

Cluster decomposition, T-duality, and gerby CFTs

Simeon Hellerman¹, André Henriques², Tony Pantev³,
Eric Sharpe^{4,5}, and Matt Ando⁶

¹School of Natural Sciences, Institute for Advanced Study,
Princeton, NJ 08540, USA

²Mathematisches Institut, Westfälische Wilhelms-Universität,
48149 Münster, Germany

³Department of Mathematics, University of Pennsylvania,
Philadelphia, PA 19104-6395, USA

⁴Department of Physics, University of Utah, Salt Lake City,
UT 84112, USA

⁵Department of Mathematics, University of Utah, Salt Lake City,
UT 84112, USA

⁶Department of Mathematics, University of Illinois, Urbana,
IL 61801, USA

simeon@ias.edu, henrique@math.uni-muenster.de,
tpantev@math.upenn.edu, ersharpe@math.utah.edu,
mando@math.uiuc.edu

Abstract

In this paper, we study conformal field theories (CFTs) associated to gerbes. These theories suffer from a lack of cluster decomposition, but this problem can be resolved: the CFTs are the same as CFTs for disconnected targets. Such theories also lack cluster decomposition, but

in that form, the lack is manifestly not very problematic. In particular, we shall see that this matching of CFTs, this duality between noneffective-gaugings and sigma models on disconnected targets, is a worldsheet duality related to T-duality. We perform a wide variety of tests of this claim, ranging from checking partition functions at arbitrary genus to D-branes to mirror symmetry. We also discuss a number of applications of these results, including predictions for quantum cohomology and Gromov–Witten theory and additional physical understanding of the geometric Langlands program.

CONTENTS

1	Introduction	754
2	Cluster decomposition	756
3	Review of gerbes	757
4	General decomposition conjecture	759
	4.1 General statement	759
	4.2 Specialization to banded G-gerbes	760
	4.3 Evidence	761
5	Examples	762
	5.1 Trivial G-gerbe	762
	5.2 First banded example	763
	5.3 Closely related example	767
	5.4 First nonbanded example	769
	5.5 Second nonbanded example	776
	5.6 Third nonbanded example	777
	5.7 General comment on partition functions	779
6	Two-dimensional gauge theoretic aspects	780

7	D-branes	781
7.1	Equivariant K -theory	781
7.2	Sheaf theory	782
7.3	A_∞ structures factorize	783
7.4	Closed string states in the B model	784
8	Mirror symmetry	785
9	Noncommutative geometry	787
9.1	Lie groupoids and C^* -algebras	787
9.2	The functor of points philosophy	787
9.3	Non-effective local orbifolds	789
9.4	Finite group gerbes and covering spaces	789
9.5	Cocycle description	790
9.6	The abelian case	791
9.7	The case when the band is trivial	791
9.8	Massless spectrum	792
10	Discrete torsion	793
10.1	First (banded) example	793
10.2	Second example	795
11	T-duality	796
12	Applications	798
12.1	Curve counting predictions	798
12.2	Gauged linear sigma model analyses	801

12.3	Geometric Langlands	802
12.4	Speculations on nonsupersymmetric orbifolds	803
13	Conclusions	804
14	Acknowledgments	806
Appendix A.	Partition functions for disconnected targets	806
Appendix B.	Miscellaneous group theory	808
Appendix C.	Group extensions and twisted K -theory (written by M. Ando)	809
C.1	The category of irreducible representations	810
C.2	Irreducible representations and equivariant vector bundles	810
C.3	Group extensions and equivariant projective bundles	810
C.4	Equivariant K -theory	813
C.5	Borel cohomology	815
	References	816

1 Introduction

In the papers [1–3], the notion of string propagation on stacks was developed. Stacks are very closely related to spaces; so, as strings can propagate on spaces, it is natural to ask whether strings can also propagate on stacks. Briefly, the idea is that nearly every stack has a presentation of the form $[X/G]$, where G is a not-necessarily-finite, not-necessarily-effectively-acting group acting on a space X , and to such presentations, one associates a G -gauged sigma model on X . The basic problem is that presentations of this form are not unique, and the physics can depend strongly on the proposed dictionary. For example, a given stack can have presentations as global quotients by both finite and nonfinite groups; the former leads immediately to a CFT, whereas the latter will give a massive nonconformal theory. The physically meaningful conjecture is that the CFT appearing at

the endpoint of renormalization-group flow is associated to the stack and not a presentation thereof.

Unfortunately, as renormalization group flow cannot be followed explicitly, to check this claim we must rely on computations in examples and indirect tests. To further complicate matters, obvious indirect tests have problems. For example, one of the first things one checks is whether mathematical deformations of the stack match physical deformations of the associated conformal field theory. Even in very simple examples, they do not. This issue and others were addressed in [1–3], where numerous tests of the conjecture were also presented, and other consequences derived, such as mirror symmetry for stacks.

For special kinds of stacks known as gerbes, there is an additional puzzle. Specifically, the naive massless spectrum computation contains multiple dimension zero operators, which manifestly violates one of the fundamental axioms of quantum field theory, known as cluster decomposition. On the face of it, this suggests that there is a significant problem here: either the massless spectrum computation is wrong or, perhaps, there is a fundamental inconsistency in the notion of strings propagating on gerbes. Perhaps, for example, renormalization group flow does not respect these mathematical equivalences, and physics is presentation-dependent.

There is a loophole, however: a sigma model describing string propagation on a disjoint union of several spaces also violates cluster decomposition, but in the mildest possible way. We shall argue in this paper that that is precisely what happens for strings on gerbes: the CFT of a string on a gerbe matches the CFT of a string on a disjoint union of spaces (which may vary in their geometry and background fields), and so both theories are consistent despite violating cluster decomposition. We refer to the claim that the CFT of a gerbe matches the CFT of a disjoint union of spaces as the decomposition conjecture.

We begin in Section 2 by reviewing how multiple-dimension zero operators violate cluster decomposition. In Section 3 we review some basics of gerbes, which will be appearing throughout this paper. In Section 4 we present the general decomposition conjecture. We claim that the CFT of a string on a gerbe is the same as the CFT of a string on a disjoint union of (in general distinct) spaces with B fields; our conjecture specifies the spaces and B fields that appear. Much of the rest of this paper is spent verifying that conjecture. In Section 5 we study several examples of gerbes presented as global quotients by finite noneffectively acting groups. In those examples, we compute partition functions at arbitrary genus and massless spectra and verify the results of the conjecture in each case. In Section 6

we discuss gerbes presented as global quotients by nonfinite noneffectively acting groups, and how the decomposition conjecture can be seen directly in the structure of nonperturbative sectors in those theories. In Section 7 we discuss the way in which this decomposition conjecture can be seen in open strings. The conjecture makes a prediction for equivariant K -theory, which we prove. This conjecture also reflects a known result concerning sheaf theory on gerbes, which we discuss, along with other checks of the open string sector of the topological B model. In Section 8 we use mirror symmetry to give additional checks of the conjecture. Mirror symmetry for gerbes (and other stacks) was worked out in [3]; here, we apply it to gerbes and use it to give another check of the decomposition conjecture. In Section 9 we give some arguments based on noncommutative-geometry that support this conjecture. In Section 10 we discuss discrete torsion in noneffective orbifolds and how it modifies the decomposition conjecture. In Section 11 we discuss why we label this decomposition result a “T-duality.”

In Section 12 we discuss applications of this decomposition. One set of applications is to Gromov–Witten theory. We discuss quantum cohomology, including the analogue of Batyrev’s conjecture for toric stacks outlined in [3], and also discuss precise predictions for Gromov–Witten theory, looking at two examples in more detail. We discuss how these same ideas can be applied to better understand the behavior of ordinary gauged linear sigma models. In particular, this should make it clear that the effects we are studying are not overspecialized, but rather occur relatively commonly. We also discuss how part of the physical picture of the geometric Langlands correspondence can be described using these results and give a physical link between some mathematical aspects of geometric Langlands not discussed in [24]. We also speculate on the possible significance of this decomposition conjecture for the interpretation of nonsupersymmetric orbifolds.

We conclude in Section 13. In appendices we review some standard results on dilaton shifts and group theory which we use in this paper. We also sketch a proof of a result on equivariant K -theory that checks one of the predictions of our decomposition conjecture.

2 Cluster decomposition

In [1–3], several examples of theories with physical fields valued in roots of unity were described. These were (mirrors of) theories in which a noneffectively acting finite group had been gauged; the physical field valued in roots of unity was a twist field for the trivially acting subgroup.

Such theories necessarily fail cluster decomposition, as we shall argue now.

Let Υ be a twist field for a trivially acting conjugacy class in a noneffectively gauged group. Because the group element acts trivially, Υ necessarily has dimension zero and charge zero — it is nearly the identity, except that it does not act trivially on fields. Because of the usual orbifold selection rules,

$$\langle \Upsilon \rangle = 0$$

as discussed in [1].

Now, on a noncompact worldsheet, if cluster decomposition holds, then

$$\langle \Upsilon \mathcal{O} \rangle = \langle \Upsilon \rangle \langle \mathcal{O} \rangle$$

when the two operators are widely separated. On the other hand, by the usual Ward identities, since Υ has dimension zero and the CFT is unitary, correlation functions are independent of the insertion position of Υ ; hence regardless of where we insert Υ , by cluster decomposition we should always have

$$\langle \Upsilon \mathcal{O} \rangle = \langle \Upsilon \rangle \langle \mathcal{O} \rangle.$$

On the other hand, since $\langle \Upsilon \rangle = 0$ by the orbifold quantum symmetry, we would be forced to conclude

$$\langle \Upsilon \mathcal{O} \rangle = 0$$

for any other operator \mathcal{O} , which ordinarily we would interpret as meaning that $\Upsilon = 0$.

For this reason, one does not usually consider theories with multiple dimension zero operators.

The theories described in [1–3] have multiple dimension zero operators, which are not equivalent to the zero operator, hence those theories cannot satisfy cluster decomposition.

3 Review of gerbes

As we shall be discussing gerbes extensively in this paper, let us take a moment to review some basic properties.

Given a manifold M and a finite group G , the easiest way of giving a G -gerbe over M is to give an action of some group H on some manifold X . We require that the stabilizer of any point $x \in X$ be isomorphic to G , and the quotient X/H to be isomorphic to M . Such a G -gerbe is denoted by $[X/H]$. (In particular, the stacky quotient $[X/H]$ knows about the existence of trivially acting G , whereas that information is lost when passing to the

ordinary quotient X/H .) However, not every G -gerbe on M comes from such a construction (at least with H a finite group).

The general way of getting a gerbe is to have an open cover $\{U_i\}$ of M . One must specify $g_{ijk} \in G$ on triple overlaps and $\varphi_{ij} \in \text{Aut}(G)$ on double overlaps, obeying the constraints

$$\varphi_{jk} \circ \varphi_{ij} = \text{Ad}(g_{ijk}) \circ \varphi_{ik} \tag{3.1}$$

on triple overlaps and

$$g_{jkl} g_{ijl} = \varphi_{kl}(g_{ijk}) g_{ikl} \tag{3.2}$$

on quadruple overlaps.

Given a G -gerbe presented as $[X/H]$, we may recover such a cocycle as follows. Pick local sections $s_i : U_i \rightarrow X$ of the projection map $X \rightarrow M$, and elements $h_{ij} \in H$ that transform s_i into s_j on double overlaps. We then have $\varphi_{ij} = \text{Ad}(h_{ij})$ and $g_{ijk} = h_{jk} h_{ij} h_{ik}^{-1}$.

In [4] an analogous description of banded $U(1)$ gerbes was given; here, we are considering G -gerbes for G finite but not necessarily banded.

Let $\text{Out}(G)$ be the quotient of $\text{Aut}(G)$ by the automorphisms of the form $\text{Ad}(g)$. Then the data above define a principal $\text{Out}(G)$ -bundle. Indeed, letting ϕ_{ij} be the images of φ_{ij} under the map $\text{Aut}(G) \rightarrow \text{Out}(G)$, we get from equation (3.1) that

$$\phi_{jk} \phi_{ij} = \phi_{ik}.$$

When that $\text{Out}(G)$ -bundle is trivial, we shall say that the gerbe is banded. Equivalently, a gerbe is banded when all the φ_{ij} can be taken in the group $\text{Inn}(G)$ of inner automorphisms of G . If G is abelian, this just means that $\varphi_{ij} = 1$, and we then see from (3.2) that a banded G -gerbe corresponds to an element of $H^2(M, G)$. (Note that we are implicitly assuming G is finite.)

Given a G -gerbe of the form $[X/H]$, there is always a short exact sequence

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1.$$

If the induced homomorphism $K \rightarrow \text{Out}(G)$ is trivial, then the gerbe is banded. For abelian G , this condition then just means that G is central in H .

We shall also be interested on gerbes on orbifolds. Given an effective orbifold $M = [X/K]$ and a short exact sequence as above, the noneffective orbifold $[X/H]$ is a G -gerbe over M . Again, this does not cover all the examples, and the same discussion about cocycles could be carried out in this new context.

