \in \mathbb{Z})$ intersect A. There is an integer q such that $\delta^q A$ intersect $A(\mathrm{id})$. Then, every $\gamma^n\delta^q A$ intersects $\delta^q A$. It implies that all the $\gamma^n\delta^q A$ are contained in the past of $\delta^3 A(\mathrm{id})$ and the future of $\delta^{-3}A(\mathrm{id})$. It follows that all the $\gamma^n A(\mathrm{id})$ are contained in the past of $\delta^6 A(\mathrm{id})$ and in the future of $\delta^{-6}A(\mathrm{id})$.

Select a representant (γ_L, γ_R) of γ as in Remark 9.4. According to Remark 9.3, we have three cases to consider:

(1) γ_L and γ'_R are parabolic or hyperbolic, but δ^k is not trivial,
(2) γ_L is elliptic and γ_R is the exponential of an element of $\mathcal{G}$, but not conjugate to γ_L,
(3) γ_L is parabolic or hyperbolic, but $\gamma_R = \gamma'_R$ is elliptic.

In the first case, $\gamma' = (\gamma_L, \gamma'_R)$ is synchronized: hence, for every integer n, $\gamma'_n A(id) \cap A(id) \neq \emptyset$. The affine domain $\gamma^n A(\mathrm{id}) = \delta^{kn}\gamma'_n A(\mathrm{id})$ intersect $\delta^{kn} A(\mathrm{id})$. Since $k \neq 0$ — let's say, $k > 0$ — if n is sufficiently big, affine domains intersecting $\delta^{kn} A(\mathrm{id})$ cannot be contained in the past of $\delta^6 A(\mathrm{id})$. Contradiction.

Consider now the second case. The first subcase is the case where γ_R is elliptic too. Moreover, after conjugacy, we can assume that γ_L and γ_R commute. Then, $\gamma' = (\gamma_R, \gamma_R)$ is synchronized, and γ is the composition of γ' with the left translation by the non-trivial elliptic element $\gamma_L\gamma_R^{-1}$. We can identify $\widetilde{\mathrm{AdS}}$ with $\mathbb{D}^2 \times \mathbb{R}$ such that the left translation by $\gamma_L\gamma_R^{-1}$ is a non-trivial translation along the $\mathbb{R}$-factor. It follows that for n sufficiently big $\gamma^n A(\mathrm{id})$ is not contained in the past of $\delta^6 A(\mathrm{id})$ and the past of $\delta^{-6} A(\mathrm{id})$. Contradiction.

Assume now that we are still in the second case, but with γ_R non-elliptic: $\gamma' = (\mathrm{id}, \gamma_R)$ is synchronized, and γ is the composition of γ' with the left translation by γ_L. We obtain a contradiction as above.

The third case reduce to the second one after composing with the involution $g \mapsto g^{-1}$, which permutes γ_R with γ_L. □

10 Invariant achronal subsets

Let Γ be a subgroup of $\widehat{\mathrm{SO}}_0(2,2)$ preserving a generic closed achronal subset $\widetilde{\Lambda}$ of $\widehat{\mathrm{Ein}}_2$. We assume that $\widetilde{\Lambda}$ is non-elementary and without lightlike pairs (we recall once more Remark 8.25).

We now consider any discrete subgroup Γ of $\widehat{\mathrm{SO}}_0(2,2)$. According to Lemmas 5.6 and 9.6, every element of Γ is synchronized. We assume, moreover, that Γ is torsion-free: it follows that Γ does not contain synchronized elements (γ_L, γ_R), where γ_L and γ_R are elliptic elements of $\widetilde{G}$ conjugate in $\widetilde{G}$. Indeed, the torsion-free hypothesis prevents γ_L, γ_R to have rational rotation angle, and if this rotation angle was irrational, Γ would not be discrete.

The action of Γ on $\widetilde{\mathrm{AdS}}$ and $\widehat{\mathrm{Ein}}_2$ preserves the invisible domains $\Omega(\widetilde{\Lambda})$ and $E(\widetilde{\Lambda})$.

Theorem 10.1. *Let $\widetilde{\Lambda}$ be a non-elementary generic achronal subset, preserved by a torsion-free discrete group $\Gamma \subset \mathrm{SO}_0(2,2)$. Then, the action of Γ on $\Omega(\widetilde{\Lambda})$ and $E(\widetilde{\Lambda})$ are free, properly discontinuous, and the quotient spacetime $M_{\widetilde{\Lambda}}(\Gamma) = \Gamma\backslash E(\widetilde{\Lambda})$ is strongly causal.*

10.1 The hyperbolic–hyperbolic case

We first consider a special case, which is truely speaking an elementary one, but which is necessary to consider for the general case:

Lemma 10.2. *Assume that Γ is cyclic, generated by some $\gamma = (\gamma_L, \gamma_R)$, and that:*

- *$\gamma_R = id$ or,*
- *$\gamma_L = id$ or,*
- *x (resp. y) is an attractive (resp. repelling) fixed point of γ.*

Then, the quotient space $M_{xy}(\Gamma) = \Gamma\backslash E(x,y)$ is strongly causal. Moreover, the projections in $M_{xy}(\Gamma)$ of the open ends $P_i \cap F_i$, and of the globally hyperbolic convex cores $P_1 \cap P_2$, $F_1 \cap F_2$, are all causally convex domains.