4 General decomposition conjecture

4.1 General statement

Suppose we have a G -gerbe over M , presented as $[X/H]$, where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1.$$

Let \hat{G} denote the set of irreps of G . Then we get a natural action of K on \hat{G} given as follows. Given $k \in K$ and $\rho \in \hat{G}$, $\rho : G \rightarrow U(V)$, we pick a lift $h \in H$ of k and define $k \cdot \rho$ to be the representation $g \mapsto \rho(h^{-1}gh) \in U(V)$. If G is abelian, this is clearly well defined since any other lift h' will be of the form $h' = g'h$, and thus $h'^{-1}gh' = h^{-1}gh$. If G is not abelian, this is a little bit more subtle. Given another lift h' of k , we now write it as $h' = hg''$. The representations $g \mapsto \rho(h^{-1}gh)$ and $g \mapsto \rho(h'^{-1}gh')$ are then intertwined by the operator $\rho(g'') : V \rightarrow V$ and thus equal in \hat{G} .

Given a G -gerbe presented as $[X/H]$, define $Y = [(X \times \hat{G})/K]$, where K acts on X in its standard effective way, and acts on \hat{G} in the way described above. Thus, Y will be a disconnected union of spaces and effective orbifolds. More precisely, Y has as many connected components as there are orbits of K on \hat{G} . If K_1, \dots, K_n are the stabilizers in K of these various orbits, then we have

$$Y = [X/K_1] \sqcup \dots \sqcup [X/K_n].$$

Furthermore, we can define a natural flat $U(1)$ -gerbe on Y as follows. For each $\rho \in \hat{G}$, we fix a vector space V_ρ with a representation of G corresponding to ρ . We can then build a vector bundle $E \rightarrow X \times \hat{G}$ such that the restriction to each $X \times \{\rho\}$ is a trivial vector bundle with fiber V_ρ . Even though dimensions differ across components, this is nevertheless a vector bundle because the fibers are the same on each individual component. Clearly, G acts on E : it acts on X by definition, it acts on \hat{G} by the trivial action, and it acts on each fiber V_ρ via ρ . But there is an obstruction to making that vector bundle H -equivariant. That obstruction is exactly the $U(1)$ -gerbe we are discussing. Namely, the group H acts on the base $X \times \hat{G}$, but not necessarily on the fibers of the bundle E . If we try to extend the action of G to an action of H , we have to make some noncanonical choices. The resulting isomorphisms will commute only up to phase (using Schur's lemma), and from there we get our $U(1)$ -gerbe.

We can be more precise. Each component $[X/K_i]$ of Y will come with some discrete torsion coming from a $U(1)$ -valued 2-cocycle on K_i . These

cocycles are computed as follows. Let ρ_1, \dots, ρ_n be elements of \hat{G} representing each one of the various K -orbits. Let K_i be the stabilizer of ρ_i and H_i be its preimage in H , so that we have exact sequences

$$1 \longrightarrow G \longrightarrow H_i \longrightarrow K_i \longrightarrow 1.$$

Pick a section $s : K_i \rightarrow H_i$, and let us fix a vector space V_i on which G acts by ρ_i . The fact that K_i fixes ρ_i means that the representations $g \mapsto \rho_i(g)$ and $g \mapsto \rho_i(h^{-1}gh)$ are equivalent for all $h \in H_i$. For h of the form $s(k)$, let us pick a map $\tilde{\rho}_i(k) : V_i \rightarrow V_i$ that intertwines the above two representations. Using Schur's lemma, one then checks that the map $gs(k) \mapsto \rho_i(g)\tilde{\rho}_i(k)$ is a projective representation of H_i on V_i . Moreover, since G has an honest action on V_i , the 2-cocycle on H_i actually comes from a 2-cocycle on K_i . This cocycle can be given explicitly by the formula

$$c(k, k') = \tilde{\rho}_i(k)\tilde{\rho}_i(k')\tilde{\rho}_i(kk')^{-1}\rho_i\left(s(k)s(k')s(kk')^{-1}\right)^{-1}.$$

The decomposition conjecture says that physically, the CFT of a string on the gerbe is the same as the CFT of a string on Y , and so is the same as a string on copies and covers of $M = [X/K]$, with variable B fields (using the relation between B fields and discrete torsion established in [5, 6]).

We have stated this conjecture in a form that appears presentation-dependent, but in fact the decomposition is independent of the choice of presentation. A presentation-independent description of Y can be given as follows. As explained in the previous section, a G -gerbe on M induces an $\text{Out}(G)$ -principal bundle $P \rightarrow M$. Since $\text{Out}(G)$ acts on \hat{G} , we can take the associated bundle

$$Y := P \times_{\text{Out}(G)} \hat{G}.$$

The gerbe on Y is then the obstruction to finding a vector bundle whose fibers are the given representations of the ineffective stabilizer G .

4.2 Specialization to banded G -gerbes

In the special case of banded gerbes, the general conjecture above simplifies somewhat. In particular, the K action on \hat{G} is trivial. (For example, if G lies in the center of H , the adjoint action of K on G is trivial, and so is its action on \hat{G} .) In this case, $[(X \times \hat{G})/K] \cong [X/K] \times \hat{G}$. Thus, our conjecture becomes the statement that the CFT of a banded G -gerbe over M (where M can be either a manifold or an effective orbifold; in the notation of the previous section, $M = [X/K]$) is T-dual to a disjoint union of CFTs, each corresponding to a CFT on M with a flat B field. The disjoint union is over irreducible representations of G , and the flat B field is determined by

the irreducible representation, which defines a map $Z(G) \rightarrow U(1)$ and hence induces a map from the characteristic class of the gerbe to flat B fields:

$$H^2(M, Z(G)) \longrightarrow H^2(M, U(1)).$$

In short,

$$\text{CFT}(\text{banded } G\text{-gerbe on } X) \xleftarrow{\text{T-dual}} \coprod_{\text{irreps of } G} \text{CFT}(X \text{ with flat } B \text{ field})$$

4.3 Evidence

We shall see several types of evidence for this conjecture in this paper:

1. For gerbes presented as global quotients by finite (noneffectively acting) groups, in Section 5 we describe some completely explicit calculations of partition functions (at arbitrary genus), massless spectra, and operator products that support the conjecture. For example, for $[T^6/D_4]$, where the \mathbf{Z}_2 acts trivially, a \mathbf{Z}_2 -gerbe over $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$, we shall see that the partition function of the $[T^6/D_4]$ orbifold is a sum of the partition functions of the $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$ orbifolds with and without discrete torsion, and also the massless spectrum of the $[T^6/D_4]$ orbifold is a sum of the massless spectra of the $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$ orbifolds with and without discrete torsion. Since discrete torsion is just a choice of B field on the orbifold, as described in detail in [5, 6], this example nicely illustrates the conjecture.
2. For gerbes presented as global quotients by nonfinite groups, we can see this decomposition directly in the structure of the nonperturbative sectors which distinguish the physical theory on the gerbe from the physical theory on the underlying space, as discussed in Section 6.
3. At the open string level, D-branes on gerbes decompose according to the irreducible representation of the band, and there are only massless open string states between D-branes in the same irreducible representation. One mathematical consequence is a statement about K -theory, whose proof is sketched in an appendix. Mathematically, this also corresponds to a standard result that sheaves on gerbes decompose into sheaves on copies of the underlying space (and covers thereof) but twisted by a flat B field determined as above. We also discuss how the A_∞ algebra of open string states in the topological B model decomposes, and the implications of that statement for the closed string states. This is discussed in Section 7.
4. Mirror symmetry for gerbes, as described in [1–3], generates Landau–Ginzburg models with discrete-valued fields, which can be

interpreted in terms of a sum over components, consistent with the decomposition conjecture. In particular, we shall see in the case of the gerby quintic that it maps banded gerbes to sums of components in which the complex structure has shifted slightly between components, so on the original side the CFT must describe a sum of components in which the complexified Kähler data has been shifted, and using the mirror map we find that it is only the B field that shifts, not the real Kähler form. This is discussed in Section 8.

5. We also have some arguments based in noncommutative geometry for this same decomposition, see Section 9.
6. From thinking about general aspects of T-duality, we get more arguments in favor of this conjecture, see Section 11.
7. From the structure of quantum cohomology rings, we get more evidence still. In particular, in [3] it was observed how Batyrev's conjecture for quantum cohomology of toric varieties naturally generalizes to toric stacks, and we find the conjectured decomposition present implicitly in that conjecture for quantum cohomology rings. This is discussed in Section 12, along with applications of these ideas.

5 Examples

5.1 Trivial G -gerbe

Consider the trivial G -gerbe over a space X . This gerbe can be presented as $[X/G]$, where all of G act trivially on X . (Also, since we assume throughout that all stacks are Deligne–Mumford, G is necessarily finite, though not necessarily abelian.)

Let us check that the massless spectrum contains precisely one copy of the cohomology of X for each irreducible representation of G , as is predicted by our general conjecture.

As in a standard effective orbifold, the twisted sectors are counted by conjugacy classes of G . Since G acts trivially, each twisted sector contains the same set of states, namely, a copy of the cohomology of X . Moreover, the number of conjugacy classes is the same as the number of irreducible representations. Thus, the massless spectrum contains precisely one copy of the cohomology of X for each irreducible representation of G .

In passing, a beautiful discussion of the twist fields in such orbifolds can be found in [7, Exercise 10.18].

5.2 First banded example

Consider the orbifold $[X/D_4]$, where the \mathbf{Z}_2 center of D_4 acts trivially. Recall there is a short exact sequence

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow D_4 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1.$$

Let the elements of D_4 be denoted by

$$\{1, z, a, b, az, bz, ab, ba = abz\},$$

where $a^2 = 1$, $b^2 = z$, and z lies in the center \mathbf{Z}_2 . Let the projection to $\mathbf{Z}_2 \times \mathbf{Z}_2$ be denoted a bar, so that the elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ are

$$\{1, \bar{a}, \bar{b}, \bar{ab}\}$$

and the projection map sends, for example,

$$a, az \mapsto \bar{a}.$$

According to our decomposition conjecture, the CFT of this noneffective orbifold should be the same as the two copies of $[X/(\mathbf{Z}_2 \times \mathbf{Z}_2)]$, one copy with discrete torsion, the other without. After all, this is a banded \mathbf{Z}_2 -gerbe, so since \mathbf{Z}_2 has two different irreducible representations, the CFT of the gerbe should involve two copies of the underlying effective orbifold $[X/(\mathbf{Z}_2 \times \mathbf{Z}_2)]$. Furthermore, since the \mathbf{Z}_2 -gerbe is nontrivial, under those two irreducible representations it induces one vanishing and one nonvanishing element of $H^2([X/(\mathbf{Z}_2 \times \mathbf{Z}_2)], U(1))$, and the nonvanishing element corresponds to discrete torsion in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold [5]. In short, the prediction of the decomposition conjecture is that

$$\text{CFT}([X/D_4]) = \text{CFT}([X/(\mathbf{Z}_2 \times \mathbf{Z}_2)]_{\text{d.t.}}) \sqcup [X/(\mathbf{Z}_2 \times \mathbf{Z}_2)]_{\text{w/od.t.}}.$$

Next, we shall verify that prediction by calculating partition functions (at arbitrary genus) and massless spectra.

Following [1], the one-loop partition function of this orbifold is

$$Z(D_4) = \frac{1}{|D_4|} \sum_{g,h \in D_4; gh=hg} Z_{g,h},$$

and each of the $Z_{g,h}$ twisted sectors that appears is the same as a $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector appearing with multiplicity $|\mathbf{Z}_2|^2 = 4$, except for the

$$\bar{a} \begin{array}{|c|} \hline \square \\ \hline \bar{b} \\ \hline \end{array}, \quad \bar{a} \begin{array}{|c|} \hline \square \\ \hline \bar{ab} \\ \hline \end{array}, \quad \bar{b} \begin{array}{|c|} \hline \square \\ \hline \bar{ab} \\ \hline \end{array}$$

twisted sectors of $[X/(\mathbf{Z}_2 \times \mathbf{Z}_2)]$. Thus, the partition function of the D_4 orbifold can be expressed as

$$\begin{aligned} Z(D_4) &= \frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|}{|D_4|} |\mathbf{Z}_2|^2 (Z(\mathbf{Z}_2 \times \mathbf{Z}_2) - (\text{some twisted sectors})) \\ &= 2 \cdot (Z(\mathbf{Z}_2 \times \mathbf{Z}_2) - (\text{some twisted sectors})). \end{aligned}$$

Now, we claim that this partition function is the sum of the one-loop partition functions for the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with and without discrete torsion. Recall from [8] that if we let ζ denote a generator of the k th roots of unity, and let the boundary conditions in a $\mathbf{Z}_k \times \mathbf{Z}_k$ orbifold be determined by the pair of group elements

$$\begin{aligned} T_\sigma &= (\zeta^a, \zeta^b), \\ T_\tau &= (\zeta^{a'}, \zeta^{b'}), \end{aligned}$$

then the k possible discrete torsion phases in the $\mathbf{Z}_k \times \mathbf{Z}_k$ orbifold are given by

$$\epsilon(T_\sigma, T_\tau) = \zeta^{m(ab' - ba')}$$

for $m \in \{0, 1, \dots, k - 1\}$. In the present case, this means the one-loop twisted sectors in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold that are multiplied by a phase are precisely

$$\bar{a} \begin{array}{|c|} \hline \square \\ \hline \bar{b} \end{array}, \quad \bar{a} \begin{array}{|c|} \hline \square \\ \hline ab \end{array}, \quad \bar{b} \begin{array}{|c|} \hline \square \\ \hline ab \end{array}$$

the same sectors that were omitted in the D_4 orbifold, and moreover, turning on discrete torsion in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold means multiplying each of the sectors above by a sign.

Thus, if we add the one-loop partition functions of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with and without discrete torsion, then it is trivial to see that we recover the one-loop partition function of the D_4 orbifold.

It is straightforward to repeat this calculation at two-loops. Let us follow the notation of [5, Section 4.3.2]. Each two-loop sector is defined by four group elements $(g_1|h_1|g_2|h_2)$, obeying the constraint

$$h_1 g_1^{-1} h_1^{-1} g_1 = g_2^{-1} h_2 g_2 h_2^{-1}.$$

Let us organize the calculation according to $\mathbf{Z}_2 \times \mathbf{Z}_2$ sectors, as before. For example, the $(\bar{a}|\bar{a}|\bar{a}|\bar{a})$ $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector is the image of the following D_4 sectors:

$$(a, az|a, az|a, az|a, az).$$

The $(\bar{a}|\bar{a}|\bar{a}|\bar{b})$ $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector, on the other hand, cannot arise from any D_4 sector. More generally, it is straightforward to check that if the pair (g_1, h_1) defines one of the excluded one-loop $\mathbf{Z}_2 \times \mathbf{Z}_2$ sectors, or if the pair (g_2, h_2) defines an excluded one-loop sector, but not both, then the set of four group

elements cannot be lifted to a consistent set of four group elements defining a two-loop D_4 sector. However, if either neither pair is excluded, or both are excluded, then the two-loop $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector does lift to two-loop D_4 sectors, in fact lifts to $2^4 = 16$ possible two-loop D_4 sectors. Thus, we can write

$$Z_{2\text{-loop}}(D_4) = \frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|^2}{|D_4|^2} |\mathbf{Z}_2|^4 (Z_{2\text{-loop}}(\mathbf{Z}_2 \times \mathbf{Z}_2) - (\text{some sectors})).$$

Furthermore, we can analyze the effects of discrete torsion as before. Because the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold is abelian and so factorizes, we can write

$$\epsilon(g_1, h_1; g_2, h_2) = \epsilon(g_1, h_1)\epsilon(g_2, h_2)$$

for the two-loop discrete-torsion phase factor, due to factorization. Thus, two-loop sectors with the property that either (g_1, h_1) or (g_2, h_2) at one-loop would have got a sign factor, but not both, will get a sign at two-loops, whereas other sectors remain invariant. So, summing over two-loop partition functions for $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds with and without discrete torsion will yield

$$2 \cdot (Z_{2\text{-loop}}(\mathbf{Z}_2 \times \mathbf{Z}_2) - (\text{some sectors})),$$

where the pattern of omitted sectors is identical. More generally, it should now be straightforward to see that at g -loops,

$$Z_{g\text{-loop}}(D_4) = \frac{|\mathbf{Z}_g \times \mathbf{Z}_2|^g}{|D_4|^2} |\mathbf{Z}_2|^{2g} (Z_{g\text{-loop}}(\mathbf{Z}_2 \times \mathbf{Z}_2) - (\text{some sectors})),$$

where the overall numerical factor is

$$\frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|^g}{|D_4|^2} |\mathbf{Z}_2|^{2g} = 2^g = 2 \cdot 2^{g-1}.$$

In other words, the g -loop partition function can be written

$$Z_{g\text{-loop}}(D_4) = (\sqrt{2})^{2g-2} \cdot Z_{g\text{-loop}} \\ \times (\text{two } \mathbf{Z}_2 \times \mathbf{Z}_2 \text{ orbifolds, w/ and w/o discrete torsion})$$

The overall factor of $(\sqrt{2})^{2g-2}$ can be absorbed into a dilaton shift (see Appendix A.), and so is of no physical significance. So, the partition function of the D_4 orbifold matches that of the two $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds, with and without discrete torsion, at arbitrary genus, exactly as predicted by the decomposition conjecture.

Next let us compute the massless spectrum in a specific example. Take $X = T^6$, with the same $\mathbf{Z}_2 \times \mathbf{Z}_2$ action described in [8]. The group D_4 has five conjugacy classes, namely,

$$\{1\}, \{z\}, \{a, az\}, \{b, bz\}, \{ab, ba\}$$

in the notation of the appendix, hence there are five sectors in the space of massless states. From [8], the D_4 -invariant (same as $\mathbf{Z}_2 \times \mathbf{Z}_2$ -invariant)

massless states in the untwisted sector of the Hilbert space are given by

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & 0 & & 0 \\
 & & & 0 & 3 & & 0 \\
 & & 1 & 3 & 3 & 3 & 1 \\
 & & 0 & 3 & & & 0 \\
 & & & 0 & & 0 & \\
 & & & & 1 & &
 \end{array}$$

and from the $\{z\}$ sector we get another copy of the same set of states, since z acts trivially. In each of the three remaining twisted sectors, we get sixteen copies of that part of

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & 0 & & 0 \\
 & & & 0 & 1 & & 0 \\
 & & 0 & 1 & 1 & 1 & 0 \\
 & & 0 & 1 & & & 0 \\
 & & & 0 & & 0 & \\
 & & & & 0 & &
 \end{array}$$

which is invariant under the centralizer of the a representative of the conjugacy class defining the twisted sector. However, the centralizer of each of a , b , ab consists only of $1, z$, the group element itself, and the other element of its conjugacy class, but from the analysis of [8] it is clear that each of the states above is invariant under all of the corresponding centralizers. Thus, in the noneffective D_4 orbifold, we have a grand total of

$$\begin{array}{ccccccc}
 & & & & 2 & & \\
 & & & & 0 & & 0 \\
 & & & 0 & 54 & & 0 \\
 & & 2 & 54 & 54 & 54 & 2 \\
 & & 0 & 54 & & & 0 \\
 & & & 0 & & 0 & \\
 & & & & 2 & &
 \end{array}$$

massless states.

Next let us compare to the massless spectrum of a CFT given by the sum of the CFTs corresponding to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with and without discrete torsion. From [8] the massless spectrum of the original $\mathbf{Z}_2 \times \mathbf{Z}_2$

orbifold, without discrete torsion, is given by

$$\begin{array}{cccc}
 & & 1 & \\
 & & 0 & 0 \\
 & 0 & 51 & 0 \\
 1 & 3 & & 3 & 1 \\
 & 0 & 51 & 0 \\
 & 0 & 0 & \\
 & & 1 &
 \end{array}$$

and the massless spectrum of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with discrete torsion turned on is given by

$$\begin{array}{cccc}
 & & 1 & \\
 & & 0 & 0 \\
 & 0 & 3 & 0 \\
 1 & 51 & 51 & 1 \\
 & 0 & 3 & 0 \\
 & 0 & 0 & \\
 & & 1 &
 \end{array}$$

so the disjoint sum of these two CFTs would have a grand total of

$$\begin{array}{cccc}
 & & 2 & \\
 & & 0 & 0 \\
 & 0 & 54 & 0 \\
 2 & 54 & 54 & 2 \\
 & 0 & 54 & 0 \\
 & 0 & 0 & \\
 & & 2 &
 \end{array}$$

massless states. Note that this exactly matches the result of the massless spectrum computation in the D_4 orbifold, hence confirming our conjecture.

One can also extract from this computation that the decomposition of the noneffective orbifold into a sum of theories is not precisely the same thing as a decomposition by twisted sectors. For example, states in the $\{a, az\}$ twisted sector contribute to both the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds, with and without discrete torsion.

5.3 Closely related example

Another \mathbf{Z}_2 -gerbe over the orbifold $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$ is given by $[T^6/\mathbf{H}]$, where \mathbf{H} is the nonabelian eight-element finite group, whose elements are the

unit quaternions:

$$\mathbf{H} = \{\pm 1, \pm i, \pm j, \pm k\}.$$

This gerbe is distinct from the \mathbf{Z}_2 -gerbe over $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$ given by $[T^6/D_4]$, though it is closely related.

In particular, although the gerbe is distinct, the prediction of the decomposition conjecture is the same in this case as for $[T^6/D_4]$, namely,

$$\text{CFT}([T^6/\mathbf{H}]) = \text{CFT}([T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]_{\text{d.t.}} \sqcup [T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]_{\text{w/o d.t.}}).$$

The derivation from the decomposition conjecture is identical to the last example, so we omit it here.

In this subsection, we shall check that the one-loop partition function and massless spectrum of the noneffective \mathbf{H} orbifold match that of the D_4 orbifold, which in turn describes the CFT of a sum of $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds with and without discrete torsion. Thus, although the gerbes are distinct, their physical descriptions are identical.

The group \mathbf{H} fits into a short exact sequence just like the one for D_4 :

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathbf{H} \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1.$$

First, let us consider the one-loop partition function of the \mathbf{H} orbifold. Just as in the D_4 orbifold, each $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector that appears, appears with multiplicity $|\mathbf{Z}_2|^2 = 4$, reflecting the fact that there is a 2-fold ambiguity in lifts of elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$, and that the ambiguity is by elements of the center of \mathbf{H} .

Thus, schematically the one-loop partition function of the \mathbf{H} orbifold is given by

$$\begin{aligned} Z_{1\text{-loop}}(\mathbf{H}) &= \frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|}{|\mathbf{H}|} |\mathbf{Z}_2|^2 (Z_{1\text{-loop}}(\mathbf{Z}_2 \times \mathbf{Z}_2) - (\text{some sectors})) \\ &= 2 \cdot (Z_{1\text{-loop}}(\mathbf{Z}_2 \times \mathbf{Z}_2) - (\text{some sectors})). \end{aligned}$$

Denote the elements of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by $\{1, \bar{i}, \bar{j}, \bar{k}\}$. Then, since for example $\pm i \in \mathbf{H}$ can only commute with ± 1 and $\pm i$, we see that the following $\mathbf{Z}_2 \times \mathbf{Z}_2$ twisted sectors are omitted from the one-loop partition function of the \mathbf{H} orbifold:

$$\bar{i} \begin{array}{|c|} \hline \square \\ \hline \bar{j} \end{array}, \quad \bar{i} \begin{array}{|c|} \hline \square \\ \hline \bar{k} \end{array}, \quad \bar{j} \begin{array}{|c|} \hline \square \\ \hline \bar{k} \end{array}.$$

These are the same one-loop sectors that were omitted in the D_4 case, and the multiplicative factors are the same between this case and the D_4 case;

hence we see that

$$Z_{1\text{-loop}}(\mathbf{H}) = Z_{1\text{-loop}}(D_4),$$

the one-loop partition function of the \mathbf{H} orbifold matches that of the D_4 orbifold, and hence can also be written as a sum of the one-loop partition functions of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds with and without discrete torsion.

Next, let us check the massless spectrum of the noneffective \mathbf{H} orbifold, and compare it to the noneffective D_4 orbifold. The group \mathbf{H} has five conjugacy classes, given by

$$\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$$

The contribution to the massless spectrum of the noneffective \mathbf{H} orbifold from the conjugacy classes above matches the contribution to the massless spectrum of the D_4 orbifold from the

$$\{1\}, \{z\}, \{a, az\}, \{b, bz\}, \{ab, ba\}$$

conjugacy classes of D_4 , respectively. Hence the massless spectrum of the noneffective \mathbf{H} orbifold matches that of the noneffective D_4 orbifold, and in particular, is a sum of the massless spectra of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds with and without discrete torsion.

5.4 First nonbanded example

5.4.1 Partition function analysis

Let \mathbf{H} denote the eight-element group of quaternions, i.e.,

$$\mathbf{H} = \{\pm 1, \pm i, \pm j, \pm k\}.$$

Consider $[X/\mathbf{H}]$, where the subgroup $\langle i \rangle \cong \mathbf{Z}_4$ acts trivially. There is a short exact sequence

$$1 \longrightarrow \langle i \rangle \longrightarrow \mathbf{H} \longrightarrow \mathbf{Z}_2 \longrightarrow 1,$$

but note that this is not a central extension — $\langle i \rangle$ does not lie in the center of \mathbf{H} .

First, let us work out the prediction of our decomposition conjecture for this case. The group $K = \mathbf{Z}_2$ acts on $G = \langle i \rangle = \mathbf{Z}_4$ by conjugation, and for the purposes of computing actions on irreducible representations, we can just consider conjugation by j representing a coset in $\mathbf{H}/\langle i \rangle$. Under such conjugation, $\pm 1 \mapsto \pm 1$ and $\pm i \mapsto \mp i$. Now, $\langle i \rangle$ has four irreducible representations, and under this action of K , two of the irreducible representations are invariant and two are exchanged. The two invariant irreducible representations correspond to two copies of $[X/\mathbf{Z}_2]$ in the decomposition, whereas the two remaining ones intertwine $[X/\mathbf{Z}_2]$ to get a 2-fold cover, which will

be X . (After all, that component is given by $[(X \times \mathbf{Z}_2)/\mathbf{Z}_2]$ and since the \mathbf{Z}_2 acts nontrivially on the \mathbf{Z}_2 , this is just X .) Furthermore, one can show that the B fields on each component will be trivial. Thus, the decomposition conjecture predicts

$$\text{CFT}([X/\mathbf{H}]) = \text{CFT}([X/\mathbf{Z}_2] \sqcup [X/\mathbf{Z}_2] \sqcup X).$$

We shall check this statement at the level of partition functions and operators.