Proof. The strong causality is proved in [7] (Proposition 4.11, cases (2) and (6)). The causal convexity is quite obvious. □

10.2 The globally hyperbolic case

According to Proposition 8.21, $E(\widetilde{\Lambda})$ is globally hyperbolic. We prove here :

Proposition 10.3. *If $\widetilde{\Lambda}$ is a topological circle, then the action of Γ on $E(\widetilde{\Lambda})$ is free and properly discontinuous, and the quotient space $\Gamma\backslash E(\widetilde{\Lambda})$ is globally hyperbolic, with regular cosmological time.*

Proof. We first prove the properness of the action:

The flat case: This is the case where $\widetilde{S}$ is a round circle, i.e., the boundary $\partial\widetilde{S}_0$ of a totally geodesic isometric copy of $\mathbb{H}^2$ in $\widetilde{\mathrm{AdS}}$. Then, Γ preserves $\tilde{x}_0^+$, the point dual to $\widetilde{S}_0$ such that $\widetilde{S}_0$ is the past boundary of the affine domain $A(\tilde{x}_0)$ (see Section 3.5). Project everything in $\mathbb{ADS}$. Select a basis on E

so that $x_0 = p(\tilde{x}_0)$ has coordinates $(1, 0, ..., 0)$. Then, the stabilizer of x_0 is $\mathrm{SO}_0(1,2)$, and Γ is a torsion-free discrete subgroup of $\mathrm{SO}_0(1,2)$. Hence, the action of Γ on $S_0 = p(\widetilde{S}_0)$ is free and properly discontinuous. Our claim then follows from the decomposition $x = \cos(\theta)y + \sin(\theta)x_0$ valid for any element x of $E(S_0)$.

The non-flat case: When $\widetilde{\Lambda}$ is not a round circle, it is proper (see Definition 8.36). We observe, as in the proof of Proposition 8.37 that $E(\Lambda) = p(E(\widetilde{\Lambda}))$ is $(\mathrm{Conv}(\Lambda)^*)^\sharp$, which is a proper convex domains in $S(E)$: the Hilbert metric on it is a well-defined metric (see [17]). It follows that the action of Γ on it is properly discontinuous. Observe that the action is free since Γ has no torsion.

Hence, in any case, the quotient $M_{\widetilde{\Lambda}}(\Gamma)$ is a well defined locally AdS spacetime. The proposition then follows immediatly from Propositions 8.21, 2.27 and Theorem 2.25. □

10.3 The general case

Even when $\widetilde{\Lambda}$ is not a topological circle, we can prove as in Section 10.2 that if $\widetilde{\Lambda}$ is non-proper, Γ acts freely and properly discontinuously on $E(\widetilde{\Lambda})$ by considering the Hilbert metric on the proper convex domains $(\mathrm{Conv}(\Lambda)^*)^\sharp$, since $E(\widetilde{\Lambda})$ is the intersection between this proper convex domain and $\mathbb{ADS}$. We leave to the reader the proof of the properness of the action in the flat case.

According to Proposition 2.8, $M_{\widetilde{\Lambda}}(\Gamma)$ is strongly causal if and only if any point x_0 in $M_{\widetilde{\Lambda}}(\Gamma)$ admits a causally convex neighborhood. If x_0 belongs to the projection of the future or the past globally hyperbolic core, this projection is the required causally convex neighborhood. If not, x_0 belongs to the projection of a closed end $\overline{\mathcal{E}}_{xy}$ (cf. Proposition 8.50).

Observe that Γ permutes the gaps, hence $\gamma\overline{\mathcal{E}}_{xy} \cap \overline{\mathcal{E}}_{xy} \neq \emptyset$ implies $\gamma\overline{\mathcal{E}}_{xy} = \overline{\mathcal{E}}_{xy}$. Moreover, in this situation, γ preserves the gap segment $[x, y]$: since the action on this segment must be free and proper, the stabilizer Γ_0 of $\overline{\mathcal{E}}_{xy}$ is a trivial or cyclic group. In the last case, since it admits two non-causally related fixed points in $\widehat{\mathrm{Ein}}_2$, elements in Γ_0 have the form (γ_L, γ_R) where γ_L, γ_R are both hyperbolic (one maybe trivial).