This example was discussed in Section 2.0.4 of [1], and we argued there that in terms of \mathbf{Z}_2 path integral twisted sectors, the one-loop partition function has the form

$$Z([X/\mathbf{H}]) = \frac{1}{|\mathbf{H}|} \left((16)_1 \square_1 + (8)_1 \square_\xi + (8)_\xi \square_\xi \right),$$

where ξ denotes the generator of the effectively acting \mathbf{Z}_2 . This can be written as

$$Z([X/\mathbf{H}]) = (2)Z([X/\mathbf{Z}_2]) + Z(X)$$

from which we conclude that the CFT of a string on this nonbanded gerbe is equivalent to the CFT of a string on the disjoint union of two copies of the orbifold $[X/\mathbf{Z}_2]$ and one copy of X .

Let us also work through a two-loop computation, following the conventions of [5, Section 4.3.2]. Each two-loop sector is defined by four group elements $(g_1|h_1|g_2|h_2)$ obeying the constraint

$$h_1 g_1^{-1} h_1^{-1} g_1 = g_2^{-1} h_2 g_2 h_2^{-1}.$$

Let us organize the calculation according to \mathbf{Z}_2 twisted sectors. For example, the $(1|1|1|1)$ two-loop sector in the \mathbf{Z}_2 orbifold comes from the

$$(\pm 1, \pm i | \pm 1, \pm i | \pm 1, \pm i | \pm 1, \pm i)$$

two-loop sectors in the noneffective \mathbf{H} orbifold. The $(1|1|1|\xi)$ two-loop sector in the \mathbf{Z}_2 orbifold (where ξ generates \mathbf{Z}_2) comes from the

$$(\pm 1, \pm i | \pm 1, \pm i | \pm 1 | \pm j, \pm k)$$

two-loop sectors in the \mathbf{H} orbifold. However, because the group constraint is not satisfied (because there is no corresponding principal \mathbf{H} -bundle),

there are no

$$(\pm 1, \pm i | \pm 1, \pm i | \pm i | \pm j, \pm k)$$

sectors. Similarly, the $(1|1|\xi|\xi)$ \mathbf{Z}_2 sectors arise from the

$$\begin{aligned} &(\pm 1, \pm i | \pm 1, \pm i | \pm j | \pm j), \\ &(\pm 1, \pm i | \pm 1, \pm i | \pm k | \pm k) \end{aligned}$$

\mathbf{H} sectors, the $(1|\xi|1|\xi)$ \mathbf{Z}_2 sectors arise from the

$$\begin{aligned} &(\pm 1 | \pm j, \pm k | \pm 1 | \pm j, \pm k), \\ &(\pm i | \pm j, \pm k | \pm i | \pm j, \pm k) \end{aligned}$$

\mathbf{H} sectors, and the $(1|\xi|\xi|\xi)$ \mathbf{Z}_2 sectors arise from the

$$\begin{aligned} &(\pm 1 | \pm j, \pm k | \pm j | \pm j), \\ &(\pm 1 | \pm j, \pm k | \pm k | \pm k), \\ &(\pm i | \pm j, \pm k | \pm j | \pm k), \\ &(\pm i | \pm j, \pm k | \pm k | \pm j) \end{aligned}$$

\mathbf{H} sectors. We shall not list other cases here, but the analysis should be clear to the reader. As a result, we can write

$$Z_{2\text{-loop}}(\mathbf{H}) = \frac{1}{|\mathbf{H}|^2} (4^4(\text{untwisted sector}) + 2 \cdot 4^3(\text{all other sectors})).$$

It should now be straightforward to check that more generally,

$$\begin{aligned} Z_{g\text{-loop}}([X/\mathbf{H}]) &= \frac{1}{|\mathbf{H}|^g} (4^{2g}(\text{untwisted sector}) + 2 \cdot 4^{2g-1}(\text{all other sectors})) \\ &= \frac{4^{2g-1}}{8g} (2Z_g(X) + 2(\text{sum over all sectors})) \\ &= 2^{g-2} (2Z_g(X) + 2|\mathbf{Z}_2|^g Z_g([X/\mathbf{Z}_2])) \\ &= (\sqrt{2})^{2g-2} Z_g(X) + 2^{2g-2} \cdot 2Z_g([X/\mathbf{Z}_2]). \end{aligned}$$

We can get rid of numerical factors of the form N^{2g-2} via dilaton shifts (see Appendix A.), involving giving a vev to the dilaton — here, since there are different types of components, one can expect different dilaton shifts on each component, corresponding to giving the dilaton a locally constant vev. Modulo such dilaton shifts, we see that again the partition function of the noneffective $[X/\mathbf{H}]$ orbifold is the same as that of the disjoint union of two copies of effective $[X/\mathbf{Z}_2]$ orbifolds and one copy of X , exactly as predicted by the general decomposition conjecture.

5.4.2 Operator analysis

In this subsection we shall analyze the operators in the special case of the CFT of a bosonic string in the orbifold $[\mathbf{R}/\mathbf{H}]$ where the effectively acting \mathbf{Z}_2 acts on $\mathbf{R} = X$ by sign flips.

By the usual logic, the twisted states correspond to conjugacy classes of group elements. There are five conjugacy classes: $[1]$, $[-1]$, $[i]$, $[j]$, and $[k]$. We will soon see that the “good” twist operators — the ones which correspond to states in a single universe — will actually correspond to certain linear combinations of those conjugacy classes.

We shall use the notation τ_g to denote the lowest weight state in a given twisted sector. This state is always unique. In the sectors $\pm 1, \pm i$ with trivially acting twist, the ground state has weight zero.

The gauge-invariant twisted ground states are then the sums of τ_g over conjugacy classes.

So we have five gauge-invariant twisted ground states:

$$\begin{aligned}\sigma_{[1]} &\equiv \tau_1, \\ \sigma_{[-1]} &\equiv \tau_{-1}, \\ \sigma_{[i]} &\equiv \frac{1}{2}(\tau_i + \tau_{-i}), \\ \sigma_{[j]} &\equiv \frac{1}{2}(\tau_j + \tau_{-j}), \\ \sigma_{[k]} &\equiv \frac{1}{2}(\tau_k + \tau_{-k}).\end{aligned}$$

The field X (we are using the same notation for the untwisted real boson as for the space) is periodic in the first three states and antiperiodic in the last two. Therefore the first three have weight $(0, 0)$ and the last two have weight $(1/16, 1/16)$.

This immediately tells us that:

- (A) There must be exactly three disconnected components (in algebraic language, mutually annihilating summands) of the CFT, since there are three linearly independent operators of weight zero.
- (B) Only two of the three components of the CFT can be orbifold CFTs, since there are only two linearly independent twisted ground states. And therefore
- (C) The third component must not be an orbifold CFT but something else. A likely candidate would be the CFT on the unorbifolded line X . But for that to happen, the states odd under the group element k would have to be restored in that factor — otherwise modular invariance could not hold. We will see, presently, that the k -odd states are indeed restored.

States, operators, OPEs and projectors. To understand how all this works, organize the three weight-zero states into mutually annihilating projectors, which we call “universe operators”, for the obvious reason that they are actually projection operators onto the Hilbert space of strings propagating in one of the three “universes”.

The lowest state in the untwisted sector is the identity, so we write $\tau_1 = 1$. The weights of operators are

$$\begin{aligned} \Delta(\tau_{\pm 1}) &= \Delta(\tau_{\pm i}) = 0, \\ \Delta(\tau_{\pm j}) &= \Delta(\tau_{\pm k}) = \frac{1}{16}. \end{aligned}$$

Note that a ground-state twist operator $\Delta(\tau_g)$ has weight zero if and only if g acts trivially on X .

The OPE of twist operators is determined as usual by conformal invariance and the consistency of boundary conditions. For appropriately arranged branch cuts, we have

$$\tau_g(z)\tau_h(0) \sim |z\bar{z}|^{\Delta(\tau_{gh})-\Delta(\tau_g)-\Delta(\tau_h)}\tau_{gh}\left(\frac{z}{2}\right).$$

In particular, the twist operators which act trivially on X have no singularities with respect to each other, and they form a ring. For g, h in the kernel K of the action on X , we have

$$\tau_g(0)\tau_h(0) = \tau_{gh}(0).$$

The $\tau_{g,h}$ are annihilated by the virasoro operators $L_{-1} = \partial_z$ and $\bar{L}_{-1} = \partial_{\bar{z}}$ for $g, h \in K$, so they are independent of position. For instance, the above equation gives

$$\tau_g(z_1)\tau_h(z_2) = \tau_{gh}(z_3)$$

for any independent choice of z_i . So when dealing with ineffectively acting twist fields, we will drop the coordinate labels.

Now, the twist operators we are talking about here are not gauge-invariant in general (except the ones which lie in the center), so the statements we are making depend on the location of branch cuts. Here we are taking the gauge in which all branch cuts of twist fields lie in the forward time direction. But when we combine the ground-state twist fields into gauge-invariant combinations (which means conjugation invariant — the gauge projectors act on the twist fields by conjugation, $\hat{g} \cdot \tau_h = \tau_{ghg^{-1}}$), we get

statements which are independent of where we put the branch cuts:

$$\begin{aligned} \sigma_{[1]}^2 &= \sigma_{[1]}, \\ \sigma_{[1]}\sigma_{[-1]} &= \sigma_{[-1]}\sigma_{[1]} = \sigma_{[-1]}, \\ \sigma_{[1]}\sigma_{[i]} &= \sigma_{[i]}\sigma_{[1]} = \sigma_{[i]}, \\ \sigma_{[-1]}^2 &= \sigma_{[1]}, \\ \sigma_{[i]}\sigma_{[-1]} &= \sigma_{[-1]}\sigma_{[i]} = 0, \\ \sigma_{[i]}^2 &= \frac{1}{2} (\sigma_{[1]} + \sigma_{[-1]}). \end{aligned}$$

The objects $\sigma_{[1,-1,i]}$ comprise the set of gauge-invariant operators of weight zero in the theory — we shall call it the “ground ring.” The other states/operators of the theory are a module over them. In particular, all gauge-invariant operators of a given weight form submodules over the ground ring. For instance, the other ground-state twist fields $\sigma_{[j,k]}$ satisfy

$$\begin{aligned} \sigma_{[1]} \cdot \sigma_{[j,k]} &= \sigma_{[j,k]}, \\ \sigma_{[-1]} \cdot \sigma_{[j,k]} &= 0, \\ \sigma_{[i]} \cdot \sigma_{[j,k]} &= \sigma_{[k,j]}. \end{aligned}$$

Now, there is a preferred basis for the ground ring, in which each basis element acts as a projection operator which annihilates the other projectors. Define

$$\begin{aligned} U^1 &\equiv \frac{1}{4}(\tau_1 + \tau_{-1} + \tau_i + \tau_{-i}) = \frac{1}{4} (\sigma_{[i]} + \sigma_{[-1]} + 2\sigma_{[i]}), \\ U^2 &\equiv \frac{1}{4}(\tau_1 + \tau_{-1} - \tau_i - \tau_{-i}) = \frac{1}{4} (\sigma_{[1]} + \sigma_{[-1]} - 2\sigma_{[i]}), \\ U^3 &\equiv \frac{1}{2}(\tau_1 - \tau_{-1}) = \frac{1}{2} (\sigma_{[1]} - \sigma_{[-1]}). \end{aligned}$$

These combinations are the “universe operators.” They are clearly gauge-invariant, since they are just linear combinations of the three gauge-invariant twisted ground states which have weight zero. They are projection operators which satisfy

$$\begin{aligned} U^1 U^1 &= U^1, \\ U^2 U^2 &= U^2, \\ U^3 U^3 &= U^3, \\ U^i U^j &= 0, \quad i \neq j. \end{aligned}$$

Note that the universe operators satisfy the completeness property, too:

$$U^1 + U^2 + U^3 = \sigma_{[1]} = \tau_1 = 1.$$

Therefore they can be used to decompose all states and operators in the theory into objects which live inside three summand CFTs. We will now see

that $U^{1,2}$ project onto orbifold CFTs $[\mathbf{R}/\mathbf{Z}_2]$ and U^3 projects onto a CFT describing the unorbifolded real line \mathbf{R} .

Physically twisted states. Next, let us look at the two gauge invariant twisted ground states $\sigma_{[j]}$ and $\sigma_{[k]}$, which are “physically twisted” — that is, there is an actual dynamical field X which is antiperiodic in those states. They are gauge-invariant and have weight $(1/16, 1/16)$.

The first thing to notice is that there are only two of them — so they can only exist in two of the three universes, a first clue that one of the three universes will not be an $[\mathbf{R}/\mathbf{Z}_2]$.

Let us see which universes they live in.

Having constructed our universe operators, it is easy to organize the physically twisted fields into eigenstates of the U^i . Define

$$T^1 \equiv \frac{1}{4}(\tau_j + \tau_{-j} + \tau_k + \tau_{-k}) = \frac{1}{2}(\sigma_{[j]} + \sigma_{[k]}),$$

$$T^2 \equiv \frac{1}{4}(\tau_j + \tau_{-j} - \tau_k - \tau_{-k}) = \frac{1}{2}(\sigma_{[j]} - \sigma_{[k]}).$$

It is straightforward to check that these twisted ground states are simultaneous eigenstates of the three universe operators:

$$U^i T^j = \delta^{ij} T^j, \quad i = 1, 2, \quad U^3 T^j = 0.$$

This shows that universe number three does not contain any physically twisted states.

States and operators in universes 1 and 2. Universes 1 and 2 are then fairly straightforward as CFT.

In particular, the operators

$$U^1 \cos pX, U^1 \partial X \bar{\partial} X, U^1 \partial X \partial^5 X (\partial^9 X)^4, T^1, (\partial X)^2 T^1, \dots$$

have exactly the same dimensions and satisfy the exact same OPEs as the corresponding operators

$$\cos pX, \partial X \bar{\partial} X, \partial X \partial^5 X (\partial^9 X)^4, \tau, (\partial X)^2 \tau, \dots$$

in a single copy of the standard effective orbifold CFT $[\mathbf{R}/\mathbf{Z}_2]$. (In this notation t is the \mathbf{Z}_2 twist field which cuts X in the effective orbifold CFT.) And obviously the same statement applies with U^1, T^1 replaced with U^2, T^2 , respectively.

This shows in particular that each of the two summands has a closed and complete OPE, and a modular-invariant partition function.

Oddness in universe 3. This presents a bit of a puzzle for universe number three. This universe would appear to be described by a CFT which contains only even states such as $U^3 \cos pX$ but no odd states such as $U^3 \sin pX$. Indeed, the latter operator is projected out by the gauge projection.

This would not be a problem if universe number 3 contained physically twisted states — but it apparently does not. So we are in danger of the third summand not being a modular-invariant CFT. And yet we know that the full partition function is modular-invariant and that universes 1 and 2 are modular-invariant CFT by themselves. So universe 3 must be modular-invariant as well.

The resolution of this puzzle is that universe number three does indeed contain “physically odd” states. Given any odd operator of the unorbifolded X CFT, such as $\sin pX$, multiply it by the ineffective twist field $\rho \equiv \frac{1}{2}(\tau_i - \tau_{-i})$ which is odd under the action of k (remember, group elements act on twist fields by conjugation). The resulting field $\rho \sin pX$ is even under the action of k and indeed invariant under all elements of the quaternion group!

Using the properties of ρ :

$$\begin{aligned}\rho^2 &= U^3, \\ U^{1,2}\rho &= 0, \\ U^3\rho &= \rho,\end{aligned}$$

we see that any ρ -dressed odd operator lives in the third universe, and that the OPE of two odd operators, each dressed with ρ , generates an even operator, dressed with U^3 . So the ρ -dressed odd operators made out of X s, play the role of odd operators made out of X s in the unorbifolded CFT on \mathbf{R} .

5.5 Second nonbanded example

Consider the nonabelian group A_4 of alternating permutations of four elements. There is a short exact sequence

$$1 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow A_4 \longrightarrow \mathbf{Z}_3 \longrightarrow 1,$$

so we can consider the orbifold $[X/A_4]$, where a $\mathbf{Z}_2 \times \mathbf{Z}_2$ subgroup acts trivially.

The analysis of the decomposition conjecture is straightforward. The \mathbf{Z}_3 fixes the trivial representation of $\mathbf{Z}_2 \times \mathbf{Z}_2$ and permutes the other three irreducible representations. So from the decomposition conjecture we should

get one copy of $[X/\mathbf{Z}_3]$ plus a 3-fold cover of $[X/\mathbf{Z}_3]$, which can be shown to be X . Also, the B fields will be trivial in this case. Thus, the decomposition conjecture predicts

$$\text{CFT}([X/A_4]) = \text{CFT}([X/\mathbf{Z}_3] \sqcup X)$$

This example was discussed in Section 2.0.5 of [1], and we argued there that in terms of \mathbf{Z}_3 path integral twisted sectors, the one-loop partition function has the form

$$Z([X/A_4]) = \frac{1}{|A_4|} \left((16)1 \square_1 + (4)1 \square_\xi + (4)1 \square_{\xi^2} + (4)\xi \square_\xi + \dots \right),$$

where ξ denotes the generator of the effectively acting \mathbf{Z}_3 . This can be written as

$$Z([X/\mathbf{A}_4]) = Z([X/\mathbf{Z}_3]) + Z(X),$$

from which we conclude that the CFT of a string on this nonbanded gerbe is equivalent to the CFT of a string on the disjoint union of the effective orbifold $[X/\mathbf{Z}_3]$ and X .

5.6 Third nonbanded example

Next we shall consider a family of nonbanded gerbes, built using the fact that the dihedral group D_n fits into the short exact sequence

$$1 \longrightarrow \mathbf{Z}_n \longrightarrow D_n \longrightarrow \mathbf{Z}_2 \longrightarrow 1.$$

(Explicitly, the group D_n is generated by the elements a, b , where $a^2 = 1, b^n = 1$, and $aba = b^{-1}$, so that the group elements are $\{1, b, b^2, \dots, b^{n-1}, a, ab, \dots, ab^{n-1}\}$.) Consider an orbifold $[X/D_n]$ in which the \mathbf{Z}_n acts trivially, so that only the \mathbf{Z}_2 quotient acts effectively.

Let us first explain the prediction of the decomposition conjecture. If n is odd, then $K = \mathbf{Z}_2$ will leave one irreducible representation of \mathbf{Z}_n invariant and switch the rest in pairs, so we expect

$$\text{CFT}([X/D_n]) = \text{CFT} \left([X/\mathbf{Z}_2] \sqcup \prod_1^{(n-1)/2} X \right).$$

If n is even, then there are two invariant irreducible representations of \mathbf{Z}_n and the other irreducible representations are switched in pairs, so we expect

$$\text{CFT}([X/D_n]) = \text{CFT} \left([X/\mathbf{Z}_2] \sqcup [X/\mathbf{Z}_2] \sqcup \prod_1^{(n-2)/2} X \right).$$

In each case, the B fields are trivial.

Now, let us check that prediction. Let ξ denote the generator of the effectively acting \mathbf{Z}_2 . The

$$1 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array}$$

sector of $[X/\mathbf{Z}_2]$ arises from

$$1, b, \dots, b^{n-1} \begin{array}{|c|} \hline \square \\ \hline 1, b, \dots, b^{n-1} \end{array}$$

one-loop twisted sectors in $[X/D_n]$, i.e., has multiplicity n^2 .

The

$$1 \begin{array}{|c|} \hline \square \\ \hline \xi \end{array}$$

one-loop twisted sector of $[X/\mathbf{Z}_2]$ arises from

$$1 \begin{array}{|c|} \hline \square \\ \hline a, ab, \dots, ab^{n-1} \end{array}$$

one-loop twisted sectors in $[X/D_n]$. In addition, if n is even, then a commutes with $b^{n/2}$, and so the

$$b^{n/2} \begin{array}{|c|} \hline \square \\ \hline a, ab, \dots, ab^{n-1} \end{array}$$

one-loop twisted sectors in $[X/D_n]$ also contribute. Thus, this $[X/\mathbf{Z}_2]$ twisted sector has multiplicity n when n is odd, and multiplicity $2n$ when n is even.

The

$$\xi \begin{array}{|c|} \hline \square \\ \hline \xi \end{array}$$

one-loop twisted sector of $[X/\mathbf{Z}_2]$ arises from

$$ab^i \begin{array}{|c|} \hline \square \\ \hline ab^i \end{array}$$

one-loop twisted sectors in $[X/D_n]$, for $i \in \{0, 1, \dots, n - 1\}$. In addition, if n is even, the

$$ab^{i+n/2} \begin{array}{|c|} \hline \square \\ \hline ab^i \end{array}$$

one-loop twisted sectors in $[X/D_n]$ also contribute. Thus, this $[X/\mathbf{Z}_2]$ twisted sector has multiplicity n when n is odd and multiplicity $2n$ when n is even.

Putting this together, the one-loop partition function for the $[X/D_n]$ orbifold can be written as

$$Z([X/D_n]) = \frac{1}{|D_n|} \left((n^2)_1 \square_1 + (n)_1 \square_\xi + (n)_\xi \square_\xi \right)$$

when n is odd, and

$$Z([X/D_n]) = \frac{1}{|D_n|} \left((n^2)_1 \square_1 + (2n)_1 \square_\xi + (2n)_\xi \square_\xi \right)$$

when n is even. This means that

$$Z([X/D_n]) = Z([X/\mathbf{Z}_2]) + \left(\frac{n}{2} - \frac{1}{2} \right) Z(X)$$

for n odd, and

$$Z([X/D_n]) = (2)Z([X/\mathbf{Z}_2]) + \left(\frac{n}{2} - 1 \right) Z(X)$$

for n even.

This suggests that for n odd, the CFT of the noneffective $[X/D_n]$ orbifold is the same as the CFT of the disjoint union of $[X/\mathbf{Z}_2]$ and $(n - 1)/2$ copies of X , and for n even, the CFT of the noneffective $[X/D_n]$ orbifold is the same as the CFT of the disjoint union of two copies of $[X/\mathbf{Z}_2]$ and $(n - 2)/2$ copies of X , exactly as the decomposition conjecture predicts.

5.7 General comment on partition functions

In the last few sections, we have studied numerous examples of partition functions of noneffective orbifolds to give evidence for our decomposition conjecture for strings on gerbes. In this section, we shall make a brief general observation concerning such partition functions.

In the case of banded G -gerbes, when comparing the noneffective orbifold $[X/H]$ to the effective orbifold $[X/K]$, the coefficient of the untwisted $(1, 1)$ sector is always equal to the number of conjugacy classes of the trivially acting subgroup $G \subset H$. Indeed, the number of ordered pairs of commuting elements of G is given by

$$\sum_g |Z(g)| = \sum_{[g]} |[g]| \left(\sum_{h \in [g]} |Z(h)| \right),$$

where $Z(g)$ denotes the centralizer of g , $[g]$ denotes a conjugacy class represented by g , and the sum over $[g]$ is a sum over conjugacy classes. Since the number of elements of any conjugacy class is given by $|[g]| = |G|/|Z(g)|$, and

the order of the centralizer is constant across elements of the same conjugacy class, we see that the number of ordered pairs of commuting elements of G is given by

$$|G|C,$$

where C is the number of conjugacy classes of G (equivalently, the number of irreducible representations of G).

Using this result, in the case of a banded gerbe (in which the theory decomposes as a sum of copies of $[X/K]$ with flat B fields), we have that the partition function of the G -gerbe can be expanded, in terms of the K -orbifold, as

$$\begin{aligned} Z &= \frac{|K|}{|H|} |G|C \cdot \left(1 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array} \right) + \dots \\ &= C \cdot \left(1 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array} \right) + \dots \end{aligned}$$

Thus, we see that in general the number of components into which a banded gerbe should decompose should equal the number of irreducible representations of the band, a result for which we have generated experimental evidence in the preceding sections. (For nonbanded gerbes, the corresponding argument is more complicated as the theory decomposes into copies not only of $[X/K]$ but also covers thereof.)

6 Two-dimensional gauge theoretic aspects

So far we have discussed numerous examples of noneffective orbifolds, but the same results also apply to presentations as global quotients by nonfinite groups. Gerbes can be described, for example, by two-dimensional abelian gauge theories with noneffectively acting groups (meaning, nonminimal charges). These were discussed in detail in [3] where, for example, a detailed discussion of how these are nonperturbatively distinct from two-dimensional gauge theories with minimal charges was presented.

The decomposition conjecture of this paper can also be seen in these gerby abelian gauge theories. One way is to use mirror symmetry ideas to dualize the theory to one in which the nonperturbative effects have become classical, as discussed in [3]. This will be reviewed in section 8. Alternately, as we shall discuss here, the effect can be seen immediately in the structure of the nonperturbative effects without dualizing.

Consider a $U(1)$ gauge theory in which the charges of all fields are a multiple of k . In such a theory, as discussed in [3], the nonperturbative

effects are equivalent to only summing over bundles whose c_1 is divisible by k . (In four dimensions, this violates cluster decomposition.) This is equivalent to summing over copies of a $U(1)$ gauge theory with minimally charged fields and variable values of the theta angle. If we let the theta angle in the formal sum of gauge theories vary through \mathbf{Z}_k , then when we sum over gauge theories, only sectors in which c_1 is a multiple of k will contribute, the others (weighted by nontrivial roots of unity) will cancel out.

More formally,

$$\sum_{n=0}^{\infty} a_{kn} q^{kn} = \frac{1}{k} \sum_{m=0}^{k-1} \left(\sum_{n=0}^{\infty} a_n q^n \zeta^{mn} \right)$$

for ζ a k th root of unity, using the identity

$$\sum_{m=0}^{k-1} \zeta^{mn} = \begin{cases} k & n = 0, \pm k, \pm 2k, \dots, \\ 0 & \text{else.} \end{cases}$$

Note that the theta angle is playing a role here closely analogous to that of discrete torsion in orbifolds — both multiply nonperturbative contributions to partition functions by phases. In effect, the theta angle is a continuous version of discrete torsion.

Thus, a $U(1)$ gauge theory in which the fields have charges that are multiples of k , is equivalent to a formal sum of gauge theories with variable theta angles, realizing our picture for banded gerbes that the physics of a banded gerbe should be equivalent to a sum of copies of the underlying space with variable B fields.