We can be slightly more precise: the projection of $\overline{\mathcal{E}}_{xy}$ is closed. Indeed, let x_n be a sequence in $\overline{\mathcal{E}}_{xy}$, and γ_n a sequence in Γ such that $\gamma_n x_n$ converge to some point $\bar{x}$ in $E(\widetilde{\Lambda})$. Then, $\bar{x}$ belongs to some closed end $\overline{\mathcal{E}}_{x'y'}$ since the complement of the union of closed ends is open (it is the union of the globally hyperbolic convex cores). If $\bar{x}$ is in the interior of $\overline{\mathcal{E}}_{x'y'}$, then $\overline{\mathcal{E}}_{x'y'} = \gamma_n \overline{\mathcal{E}}_{xy}$ for every sufficiently great n. The claim follows. If $\bar{x}$ is on the boundary of $\overline{\mathcal{E}}_{x'y'}$, then it is in the lightcone of some corner point z' of a upper or lower tent. In other words, $\bar{x} = ax' + by' + cz'$ with $a, b > 0, c \geq 0$. But every x_n can be written: $x_n = a_n x + b_n y + c_n z^+ + d_n z^-$. Hence, $\langle z \mid \gamma_n x_n \rangle$ is the sum of the non-positive terms $a_n \langle \gamma_n x \mid z' \rangle$, $b_n \langle \gamma_n y \mid z' \rangle$, $c_n \langle \gamma_n z^+ \mid z' \rangle$ and $d_n \langle \gamma_n z^- \mid z' \rangle$. All these terms have to tend to 0: it follows that $\gamma_n^{-1} z'$ tends to z^+ or z^-. But corner points are isolated in $\widetilde{\Lambda}^{\pm}$. Hence, $\overline{\mathcal{E}}_{x'y'} = \gamma_n \overline{\mathcal{E}}_{xy}$ for every sufficiently great n, and the claim follows in this case too.

Hence, we can associate to every closed end $\overline{\mathcal{E}}_{xy}$ an open neighborhood W_{xy} in $E(\widetilde{\Lambda})$ such that $\gamma W_{xy} \cap W_{xy} \neq \emptyset$ implies $\gamma W_{xy} = W_{xy}$.

Now, $\Omega(\widetilde{\Lambda})$ is contained in $\Omega(x, y) = \Delta_1 \cup \Delta_2$, where we can consider that Δ_1 is the conformal boundary of the end $\overline{\mathcal{E}}_{xy}$. Since (x, y) is a gap pair, $\widetilde{\Lambda}$ is a Γ_0-invariant closed achronal subset contained in the closure of Δ_2. Moreover, since it is non-elementary, $\widetilde{\Lambda}$ has a non-trivial intersection with Δ_1. Since it is achronal, it follows that the hypothesis of Lemma 10.2 are fulfilled: either Γ_0 is a subgroup of G_L or G_R, or x, y are attractive or repulsive fixed points of every element of Γ_0.

Finally, there is a neighborhood W'_{xy} of $\overline{\mathcal{E}}_{xy}$ in $E(x, y)$ such that W_{xy} contains the intersection $W'_{xy} \cap E(\widetilde{\Lambda})$. According to Lemma 10.2, the neighborhood W'_{xy} can be selected so that its projection in $\Gamma_0 \backslash E(x, y)$ is a causally convex domain. Then, the projection of $W'_{xy} \cap E(\widetilde{\Lambda})$ in $M_{\widetilde{\Lambda}}(\Gamma)$ is a causally convex neighborhood of x_0.

Remark 10.4. (The case with torsion) The results above can be extended to the case with torsion: the action of a discrete group Γ on the invisibility domain of a non-elementary achronal subset $\widetilde{\Lambda}$ is properly discontinuous and *strongly causal* in the following meaning:

Definition 10.5. The action of a group $\Gamma \subset \widehat{\mathrm{SO}}_0(2,2)$ on an open subset E of $\widetilde{\mathrm{AdS}}$ is strongly causal if every element x of E admits an open neighborhood U such that, for every element γ of Γ_0, either $\mathbb{ADS}$ is a fixed point of γ, or no element of U is causally related to an element $\gamma(U)$.

The quotient $\Gamma \backslash E(\widetilde{\Lambda})$ is a AdS-spacetimes with singularities, the singularities ("particles") being timelike lines. Observe that when Γ is finitely generated this quotient is finitely covered by a AdS spacetime without singularity, since according to Selberg lemma the discrete group Γ contains then a finite index torsion-free subgroup.

10.4 Existence of invariant achronal subsets

We have proved that in most cases, if a discrete group $\Gamma \subset \widehat{\mathrm{SO}}_0(2,2)$ preserves a generic closed achronal subset $\widetilde{\Lambda}$ containing at least two points, then the action of Γ on $E(\widetilde{\Lambda})$ is proper and strongly causal. We now try to answer to the question: *given a torsion-free discrete subgroup* Γ *of* $\widehat{\mathrm{SO}}_0(2,2)$, *is there a* Γ*-invariant generic closed achronal subset of* $\widehat{\mathrm{Ein}}_2$? According to Lemmas 5.6 and 9.6, in order to preserve such an achronal subset, every element of Γ must be synchronized, and Γ projects injectively in $G \times G$, with $G = \mathrm{PSL}(2,\mathbb{R})$. Moreover, this projection must be faithfull, with discrete image. Hence, Γ has to be the image of some faithfull morphism $\rho : \Gamma \to G \times G$.

We then reformulate the question above in the following way:

Definition 10.6. Let $\rho_L : \Gamma \to G$ and $\rho_R : \Gamma \to G$ two morphisms. The representation $\rho = (\rho_L, \rho_R)$ is admissible if and only if it is faithfull, has discrete image, and lifts to some representation $\tilde{\rho}\colon \Gamma \to (\widetilde{G} \times \widetilde{G})_{/\mathcal{Z}}$ preserving a generic closed achronal subset of $\widehat{\mathrm{Ein}}_2$ containing at least two points.