7 D-branes

7.1 Equivariant K -theory

Since D-brane charges are classified by K -theory, our decomposition conjecture relating conformal field theories of strings on gerbes to conformal field theories of strings on spaces has a specific prediction for equivariant K -theory. Specifically, consider the quotient $[X/H]$, where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially on X (and H need not be finite); then

$$\begin{aligned} &\text{the } H\text{-equivariant } K\text{-theory of } X \\ &\quad \text{should be the same as} \\ &\text{the twisted } K\text{-equivariant } K\text{-theory of } X \times \hat{G}, \end{aligned}$$

where the twisting is defined by the B field, as in our decomposition conjecture.

In Appendix C we sketch a proof of the mathematical prediction above for equivariant K -theory.

7.2 Sheaf theory

It is a standard mathematical result that the category of sheaves on a gerbe decomposes in a fashion consistent with our conjecture of this paper. For example, the category of sheaves on a banded G -gerbe decomposes into subcategories indexed by irreducible representations of G , and Ext groups between distinct components vanish. This mathematical fact corresponds to a prediction for D-branes in the topological B model, which are modeled by coherent sheaves.

To help illustrate the general remarks about decomposition of sheaves and Ext groups, let us consider an easy example, involving D0-branes in the trivially acting \mathbf{Z}_n orbifold of \mathbf{C} .

Although the \mathbf{Z}_n acts trivially on \mathbf{C} , it can act nontrivially on Chan-Paton factors, as discussed in more detail in [1].

Let ρ_i denote the i th one-dimensional irreducible representation of \mathbf{Z}_n , where ρ_0 is the trivial representation.

Consider two sets of D0-branes on \mathbf{C} , one set containing

$$a_0 + a_1 + \cdots + a_{n-1}$$

D0-branes, all at the origin, with the i th set of branes in the ρ_i representation of \mathbf{Z}_n , and the other set containing

$$b_0 + b_1 + \cdots + b_{n-1}$$

D0-branes, all at the origin, with the i set of branes again in the ρ_i representation of \mathbf{Z}_n .

First, note that this decomposition is precisely the decomposition by characters of the band described formally earlier. Although the \mathbf{Z}_n acts trivially on the space, it can act nontrivially on the D-branes, and its possible actions can be classified by characters of \mathbf{Z}_n .

Although reference [9] discussed D-branes in orbifolds by effectively acting finite groups, the same analysis applies to noneffectively acting finite groups. As in [9], open string spectra are calculated by Ext groups between

corresponding sheaves. In this case, the spectral sequence of [9] is trivial, and so if \mathcal{E}, \mathcal{F} denote the sheaves over the origin corresponding to the two sets of D-branes, then

$$\text{Ext}_{[\mathbf{C}/\mathbf{Z}_n]}^p(\mathcal{E}, \mathcal{F}) = H^0(\text{pt}, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^p \mathcal{N}_{\text{pt}/\mathbf{C}})^{\mathbf{Z}_n}.$$

As in [9], the normal bundle $\mathcal{N}_{\text{pt}/\mathbf{C}} = \mathcal{O}$ has a \mathbf{Z}_n -equivariant structure induced by the \mathbf{Z}_n action on \mathbf{C} , but since the \mathbf{Z}_n action on \mathbf{C} is trivial, the equivariant structure is defined by the trivial representation ρ_0 .

Using the fact that $\rho_i^\vee = \rho_{n-i}$, it is straightforward to calculate that

$$\begin{aligned} h^0(\text{pt}, \mathcal{E}^\vee \otimes \mathcal{F})^{\mathbf{Z}_n} &= h^0(\text{pt}, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \mathcal{N}_{\text{pt}/\mathbf{C}})^{\mathbf{Z}_n} \\ &= a_0 b_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1}. \end{aligned}$$

There are no cross-terms because we take \mathbf{Z}_n -invariants, and the \mathbf{Z}_n acts nontrivially on Chan–Paton factors. As a result,

$$\dim \text{Ext}_{[\mathbf{C}/\mathbf{Z}_n]}^p(\mathcal{E}, \mathcal{F}) = a_0 b_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1}$$

for $p = 0, 1$ and vanishes for other p .

7.3 A_∞ structures factorize

We have just explained that Ext groups decompose, in the sense that any Ext group between sheaves associated to distinct irreducible representations necessarily vanishes. This implies that two-point functions in the open string B model factorize between irreducible representations.

We shall argue here that it also implies that all higher n -point functions also factorize, in the sense that an n -point function will only be nonzero if all of the boundaries are associated to the same irreducible representation.

To see this, we can apply the methods of [10], who showed¹ how to compute superpotentials from D-branes in the open string B model.

The basic idea is that computation of n -point functions amounts to understanding the structure of an A_∞ algebra describing the disc-level open string field theory. In the case of the open string B model, there is an easy description of that A_∞ algebra, in terms of Čech cocycles valued in local Homs, on which the various products can all be easily defined in terms of compositions. That particular description is not as useful as one would like, because one

¹Strictly speaking, there is an implicit assumption that boundary renormalization group flow respects the mathematical notion of quasi-isomorphism, and so a more nearly direct calculation would still be very interesting, but for the purposes of this paper we shall gloss over that subtlety.

would like to phrase it in terms of multiplications between Ext group elements, which can be described as the cohomology (with respect to m_1) of A .

Now, an A_∞ structure on A induces an A_∞ structure on $H^*(A)$, and it is that latter A_∞ structure which most directly computes superpotential terms.

In the case at hand, it is a standard fact that Ext groups between sheaves on gerbes factorize, and similarly the algebra A of [10] also necessarily factorizes. After all, if \mathcal{E} and \mathcal{F} are sheaves on the gerbe associated to distinct components, then $\text{Ext}^i(\mathcal{E}, \mathcal{F}) = 0$, and so there is no way to get a nonzero product in A . The method of [10] for constructing product structures on $H^*(A)$ involves constructing, level by level, products on $H^*(A)$ obeying the A_∞ algebra and latching onto corresponding products in A under the map $i : H^*(A) \rightarrow A$, but if a set of products in A vanishes identically, then in order to be consistent, the corresponding products in $H^*(A)$ must also vanish.

7.4 Closed string states in the B model

The fact that the open string states factorize also implies that the closed string states factorize. In the topological B model, the closed string states can be interpreted as Hochschild cohomology. However, Hochschild cohomology can be interpreted as deformations of an A_∞ algebra, and in particular the Hochschild cohomology of closed string B model states can be interpreted as infinitesimal deformations of the open string A_∞ algebra.

Let us briefly recall how this procedure works, at least in general terms. Let A be an associative algebra over the complex numbers. The Hochschild cochain complex (with coefficients in A) is the sequence of vector spaces

$$C^n(A) = \text{Hom}_{\mathbf{C}}(A^{\otimes n}, A), \quad n = 0, 1, 2, \dots$$

with differential $\delta : C^n(A) \rightarrow C^{n+1}(A)$ defined by

$$\begin{aligned} (\delta f)(a_1, \dots, a_{n+1}) &= a_1 f(a_2, \dots, a_n) + (-)^{n+1} f(a_1, \dots, a_n) a_{n+1} \\ &\quad + \sum_{i=1}^n (-)^i f(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_n). \end{aligned}$$

The cohomology of δ in degree n will be denoted by $HH^n(A, A)$ and called the Hochschild cohomology of A (with coefficients in A).

Each 2-cocycle $f(a_1, a_2)$ defined an infinitesimal deformation of the associative product on A . Specifically, if one defines a new product by

$$a * b = ab + tf(a, b),$$

then the product will be associative to first-order in t if and only if $\delta f = 0$. Furthermore, δ -coboundaries correspond to deformations which lead to isomorphic algebras, so one is only interested in δ -closed objects modulo δ -exact objects. More generally, the total Hochschild cohomology $HH^*(A)$ can be understood as classifying infinitesimal deformations of A in the class of A_∞ algebras, see for example [11, Section 5].

In the present case, the fact that the open string A_∞ algebra factorizes trivially implies that the closed string states also factorize, giving us another perspective on the decomposition statement for the closed string B model.

8 Mirror symmetry

Let us now examine how our decomposition conjecture is consistent with mirror symmetry for gerbes and stacks, as discussed in [1–3].

Mirror symmetry for toric stacks and complete intersections therein was discussed in [3]. Specifically, it was discussed how to build Landau–Ginzburg point “mirrors” to the A model on stacks, following ideas in [12, 13].

For example, the Toda-dual to the A model on a \mathbf{Z}_k -gerbe over \mathbf{P}^{N-1} with characteristic class $-n \bmod k \in H^2(\mathbf{P}^{N-1}, \mathbf{Z}_k)$ is defined by the superpotential

$$W = \exp(-Y_1) + \cdots + \exp(-Y_{N-1}) + \Upsilon^n \exp(Y_1 + \cdots + Y_{N-1}),$$

where Υ is a field valued in k th roots of unity, rather than the complex numbers. In effect, mirror symmetry mapped the nonperturbative effects that distinguish the A model on a gerbe from the A model on the underlying space, to a purely classical effect, namely a discrete- or character-valued field. (More formally, mirror symmetry seems to be exchanging the class algebra of the band with the representation ring.)

Such a discrete-valued field can be re-interpreted in terms of a sum over components, as our decomposition statement predicts. In particular, the

path integral measure contains a discrete sum, over values of Υ , and that sum can be pulled out of the path integral:

$$\int [DY_i] \sum_{\Upsilon} \exp(-S(Y_i, \Upsilon)) = \sum_{\Upsilon} \int [DY_i] \exp(-S(Y_i, \Upsilon)).$$

The right-hand side of the equation above can be interpreted as the partition function of a disjoint union of theories, indexed by the value of Υ , in line with the general prediction of our decomposition conjecture.

Consider the gerby quintic discussed in [3], that is a banded \mathbf{Z}_k -gerbe over the quintic hypersurface in \mathbf{P}^4 , described as a hypersurface in the gerbe over \mathbf{P}^4 with characteristic class $-1 \pmod k$. It was argued there that, so long as 5 does not divide k , the mirror Landau–Ginzburg point should be a \mathbf{Z}_5^3 orbifold of the Landau–Ginzburg model with superpotential

$$W = x_0^5 + \dots + x_4^5 + \psi \Upsilon x_0 x_1 x_2 x_3 x_4,$$

where x_0, x_1, x_2, x_3, x_4 are ordinary chiral superfields, Υ is a field valued in k th roots of unity, and ψ is a complex number, mirror to the complexified Kähler parameter of the original theory.

Having a field valued in roots of unity is equivalent to having a partition function given by a sum over theories deformed by roots of unity, so already from this description, we see that the CFT of the gerby quintic can be described as a sum over theories in which the complexified Kähler parameter has been shifted by k th roots of unity.

Now, let us understand the meaning of that shift more systematically. Recall from [14, equation 5.9] that the relation between the complexified Kähler parameter t of the original quintic and the complex structure parameter ψ of the mirror quintic is given by

$$t = -\frac{5}{2\pi i} \left\{ \log(5\psi) - \frac{1}{\omega_0(\psi)} \sum_{m=1}^{\infty} \frac{(5m)!}{(m!)^5 (5\psi)^{5m}} [\Psi(1 + 5m) - \Psi(1 + m)] \right\},$$

where $2\pi ikt$ is the value of the action evaluated on a rational curve of degree k .

Close to large radius, the mirror map is given by

$$t \cong -\frac{5}{2\pi i} \log(5\psi),$$

from which we see that if we multiply ψ by a k th root of unity, where 5 does not divide k , the result is to shift the purely real part of t , not the imaginary part. Since $t \cong B + iJ$, shifting only the real part of t corresponds to shifting B only, not J , exactly our proposed interpretation. (Away from large radius, geometric meanings can no longer be meaningfully assigned.)

9 Noncommutative geometry

In this section we shall give a derivation of the full decomposition conjecture based on noncommutative geometry. We shall write it in the language of local orbifolds, this having the advantage of showing that the decomposition is presentation-independent.

9.1 Lie groupoids and C^* -algebras

A local orbifold is often modeled by a Lie groupoid \mathcal{G} [20]. The corresponding non-commutative space is then modeled by the convolution C^* -algebra $C^*(\mathcal{G})$. If the stack \mathcal{X} is of the form $[X/H]$, then the groupoid \mathcal{G} has X as space of objects, and has an arrow $x \rightarrow y$ for each group element g sending x to y . The corresponding C^* -algebra is then the crossed product of the group H with the algebra of functions on X . A same stack \mathcal{X} can however be presented by many different groupoids \mathcal{G} . The corresponding C^* -algebras then will not be isomorphic but merely Morita-equivalent. We shall write $\text{nc}(\mathcal{X})$ for the above ‘‘Morita-equivalence class’’ and call it the noncommutative space associated to \mathcal{X} .

Given a $U(1)$ -gerbe τ on \mathcal{X} , we can twist the above construction. Such a gerbe associates a phase $c(g_1, g_2) \in U(1)$ to each pair of composable arrows of \mathcal{G} , where \mathcal{G} is some groupoid representing \mathcal{X} . This data represents a class in $H^2(\mathcal{X}, U(1))$. One can then form the twisted algebra $C^*(\mathcal{G}, \tau)$ as follows. As for $C^*(\mathcal{G})$, the elements of $C^*(\mathcal{G}, \tau)$ are complex-valued functions on the space of arrows of \mathcal{G} , but their product is now given by

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} c(g_1, g_2) f_1(g_1) f_2(g_2).$$

Once again, the C^* -algebra $C^*(\mathcal{G}, \tau)$ depends on the particular choice of groupoid \mathcal{G} , but its Morita-equivalence class $\text{nc}(\mathcal{X}, \tau)$ only depends on \mathcal{X} and $\tau \in H^2(\mathcal{X}, U(1))$.

9.2 The functor of points philosophy

We quickly review the formalism of functor of points [34], which allows one to talk about a stack \mathcal{X} without having to mention the groupoid \mathcal{G} , (or the action of H on X in the case $\mathcal{X} \simeq [X/H]$). This philosophy says that to give

a stack, it is enough to give its set of points, and the following two additional pieces of data:

- One should say what it means to have an isomorphism between two points.
- One should say what it means to have a family of points parametrized by a given manifold.

As an example, if $\mathcal{X} = [X/H]$, then a point of \mathcal{X} is an H -equivariant map from H to X . An isomorphism between two points p and q is then given by an element $h \in H$ such that $p(gh) = q(g)$ for all $g \in H$. And an M -parametrized family of points is an H -principal bundle over M , along with an H -equivariant map to X .

One can also treat noncommutative spaces via the functor-of-points point of view, even though this is less standard and less mathematically rigorous. Given a C^* -algebra A , a *point* is an irreducible representation of A (not an isomorphism class!). Note that every irreducible representation has $U(1)$ as automorphism group, so noncommutative geometry leads naturally to considering $U(1)$ -gerbes. Unfortunately, irreducible representations do not behave well in families. Namely, a limit of irreducible representations might be suddenly reducible, this is the phenomenon known as “brane fractionation”. We shall decide to ignore this technical subtlety.

Using the functor of points language, we now explain how one associates a noncommutative space to a local orbifold \mathcal{X} . We just declare a point of $\text{nc}(\mathcal{X})$ to be a pair (p, ρ) , where p is a point of \mathcal{X} , and $\rho : \text{Aut}(p) \rightarrow U(V)$ is an irreducible representation of the automorphism group of p (= stabilizer group) on some vector space V .

$U(1)$ -gerbes can also be described in that language. Namely, a $U(1)$ -gerbe over some stack \mathcal{X} is a stack τ , equipped with a map $\tau \rightarrow \mathcal{X}$, satisfying the following properties. Any two points $\mathcal{L}, \mathcal{L}'$, of τ sitting over a given point p of \mathcal{X} are isomorphic, and the map $\text{Aut}(\mathcal{L}) \rightarrow \text{Aut}(p)$ is a central extension by $U(1)$. One should think of \mathcal{L} as objects that want to be one-dimensional \mathbf{C} -vector spaces, but only manage to be so locally in \mathcal{X} .

Given a $U(1)$ -gerbe τ on a stack \mathcal{X} , then $\text{nc}(\mathcal{X}, \tau)$ is given as follows. A point of $\text{nc}(\mathcal{X}, \tau)$ is a pair (p, ρ) , where p is a point of \mathcal{X} , and ρ is an irreducible twisted representation of $\text{Aut}(p)$ on some $V \otimes \mathcal{L}$. Here, V is an ordinary vector space, \mathcal{L} is a point of the $U(1)$ -gerbe sitting over p , and symbol \otimes means that we identify the two copies of $U(1)$ sitting in $U(V)$ and in $\text{Aut}(\mathcal{L})$. A twisted representation assigns to each element $g \in \text{Aut}(p)$ an automorphism $\rho(g) = \alpha(g) \otimes \phi(g)$ of $V \otimes \mathcal{L}$ with the property

that $\phi(g) \mapsto g$ under the map $\tau \rightarrow \mathcal{X}$. We then have the usual condition that $\rho(gg') = \rho(g)\rho(g')$. It should be noted that a twisted representation on $V \otimes \mathcal{L}$ always induces a projective representation $g \mapsto \alpha(g)$ of $\text{Aut}(p)$ on V .

If the stack \mathcal{X} is presented by a groupoid \mathcal{G} , then $nc(\mathcal{X}, \tau)$ is presented by the algebra $C^*(\mathcal{G}, \tau)$ introduced in the previous section.

9.3 Non-effective local orbifolds

A local orbifold \mathcal{X} is called noneffective if every point p of \mathcal{X} has a nontrivial stabilizer group. In that case one can single out a subgroup of $\text{Aut}_0(p) \subset \text{Aut}(p)$ called the ineffective stabilizer group. It is the kernel of the natural action of $\text{Aut}(p)$ on the tangent space $T_p\mathcal{X}$. The quotient of \mathcal{X} , where we mod out all stabilizer groups by their ineffective subgroups is called \mathcal{X}_{eff} . It is again a local orbifold, and \mathcal{X} sits as a gerbe over \mathcal{X}_{eff} with band $G := \text{Aut}_0(p)$. In other terms, \mathcal{X} is a bundle over \mathcal{X}_{eff} whose fibers are BG . By definition, the gerbe is then banded if that bundle is trivial.

9.4 Finite group gerbes and covering spaces

Let \mathcal{X} be a local orbifold with ineffective stabilizer G , and with effective quotient $M := \mathcal{X}_{\text{eff}}$.

From the fiber sequence $BG \rightarrow \mathcal{X} \rightarrow M$, we expect to get a fiber sequence

$$nc(BG) \rightarrow nc(\mathcal{X}) \rightarrow nc(M).$$

As a space, $nc(BG)$ looks like a disjoint union of d points, one for each irreducible representation of G . So one expects $nc(\mathcal{X})$ to be a d -fold cover of $nc(M)$. This is almost the case, but not quite. Indeed $nc(BG)$ is not exactly d points, it is rather d copies of $BU(1)$. So $nc(\mathcal{X})$ is a bundle over M with fiber $\coprod^d BU(1)$. In other words, $nc(\mathcal{X})$ is a d -fold cover of $nc(M)$ but twisted by a $U(1)$ -gerbe.

The underlying cover is easy to describe: Let $\hat{G} := \pi_0(nc(BG))$ denote the set of isomorphism classes of irreducible representations of G . The monodromy representation $\pi_1(M) \rightarrow \text{Out}(BG)$ induces an action of $\pi_1(M)$ on \hat{G} . From it, we then get a cover

$$Y := \widetilde{M} \times_{\pi_1(M)} \hat{G},$$

where \widetilde{M} denotes the universal cover of M . The local orbifold Y can also be described as follows: a point of Y is a point p of \mathcal{X} , and an isomorphism class

of irreducible representation of $\text{Aut}_0(p)$. The stack Y comes with a canonical $U(1)$ -gerbe ν whose points are pairs (p, V) , where p is a point of \mathcal{X} and V an irreducible representation of $\text{Aut}_0(p)$. The map $\nu \rightarrow Y$ then forgets the irreducible representation and replaces it by its isomorphism class. Given this gerbe ν , we can state the result:

The bundle $BG \rightarrow \mathcal{X} \rightarrow M$ induces a bundle $\hat{G} \rightarrow Y \rightarrow M$
and we have $\text{nc}(\mathcal{X}) = \text{nc}(Y, \nu)$.

Clearly, Y decomposes into as many connected components as there are orbits of $\pi_1(M)$ on \hat{G} . It follows from the above statements that the same holds for $\text{nc}(\mathcal{X})$. So the above argument proves the decomposition conjecture for the part of the CFT that can be expressed in terms of $\text{nc}(\mathcal{X})$. This includes for example sheaf theory, D-branes, and K -theory.

Note that if \mathcal{X} came initially with a $U(1)$ -gerbe τ , then Y would be replaced by

$$Y^\tau := \widetilde{M} \times_{\pi_1(M)} \hat{G}^\tau,$$

where $\hat{G}^\tau = \pi_0(\text{nc}(BG, \tau))$ denotes the set of isomorphism classes of projective representations of G with cocycle fixed by τ . This space has now as many components as there are orbits of $\pi_1(M)$ on \hat{G}^τ .

9.5 Cocycle description

Recall that in terms of Čech cocycles, a G -gerbe $\mathcal{X} \rightarrow M$ is given by an open cover $\{U_i\}$ of M , functions $\varphi_{i,j} \in \text{Aut}(G)$ on double intersections, and $g_{ijk} \in G$ on triple intersections, satisfying

$$\begin{aligned} \varphi_{jk} \circ \varphi_{ij} &= \text{Ad}(g_{ijk}) \circ \varphi_{ik}, \\ g_{jkl} g_{ijl} &= \varphi_{kl}(g_{ijk}) g_{ikl}. \end{aligned} \tag{9.1}$$

The cover $Y \rightarrow M$ is then locally $U_i \times \hat{G}$, and the glueing maps on the double intersections are given by $\varphi_{ij}^* : \hat{G} \rightarrow \hat{G}$.

We now describe the $U(1)$ -gerbe ν on Y in terms of cocycles. That gerbe is the obstruction to finding a vector bundle on Y whose fibers are the required representations of the ineffective stabilizer groups (pulled back to Y). So in order to write down the cocycle, we try to build that vector bundle and when the construction does not work any more, we read out our cocycle.

The cover $\{U_i\}$ used in (9.1) induces a cover of Y whose elements are pairs $(U_i, [\rho_i])$, $[\rho_i] \in \hat{G}$. Pick representatives $(V_i, \rho_i : G \rightarrow U(V_i))$ of the isomorphism classes $[\rho_i]$. Over double intersections, we then pick φ_{ij} -equivariant

homomorphisms $f_{ij} : V_i \rightarrow V_j$. This would glue into a vector bundle on Y if we had

$$f_{jk} \circ f_{ij} = \rho_k(g_{ijk}) \circ f_{ik}.$$

So our cocycle is

$$c_{ijk} = \rho_k(g_{ijk}) \circ f_{ik} \circ f_{ij}^{-1} \circ f_{jk}^{-1}, \tag{9.2}$$

which is a $U(1)$ -valued function by Schur's lemma. Note also that the f_{ij} , and hence c_{ijk} , are constant functions: this means that ν is a flat $U(1)$ -gerbe.

9.6 The abelian case

If G is abelian, we can be more explicit in identifying the $U(1)$ -gerbe on Y . In that case, all the irreducible representations of G are one-dimensional, so we may pick our representatives of \hat{G} to be of the form $(\mathbf{C}, \rho : G \rightarrow U(1))$. By Schur's lemma, if (\mathbf{C}, ρ_i) is isomorphic to $\varphi_{ij}^*(\mathbf{C}, \rho_j)$, then $1 : \mathbf{C} \rightarrow \mathbf{C}$ is φ_{ij} -equivariant. So we can take all our f_{ij} to be 1. Our cocycle formula (9.2) then simplifies to

$$c_{ijk} = \rho_k(g_{ijk}).$$

In other words, the $U(1)$ -gerbe on Y is the image under the representation $\rho : G \rightarrow U(1)$ of the G -gerbe \mathcal{X} over M . More precisely, the band of the G -gerbe $\mathcal{X} \rightarrow M$ is a local system on M . The pull-back to Y of that local system admits a homomorphism to the constant local system $U(1)$. The $U(1)$ -gerbe on Y is then obtained from the gerbe $\mathcal{X} \times_M Y \rightarrow Y$ by changing coefficients along the above homomorphism.

For arbitrary G , we get a similar description of the $U(1)$ -gerbe on the subspace of Y corresponding to the one-dimensional representations of G .

9.7 The case when the band is trivial

Similarly to the abelian case, if the band is trivial, we can simplify our cocycle formula (9.2). In that case we may assume that the φ_{ij} in (9.1) are all 1. It then follows that all the g_{ijk} are central in G . The isomorphism classes of G -gerbes with trivial band are thus parametrized by $H^2(M, Z(G))$, a theorem of Giraud [35].

In the construction of our cocycle, we may take all the f_{ij} to be identity maps, and once again we get

$$c_{ijk} = \rho_k(g_{ijk}).$$

So the cocycle for the $U(1)$ -gerbe on Y is the image under the representation $\rho : Z(G) \rightarrow U(1)$ of the cocycle in $H^2(M, Z(G))$ for the G -gerbe $\mathcal{X} \rightarrow M$.

9.8 Massless spectrum

To compute the cohomology of a space X , one often uses a cellular decomposition of X to write down the cochain complex $C^*(X) = \mathbf{C}^{\{\text{cells of } X\}}$. The coboundary operator $C^n(X) \rightarrow C^{n+1}(X)$ then records which cells are in the boundary of which other cells.

Given a stack \mathcal{X} , its free loop space $L\mathcal{X}$ contains a substack $I\mathcal{X}$ of loops of length zero, commonly called the inertia stack of \mathcal{X} . Let $\underline{I\mathcal{X}}$ be the quotient topological space of $I\mathcal{X}$. A point of $\underline{I\mathcal{X}}$ is then the same thing as a pair $(p, [g])$, where p is a point of \mathcal{X} and $[g]$ is a conjugacy class of $\text{Aut}(p)$, two such pairs $(p, [g]), (p', [g'])$ being identified if p and p' are isomorphic and the isomorphism sends $[g]$ to $[g']$. We shall take the point of view that the massless spectrum of \mathcal{X} agrees with the cohomology of $\underline{I\mathcal{X}}$ (which is the same as the cohomology of $I\mathcal{X}$ since we are taking complex coefficients), and use it to check the decomposition conjecture. The complex that computes the massless spectrum of \mathcal{X} is then

$$\bigoplus_{\text{cells of } X} \mathbf{C}^{\{\text{conjugacy classes of stabilizer group}\}}$$

and the differential is a combination of the usual cellular coboundary, and of the adjoint of the map $\text{Conj}(G) \rightarrow \text{Conj}(H)$ induced by a group homomorphism $G \rightarrow H$.