A ρ-admissible closed subset for an admissible representation ρ is the projection in $\overline{\mathrm{Ein}}_2$ of $\tilde{\rho}$-invariant generic closed achronal subset of $\widehat{\mathrm{Ein}}_2$ containing at least two points.

Problem: characterize admissible representations.

Theorem 10.7. *Let* Γ *be a torsion-free group, and* $\rho : \Gamma \to G \times G$ *a faithfull representation. Then,* ρ *is admissible if and only if one the following occurs:*

(1) The abelian case: $\rho(\Gamma)$ *is a discrete subgroup of* A_{hyp}, A_{ext} *or* A_{par} *where:*
  - $A_{\mathrm{hyp}} = \{(\exp(\lambda\Delta), \exp(\mu\Delta))/\lambda, \mu \in \mathbb{R}\}$,
  - $A_{\mathrm{ext}} = \{(\exp(\lambda\Delta), \exp(\eta H))/\lambda, \eta \in \mathbb{R}\}$,
  - $A_{\mathrm{par}} = \{(\exp(\lambda H), \exp(\lambda H))/\lambda \in \mathbb{R}\}$.

(2) The non-abelian case: *The left and right morphisms* ρ_L, ρ_R *are faithfull with discrete image, and the marked surfaces* $\rho_L(\Gamma)\backslash\mathbb{H}^2$, $\rho_R(\Gamma)\backslash\mathbb{H}^2$ *are homeomorphic, i.e., there is a* Γ*-equivariant homeomorphism* $f : \mathbb{H}^2 \to \mathbb{H}^2$ *satisfying:*

$$\forall \gamma \in \Gamma,\ f \circ \rho_L(\gamma) = \rho_R(\gamma) \circ f$$

Observe that the restriction of an admissible representation to any non-trivial subgroup of Γ is still admissible.

The abelian case needs a study of the elementary case: hence we postpone its proof to [7], and assume from now that Γ is not abelian. The main step in the proof of Theorem 10.7 is to prove:

Proposition 10.8. *If ρ is admissible, then the representations ρ_L, ρ_R are faithfull.*

Proof. Assume by contradiction that the kernel Γ_L of ρ_L is not trivial, and that ρ is admissible. Let $\widetilde{\Lambda}$ be a generic achronal $\tilde{\rho}(\Gamma)$-invariant closed subset of $\widehat{\mathrm{Ein}_2}$. Let $\widetilde{\Lambda}^{\pm}$ be the up and low completions of $\widetilde{\Lambda}$. Recall the description of the left and right foliations $\widehat{\mathcal{G}}_L$, $\widehat{\mathcal{G}}_R$ in Remark 4.10. Then, every $\tilde{\rho}(\gamma)$, for every γ in Γ_L preserves individually every leaf of $\widehat{\mathcal{G}}_L$. Hence, for every such a leaf, the intersection $\widetilde{\Lambda}^{\pm} \cap l$, if non-empty, is a $\tilde{\rho}(\Gamma_L)$-invariant closed interval (this intersection is connected since $\widetilde{\Lambda}^{\pm}$ is a topological circle). The extremities of this interval — maybe reduced to one point — project in $\mathbb{R}P^1_R$ as fixed points for every element of $\rho_R(\Gamma_L)$. Hence, the fixed point set F_R of $\rho_R(\Gamma_L)$ in $\mathbb{R}P^1_R$ is not empty. On the other hand, an element of G with three fixed point in $\mathbb{R}P^1$ is trivial, and the restriction of ρ_R to Γ_L is faithfull, since ρ is faithfull: F_R contains at most two points. The action of $\rho_R(\Gamma)$ on $\rho_R(\Gamma_L)$ permutes these two points, and an instant of reflexion is enough to realize that, since $\tilde{\rho}(\Gamma)$ preserves the chronological orientation, every element of $\rho_R(\Gamma)$ must preserves every element of F_R.

Assume that if F_R is reduced to one point. Let r_0 be the corresponding leaf of the right foliation $\widehat{\mathcal{G}}_R$. The argument above implies that for every leaf l of $\widehat{\mathcal{G}}_L$, $l \cap \widetilde{\Lambda}^{\pm}$ is either empty, either a point in $l \cap r_0$, or a closed interval projecting on the entire $\mathbb{R}P^1_R$. If the last case occurs, then $\widetilde{\Lambda}^{\pm}$ is pure lightlike: it means that $\widetilde{\Lambda}$ is elementary, more precisely, that it is conical or extreme. Actually, the conical case would imply that F_R contains two points (the non-causally related extremities). Hence, $\widetilde{\Lambda}$ is a lightlike segment contained in r_0. The projections in $\mathbb{R}P^1_L$ of the two extremities of this lightlike segment are distinct $\rho_L(\Gamma)$-fixed points. It follows that $\rho(\Gamma)$ is contained in a conjugate of A_{ext}. It is absurd since Γ is not abelian.

Therefore, F_R contains two points. After conjugacy, $\rho_R(\Gamma)$ is contained in the 1-parameter group $\{\exp(\lambda\Delta)/\lambda \in \mathbb{R}\}$. Since Γ is not abelian, it means that ρ_R is not injective too! Apply once more all the arguments above: it follows that $\rho_L(\Gamma)$ admits two distinct fixed points in $\mathbb{R}P^1_L$. In other words, $\rho(\Gamma)$ is contained in the abelian group A_{hyp}. Contradiction. □

Corollary 10.9. *If ρ is admissible, ρ_L and ρ_R have discrete image in G.*

Proof. Since ρ is admissible, $\rho_L(\Gamma)$ has no elliptic element. Hence, if $\rho_L(\Gamma)$ is not discrete, the neutral component of its closure is a the stabilizer of one or two points in $\mathbb{R}P^1_L$. These fixed points are permuted, and actually preserved, by every $\rho_L(\gamma)$. Since ρ_L is faithfull and Γ is not abelian, it means that there is only one fixed point, i.e., $\rho_L(\Gamma)$ is contained in a solvable group conjugate to Aff. It follows that $\rho_R(\Gamma)$ is solvable too, hence contained also up to conjugacy in Aff. The elements of the commutator subgroup $\rho([\Gamma,\Gamma])$, which is not trivial since Γ is not abelian, are parabolic elements. The representation $\rho : [\Gamma,\Gamma] \to G \times G$ is then an admissible representation of an abelian group, case which is studied in [7]. It follows that up to conjugacy $\rho([\Gamma,\Gamma])$ is contained in A_{par}. Since ρ has discrete image, $\rho_L([\Gamma,\Gamma]) \approx \rho_R([\Gamma,\Gamma])$ is a cyclic group, preserving a copy of the affine line $\mathbb{R}$ in $\mathbb{R}P^1_{L,R}$, and acting on this line as translations. But the action by conjugacy of Γ on $[\Gamma,\Gamma]$ induces an action by homotheties on this discrete goup of translations; hence either Γ is contained in A_{par}, or $[\Gamma,\Gamma]$ is trivial. In the later case, Γ is abelian: contradiction. In the former case, we can apply the arguments above: $\rho_L(\Gamma) = \rho_R(\Gamma)$ are cyclic groups of translations. □

Observe that if ρ is admissible, then the future extension $\widetilde{\Lambda}^+$ (for example) of a $\tilde{\rho}(\Gamma)$-invariant closed generic achronal subset is a topological circle which defined a *monotone* map $f : \mathbb{R}P^1_L \to \mathbb{R}P^1_R$ which is equivariant:

$$\forall \gamma \in \Gamma, \quad f \circ \rho_L(\gamma) = \rho_R(\gamma) \circ f$$

Here, by monotone map, we mean a relation which can send a single point in $\mathbb{R}P^1_L$ on a closed segment of $\mathbb{R}P^1_R$, and such that every $f^{-1}(x)$ is a point or a closed segment of $\mathbb{R}P^1_L$. Equivalently, a monotone map is the quotient of a monotone relation $\tilde{f} : \widetilde{\mathbb{R}P}^1_L \to \widetilde{\mathbb{R}P}^1_R$ which commutes with the Galois groups:

$$\tilde{f} \circ \delta_L = \delta_R \circ \tilde{f}$$

The projective line $\mathbb{R}P^1_L$, $\mathbb{R}P^1_R$ has to be considered as the conformal boundary of the hyperbolic plane $\mathbb{H}^2$ on which are acting, respectively, $\rho_L(\Gamma)$, $\rho_R(\Gamma)$. Theorem 10.7 follows from the well-known fact that the existence of a homeomorphism between the marked surfaces $\Sigma_L = \rho_L(\Gamma)\backslash\mathbb{H}^2$, $\Sigma_R = \rho_R(\Gamma)\backslash\mathbb{H}^2$ is equivalent to the existence of an equivariant monotone map as above. □

Remark 10.10. In the non-abelian case, there is a much more elegant and concise formulation of Theorem 10.7, using the notion of bounded Euler cohomology class, which is exactly the obstruction for the existence of a equivariant monotone map semi-conjugating two actions of a given group on the circle (see [24, 25]):

Theorem 10.11. *Let Γ a non-abelian group without torsion, and $\rho : \Gamma \to G \times G$ a faithfull representation. Then, ρ is admissible if and only if the left* *and right representations ρ_L, ρ_R are faithfull discrete representations with the same Euler bounded cohomology class.*

10.5 Minimal invariant achronal subsets

In almost all this section, Γ is a non-abelian group, and $\rho : \Gamma \to G \times G$ an admissible representation.

Definition 10.12. $\overline{\Lambda}(\rho)$ is the closure of the set of attractive fixed points in $P(E)$.

Observe that attractive fixed points in $P(E)$ of elements of G belong to $\overline{\mathrm{Ein}_2}$. Hence, $\overline{\Lambda}(\rho)$ is contained in $\overline{\mathrm{Ein}_2}$.

Theorem 10.13. *Let Γ be a non-abelian torsion-free group, and $\rho : \Gamma \to G \times G$. Then, every $\rho(\Gamma)$-invariant closed subset of $P(E)$ contains $\overline{\Lambda}(\rho)$.*

Corollary 10.14. *Let (Γ, ρ) be pair satisfying the hypothesis of Theorem* 10.13*. Then, $\overline{\Lambda}(\rho)$ is a $\rho(\Gamma)$-invariant generic nonelementary achronal subset of $\overline{Ein_2}$. Furthermore, for every $\rho(\Gamma)$-invariant closed achronal subset Λ in Ein_2, the invisibility domain $E(\Lambda)$ projects injectively in $\overline{\mathbb{ADS}}$ inside $E(\overline{\Lambda}(\rho))$.*

The essential step for the proof of Theorem 10.13 is:

Lemma 10.15. *If $\rho(\Gamma)$ is admissible, and does not preserve a point in $\mathbb{ADS}$, then $\rho(\Gamma)$ is strongly irreducible, i.e., for every finite index subgroup $\Gamma' \subset \Gamma$, there is not $\rho(\Gamma')$-invariant proper projective subspace in $S(E)$.*

Proof. Assume by contradiction that ρ is admissible, and that some finite index subgroup $\Gamma' \subset \Gamma$ preserves a projective subspace $S(F) \subset S(F)$, where $F \neq E$ is a non-trivial linear subspace of E. Observe that $F^{\perp}$ is also preserved by $\rho(\Gamma')$.

If F and $F^{\perp}$ does not contain Q-isotropic vectors, then Γ' is contained in $O(2) \times O(2)$. It is impossible since elements of Γ are non-elliptic synchronized.

Hence, $S(F) \cap \mathrm{Ein}_2$ or $S(F^{\perp}) \cap \mathrm{Ein}_2$ is not empty. Such an intersection is either a $\rho(\Gamma')$-fixed point in $\overline{\mathrm{Ein}_2} \approx \mathbb{R}P^1_L \times \mathbb{R}P^1_R$, a lightlike geodesic, or an invariant round circle ∂x_0^*. The last case is impossible, since x_0 would be a $\rho(\Gamma)$ fixed point, situation that we have excluded by hypothesis. In the two other remaining cases, after switching if necessary the left and right factors, we obtain that $\rho_L(\Gamma')$ admits a global fixed point in $\mathbb{R}P^1_L$. It is impossible since $\rho_L(\Gamma)$ is a non-abelian discrete subgroup of G. □

Proof of Theorem 10.13. We first observe that, according to Theorem 10.7, and since any non abelian discrete subgroup of G admits at least one hyperbolic element, $\rho(\Gamma)$ is proximal, i.e., $\overline{\Lambda}(\rho)$ is non-empty. Then, Theorem 10.13 is an immediate corollary of Lemma 10.15 and Lemma 2.5–(2) of [12]. □

10.6 Convexity and causality

In this Section, we consider a representation $\rho : \Gamma \to G \times G$, where Γ is not abelian.

Definition 10.16. An element of $\mathrm{GL}(E)$ is proximal if its action on $P(E)$ admits an attractive fixed point. It is positively proximal if its action on $S(E)$ has two attractive fixed points (one opposite to the other).

A faithfull representation $\rho : \Gamma \to \mathrm{GL}(E)$ is positively proximal if $\rho(\Gamma)$ contains at least a proximal element, and that every proximal element of $\rho(\Gamma)$ is positively proximal.

The main result of [12] (Propositions 1.1, 1.2) is:

Theorem 10.17. *A strongly irreducible representation is positively proximal if and only if it preserves a proper convex domains in $P(E)$.*

It follows that in our case, if ρ is admissible, then it is positively proximal (observe that this statement is true, even if ρ is not strongly irreducible, i.e., preserves a point in AdS). We wonder here about the inverse statement: is any positively proximal ρ admissible?

In the following, we are using results in [12] which are established for strongly irreducible representations, but these results are easily checked when $\rho(\Gamma) \subset \mathrm{SO}_0(2,2)$ admits a fixed point in AdS.

Let $\mathcal{F}(E)$ be the flag variety, i.e., the space of pairs $([u],[u^*])$ in $P(E) \times P(E^*)$ such that $u^*(u) = 0$. Here, we can of course define $\mathcal{F}(E)$ as the space of pairs of Q-orthogonal elements of $P(E)$: $\mathcal{F}(E) = \{([u],[v]) \in P(E) \times P(E)/\langle u \mid v\rangle = 0\}$. The group $\mathrm{SO}_0(2,2)$ acts naturally on it, by the diagonal action. The closure in $\mathcal{F}(E)$ of the set of attractive fixed points of elements of $\rho(\Gamma)$ is: $\Lambda^{\mathcal{F}} = \{([u],[u]) \in \mathcal{F}(E)/[u] \in \Lambda^{\mathbb{P}}\}$.

Proposition 10.18 (Lemma 2.5 — (3) of [12]). *Any $\rho(\Gamma)$-invariant subset of $\mathcal{F}(E)$ contains $\Lambda^{\mathcal{F}}$.*

Moreover, the statement (3) — d expresses here:

Proposition 10.19. *$\Lambda^{\mathbb{P}} \times \Lambda^{\mathbb{P}}$ contains a dense subset Y, which is* transverse*, i.e., for every $([u],[v])$ in Y, the scalar product $\langle u \mid v\rangle$ is nonzero.*

Observe that in the statement above, we cannot define the sign of $\langle u \mid v\rangle$, since $[u]$, $[v]$ are only elements of $P(E)$. But there is a sign if we lift all the picture in $S(E)$. Define Λ^S as the preimage in $S(E)$ of $P(E)$. It can also be defined as the closure of the set of attractive fixed points of elements of $\rho(\Gamma)$.

Lemma 10.20 (Proposition 3.15 in [12]). *A strongly irreducible representation ρ is positively proximal if and only if the action of $\rho(\Gamma)$ on Λ^S is not minimal. If it is the case, Λ^S is the union of two disjoint minimal closed invariant subset Λ^S_1 and $\Lambda^S_2 = -\Lambda^S_1$.*

Lemma 10.21. *The closed minimal subset Λ^S_1 is* positive, *i.e., we have the following alternative:*

(1) *for every element $([u],[v])$ in $\Lambda^S_1 \times \Lambda^S_1$, we have $\langle u \mid v\rangle \leq 0$, or*
(2) *for every element $([u],[v])$ in $\Lambda^S_1 \times \Lambda^S_1$, we have $\langle u \mid v\rangle \geq 0$.*

Proof. It follows from the proof of Proposition 3.11 in [12]. □

In the first case of the alternative of Lemma 10.21, Λ^S_1 is an achronal closed subset of Ein_2. According to Proposition 10.19, it contains at least a pair of non-causally related points: Λ^S_1 is generic. It follows that ρ is admissible.

But, in the second case, ρ is not admissible! Every pair of points in Λ^S_1 is causally related, and most of these pairs are strictly causally related. The situation can be entirely understood in the light of Remarks 3.1, 4.3: we have to consider $\overline{\mathrm{Ein}}_2$ not as the Klein boundary of $\overline{\mathbb{ADS}}$, but as the Klein boundary of the complementary AdS copy, which is the projection of $\{Q = +1\}$ equipped with the restriction of $-Q$. We obtain the notion of — admissible representations $\rho : \Gamma \to \mathrm{SO}_0(2,2)$: the representations conjugate to admissible representations in the previous meaning by an anti-isometry of E permuting $\{Q > 0\}$ and $\{Q < 0\}$.

Proposition 10.22. *Let Γ be a torsion-free non-abelian group, and $\rho : \Gamma \to \mathrm{SO}_0(2,2)$ a faithfull representation with discrete image. Then, ρ is positively proximal if and only if it is admissible or — admissible.*

11 Cauchy complete globally hyperbolic AdS spacetimes

Let M be a 3-dimensional manifold equipped with a lorentzian metric of constant curvature -1, i.e., locally modeled on AdS. Let $p : \widetilde{M} \to M$ be the universal covering, and Γ the fundamental group of M, considered as the group of desk transformations of p. We recall some basic facts of (G, X)-structure's theory, applied in our context:

- There is a developing map $\mathcal{D} : \widetilde{M} \to \widetilde{\mathrm{AdS}}$, which is a local homeomorphism;
- There is a holonomy morphism $\widetilde{\rho} : \Gamma \to \widehat{\mathrm{SO}}(2,2)$ for which $\mathcal{D}$ is equivariant (here, $\widehat{\mathrm{SO}}(2,2)$ is the isometry group of $\widetilde{\mathrm{AdS}}$). More precisely, for every γ in Γ, $\rho(\gamma) \circ \mathcal{D} = \mathcal{D} \circ \gamma$.

Assume that M is globally hyperbolic, and *Cauchy complete*, i.e., admits a Cauchy surface S such that the restriction of the ambient lorentzian metric on S is a complete Riemannian metric. Then, the preimage $\widetilde{S}$ in $\widetilde{M}$ is a Cauchy surface for $\widetilde{M}$. According to Proposition 7.2, the restriction to $\widetilde{S}$ of the developping map $\mathcal{D}$ is an embedding, and the image $\mathcal{D}(\widetilde{S})$ is the graph of a contracting map from $\mathbb{D}^2$ into $\mathbb{R}$. Since $\widetilde{S}$ is a Cauchy surface of $\widetilde{M}$, the image of $\mathcal{D}$ must be contained in the Cauchy development $T(\mathcal{D}(\widetilde{S})) = \mathcal{C}(\mathcal{D}(\widetilde{S}))$. Since local homeomorphisms between subintervals of $\mathbb{R}$ are always injective, the restriction of $\mathcal{D}$ to any timelike geodesic is injective: it follows that $\mathcal{D}$ is injective.

Now, observe that the holonomy group $\widetilde{\rho}$ preserves $\mathcal{D}(\widetilde{S})$ and $E(\partial\mathcal{D}(\widetilde{S}))$. Hence, according to Theorem 10.1 and Proposition 10.3:

Proposition 11.1. *Any globally hyperbolic AdS spacetime M,with complete Cauchy surface S, embeds isometrically in the spacetime $M_{\partial\widetilde{S}}(\widetilde{\rho}(\Gamma))$, where $\widetilde{\rho} : \Gamma \to \widehat{\mathrm{SO}}(2,2)$ is the holonomy morphism, and $\widetilde{S}$ the image by the developping map of a Cauchy surface in $\widetilde{M}$.*

Recall the notion of maximal global hyperbolicity (Definition 2.18).

Corollary 11.2. *The maximal Cauchy complete globally hyperbolic spacetimes are the quotient spacetimes $M_{\widetilde{\Lambda}}(\Gamma)$, where $\widetilde{\Lambda}$ is a generic achronal topological circle in $\widehat{Ein}_2$, and Γ a discrete torsion-free subgroup of $\widehat{\mathrm{SO}}(2,2)$ preserving $\widetilde{\Lambda}$.*

Proof. In the light of Propositions 10.3 and 11.1, the only remaining point to check is the fact that every spacetime $M_{\widetilde{\Lambda}}(\Gamma)$ is indeed Cauchy complete.

Let $\mathcal{N}(\Lambda)$ be the subset of $\mathcal{T}^+$ formed by pairs (x, y) such that y belongs to the future connected component $C^+(\Lambda)$ of $\partial\mathrm{Conv}(\Lambda) \setminus \Lambda$. Then, x must belong the past connected component $\partial^- E(\Lambda)$ of $\partial E(\Lambda) \setminus \Lambda$ (see Proposition 8.20). Moreover, y must belong to the spacelike part $C^+_{\mathrm{spa}}(\Lambda)$ of $C^+(\Lambda)$.

Equivalently, $\mathcal{N}(\Lambda)$ is the Gauss graph of $C^+_{\mathrm{spa}}(\Lambda)$ (see Definition 7.18; the surface $C^+_{\mathrm{spa}}(\Lambda)$ is actually past-convex, i.e., the roles of x, y has been permuted, but without incidence on the results of Section 7.3).

Consider now the cosmological time τ on $E(\Lambda)$; more precisely, the level set $\Sigma = \tau^{-1}(\pi/4)$. For any p in Σ, the surface formed by points x in the past of p and at AdS-distance $\pi/4$ is concave. Since $\partial^- E(\Lambda)$ is convex, it follows that there is one and only one point $x(p)$ in $\partial^- E(\Lambda)$ at distance $\pi/4$ from p. Let $y(p)$ be the first intersection in the future of p of the future oriented geodesic $(p, x(p))$ with $\partial\mathrm{Conv}(\Lambda)$. Then, $y(p)$ belongs to $C^+_{\mathrm{spa}}(\Lambda)$, and $(x(p), y(p))$ belongs to $\mathcal{N}(\Lambda)$. Hence, p is equal to $\frac{1}{2}(x(p) + y(p))$. Inversely, for every (x, y) in $\mathcal{N}(\Lambda)$, $\frac{1}{2}(x + y)$ belongs to Σ.

We have thus defined a homeomorphism $(x, y) : \Sigma \to \mathcal{N}(\Lambda)$, differentiable almost everywhere[2] such that $p = \frac{1}{2}(x(p) + y(p))$. The norm in AdS of a tangent vector $\frac{1}{2}(u + v)$ of Σ is $\frac{1}{4}(\mid u \mid^2 + \mid v \mid^2) + \frac{1}{2}\langle u \mid v\rangle$, whereas the norm of the image of this tangent vector by the differential of (x, y) is $\frac{1}{4}(\mid u \mid^2 + \mid v \mid^2)$. We therefore obtain as a corollary[3] of Lemma 7.22 the following key point:

The map $(x, y) : \Sigma \to \mathcal{N}(\Lambda)$ decreases the distance.

Consider a Cauchy sequence p_n in Σ. Denote $(x_n, y_n) = (x(p_n), y(p_n))$. Then, according to the key point we have just established, (x_n, y_n) is a Cauchy sequence in $\mathcal{N}(\Lambda)$. According to Lemma 7.21, the π_G-projection of (x_n, y_n) is also a Cauchy sequence in Q_G. But, according to Proposition 7.11, Q_G is complete. Hence, if Δ_n denotes the timelike geodesic containing x_n and y_n, the Δ_n converges to a timelike geodesic Δ_∞. This geodesic intersects $\partial^- E(\Lambda)$ at an unique point x_∞, and intersects $C^+(\Lambda)$ at an unique point y_∞. Then, the x_n converge to x_∞, and the y_n converge to y_∞. It follows that the p_n converge to $p_\infty = \frac{1}{2}(x_\infty + y_\infty)$. This point p_∞ belongs to Σ.

Therefore, the spacelike surface Σ is Cauchy complete. Since it is Γ-invariant, its projection in $M_\Lambda(\Gamma)$ is a spacelike Cauchy complete surface.

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[2]This map is actually C^1. See [11].

[3]The reader can now understand the choice of the factor 1/4 introduced in the definition of the lorentzian metric on $\mathcal{T}$.

According to Proposition 8.21, Σ is also a Cauchy surface in $M_\Lambda(\Gamma)$. The proposition is proved. □

Remark 11.3. An interesting — and difficult — problem is to find a *smooth* Cauchy complete Cauchy surface in $M_\Lambda(\Gamma)$.

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Additionnal comments. Besides all the references specific to BTZ blackholes themselves, this work is based on many ideas present in the unpublished preprint [30]. The elaboration of this paper has been announced in [5] with the title "Limit sets of discrete Lorentzian groups".

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