The space $\underline{I\mathcal{X}}$ however had the disadvantage that it has too many connected components (even if \mathcal{X} is an effective orbifold, $\underline{I\mathcal{X}}$ is typically disconnected), and thus the conjectured decomposition is not apparent from the geometry of \mathcal{X} . For that purpose, $\text{nc}(\mathcal{X})$ is better behaved. Let $\underline{\text{nc}}(\mathcal{X})$ be the topological quotient of $\text{nc}(\mathcal{X})$. A point of $\underline{\text{nc}}(\mathcal{X})$ is then a pair $(p, [\rho])$, where p is a point of \mathcal{X} and $[\rho]$ is an isomorphism class of irreducible representation of the stabilizer group of p , two such pairs $(p, [\rho]), (p', [\rho'])$ being identified if p and p' are isomorphic and the isomorphism sends $[\rho]$ to $[\rho']$. The space $\underline{\text{nc}}(\mathcal{X})$ can also be described as the spectrum of any \mathbf{C}^* -algebra representing $\text{nc}(\mathcal{X})$.

A cell of $\underline{\text{nc}}(\mathcal{X})$ then corresponds to a cell of X , along with an isomorphism class of irreducible representation of its stabilizer. The topology of $\underline{\text{nc}}(\mathcal{X})$ is typically not Hausdorff, but we can nevertheless write down the complex computing its cohomology:

$$\bigoplus_{\text{cells of } X} \mathbf{C}^{\{\text{iso. classes of irreps of stabilizer group}\}}$$

The differential is induced by the cellular coboundary operator, and by the map $\text{Rep}(H) \rightarrow \text{Rep}(G)$ coming from a group homomorphism $G \rightarrow H$. It should be noted that the above two complexes are isomorphisms, where the isomorphism

$$\mathbf{C}^{\{\text{conjugacy classes}\}} = \mathbf{C}^{\{\text{iso. classes of irreps}\}}$$

is given by the character table. This isomorphism is compatible with the differentials, and so we are indeed computing the same cohomology in two different ways.

Now we may use the isomorphism $\underline{\text{nc}}(\mathcal{X}) = \underline{\text{nc}}(Y, \nu)$ and the decomposition of $Y = \widetilde{M} \times_{\pi_1(M)} \hat{G}$ coming from the orbits of $\pi_1(M)$ on \hat{G} . Clearly, the cohomology of a disjoint union of spaces is the direct sum of the cohomologies, and so this provides the desired decomposition of the massless spectrum.

10 Discrete torsion

So far, apart from a brief mention in Section 9.4, discrete torsion has appeared solely in the context of effective orbifolds, as a summand in the decomposition of a string on a gerbe. However, in principle, given a non-effective orbifold, one could imagine turning on discrete torsion in the non-effectively acting group. In this section we study a few examples where we turn on discrete torsion in the noneffectively acting group, and examine how the decomposition conjecture is modified.

10.1 First (banded) example

For our first example, consider the noneffective orbifold $[X/(\mathbf{Z}_2 \times \mathbf{Z}_2)]$, where the first \mathbf{Z}_2 acts trivially but the second acts effectively. Let a denote the generator of the first (trivial) \mathbf{Z}_2 , and b the generator of the second. Let us also turn on discrete torsion in this model; since $H^2(\mathbf{Z}_2 \times \mathbf{Z}_2, U(1)) = \mathbf{Z}_2$, there is only one choice. Let us try breaking up the one-loop twisted sectors

in the noneffective $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold into effective \mathbf{Z}_2 twisted sectors. The

$$1 \begin{array}{|c|} \hline \square \\ \hline 1 \end{array}$$

sector in the effective \mathbf{Z}_2 orbifold comes from any of

$$1,a \begin{array}{|c|} \hline \square \\ \hline 1,a \end{array}$$

and so has multiplicity $2^2 = 4$. The

$$1 \begin{array}{|c|} \hline \square \\ \hline b \end{array}$$

sector in the effective \mathbf{Z}_2 orbifold comes from any of

$$1,a \begin{array}{|c|} \hline \square \\ \hline b,ab \end{array}$$

but although the

$$1 \begin{array}{|c|} \hline \square \\ \hline b,ab \end{array}$$

sectors contribute with a positive sign, because of discrete torsion in the noneffective $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold, the sectors

$$a \begin{array}{|c|} \hline \square \\ \hline b,ab \end{array}$$

contribute with a minus sign, and so cancel out. Thus, the partition function for the noneffective $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold contains no net contribution that appears as a

$$1 \begin{array}{|c|} \hline \square \\ \hline b \end{array}$$

sector in the effective \mathbf{Z}_2 orbifold. Similarly, the

$$b \begin{array}{|c|} \hline \square \\ \hline b \end{array}$$

sector of the effective \mathbf{Z}_2 orbifold comes from

$$b,ab \begin{array}{|c|} \hline \square \\ \hline b,ab \end{array}$$

sectors in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold, but the contribution of the sectors

$$b \begin{array}{|c|} \hline \square \\ \hline b \end{array}, \quad ab \begin{array}{|c|} \hline \square \\ \hline ab \end{array}$$

is canceled by the contribution of the sectors

$$ab \begin{array}{|c|} \hline \square \\ \hline b \end{array}, \quad b \begin{array}{|c|} \hline \square \\ \hline ab \end{array}$$

which because of discrete torsion contribute with a sign.

The final result for the one-loop partition functions has the form

$$\begin{aligned} Z_{1\text{-loop}}([X/\mathbf{Z}_2 \times \mathbf{Z}_2]) &= \frac{4}{|\mathbf{Z}_2 \times \mathbf{Z}_2|} 1 \square_1 \\ &= Z_{1\text{-loop}}(X), \end{aligned}$$

from which we conclude that the CFT of the noneffective $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with discrete torsion is identical to the CFT of X .

10.2 Second example

Next, let us consider a noneffective $\mathbf{Z}_k \times \mathbf{Z}_k$ orbifold, where we turn on an element of discrete torsion, $H^2(\mathbf{Z}_k \times \mathbf{Z}_k, U(1)) = \mathbf{Z}_k$.

Let g denote the generator of the first, trivially acting, \mathbf{Z}_k , and h denote the generator of the second, effectively acting, \mathbf{Z}_k .

Here, there is more than a single value of discrete torsion, as $H^2(\mathbf{Z}_k \times \mathbf{Z}_k, U(1)) = \mathbf{Z}_k$. Let ζ denote a generator of k th roots of unity; then discrete torsion assigns to the sector

$$\begin{matrix} g^a h^b & \square \\ & g^{a'} h^{b'} \end{matrix}$$

the phase factor $\zeta^{m(ab' - ba')}$, where $m \in \{0, \dots, k - 1\}$ indexes the possible values of discrete torsion.

As before, let us calculate the one-loop partition function of the noneffective orbifold $[X/\mathbf{Z}_k \times \mathbf{Z}_k]$, relating it to the partition function of the effective $[X/\mathbf{Z}_k]$ orbifold.

The untwisted sector of the effective orbifold arises from the

$$\begin{matrix} g^a & \square \\ & g^{a'} \end{matrix}$$

sectors of the noneffective $\mathbf{Z}_k \times \mathbf{Z}_k$. Discrete torsion leaves these sectors invariant, so the result is that the untwisted sector of the effective $[X/\mathbf{Z}_k]$ orbifold appears with multiplicity k^2 .

The

$$\begin{matrix} 1 & \square \\ & h^{b'} \end{matrix}$$

sector of the $[X/\mathbf{Z}_k]$ orbifold arises from sectors of the form

$$\begin{matrix} g^a & \square \\ & g^{a'} h^{b'} \end{matrix}$$

in the noneffective $\mathbf{Z}_k \times \mathbf{Z}_k$ orbifold. These sectors each pick up phases; the result is that this sector of the $[X/\mathbf{Z}_k]$ orbifold arises with a factor

$$\sum_{a,a'} \zeta^{m(ab')} = k \sum_a \left(\zeta^{mb'}\right)^a = \begin{cases} 0 & mb' \not\equiv 0 \pmod k \\ k^2 & mb' \equiv 0 \pmod k \end{cases}$$

Finally, the

$$h^b \begin{matrix} \square \\ h^{b'} \end{matrix}$$

sectors of the effective $[X/\mathbf{Z}_k]$ orbifold arise from sectors of the form

$$g^a h^b \begin{matrix} \square \\ g^{a'} h^{b'} \end{matrix}$$

in the noneffective $\mathbf{Z}_k \times \mathbf{Z}_k$ orbifold. These sectors each pick up phases; the result is that these sectors of the $[X/\mathbf{Z}_k]$ orbifold arise with a factor

$$\begin{aligned} \sum_{a,a'} \zeta^{m(ab'-ba')} &= \left[\sum_a \left(\zeta^{mb'}\right)^a \right] \left[\sum_{a'} \left(\zeta^{-mb}\right)^{a'} \right] \\ &= \begin{cases} 0 & mb' \not\equiv 0 \pmod k \quad \text{or} \quad mb \not\equiv 0 \pmod k, \\ k^2 & mb' \equiv 0 \pmod k \quad \text{and} \quad mb \equiv 0 \pmod k. \end{cases} \end{aligned}$$

When k is prime, the product mb can never be divisible by k , hence the one-loop partition function of the $\mathbf{Z}_k \times \mathbf{Z}_k$ orbifold is given by

$$\begin{aligned} Z_{1\text{-loop}}([X/\mathbf{Z}_k \times \mathbf{Z}_k]) &= \frac{k^2}{|\mathbf{Z}_k \times \mathbf{Z}_k|} \mathbb{1} \begin{matrix} \square \\ \mathbb{1} \end{matrix} \\ &= Z(X) \end{aligned}$$

for any nonzero discrete torsion.

11 T-duality

So far in this paper we have described a duality between strings on gerbes and strings on disjoint unions of spaces. We believe that this duality should be interpreted as a T-duality, for reasons we shall explain here.

First, the duality in question is an isomorphism of conformal field theories, which excludes, for example, S-duality.

Second, this duality can often be understood as a Fourier–Mukai transform, another hallmark of T-duality. This is exactly what happens in [15], for example. If you fix an elliptic fibration $X \rightarrow B$ and a class $\alpha \in H^2(\mathcal{O}_X^\times)$, then the derived category of the gerbe on X classified by α is Fourier–Mukai dual to the derived category of the disconnected union of spaces Y_k , k is an integer, where Y_k is a deformation of the dual elliptic fibration X corresponding to the class $k\alpha$. The kernel of the Fourier–Mukai transform is essentially the usual Poincare sheaf. More precisely, say that we have a trivial \mathbf{Z}_k -gerbe on E . Denote this gerbe by X . Consider the disconnected space $E \times (\mathbf{Z}_k)^\vee$, where $(\mathbf{Z}_k)^\vee$ is the group of characters of \mathbf{Z}_k . Now on the product $X \times (E \times (\mathbf{Z}_k)^\vee)$, we have a line bundle L which on each component $X \times (E, \chi)$ is simply the Poincare bundle on $E \times E$ but lifted to the gerbe as a weight χ sheaf.

Next, let us briefly outline another, much more handwaved, method to relate our duality to T-duality. Begin with an ordinary T-duality relating S^1 of radius R to an S^1 of radius $1/R$. Now, gauge $U(1)$ rotations on the first S^1 that rotate the S^1 k times instead of once. Such a gauging should kill all perturbative (momentum) modes, but should leave k winding modes: after all, if we quotient by windings that wrap k times, then a string that wraps only once should survive. Mathematically, gauging an S^1 by rotations that wind k times is equivalent to the stack $[\text{point}/\mathbf{Z}_k] = B\mathbf{Z}_k$. Now, consider the T-dual gauging. We will not attempt to write down the detailed form of that gauging, but as T-duality exchanges momentum and winding modes, the effect should be to kill off all winding modes while leaving k momentum modes. This we would like to interpret as a set of k points. Thus, in other words, we have a commuting diagram

$$\begin{array}{ccc}
 S^1 & \xrightarrow{\text{T-duality}} & S^1 \\
 \downarrow /U(1) & & \downarrow /\text{dual } U(1) \\
 [\text{pt}/\mathbf{Z}_k] & \xrightarrow{\text{our duality}} & \coprod_1^k \text{pt}
 \end{array}$$

At this same level of handwaving, we can also relate our duality to T-duality for fibered S^1 s as presented in [16]. That reference argues that T-duality will exchange c_1 of an S^1 bundle with (the pushforward along fibers of) H flux. For example, if we start with a nontrivial S^1 bundle with $H = 0$, then the T-dual will be a trivial S^1 bundle with $H \neq 0$. Given that picture, if we quotient the S^1 fibers by $U(1)$ rotations, then one is led to the picture that \mathbf{Z}_k -gerbes (which are $B\mathbf{Z}_k$ -bundles) should be T-dual to disconnected sums of spaces with B field flux, which is certainly part of our decomposition conjecture.

12 Applications

12.1 Curve counting predictions

12.1.1 Quantum cohomology

In our previous work [3] we observed that Batyrev’s conjecture for the quantum cohomology ring of toric varieties, easily generalizes to toric stacks. In particular, Batyrev’s conjecture has a precise physical meaning in terms of the effective action of a UV theory, the gauged linear sigma model. If the toric stack is described in the form

$$\left[\frac{\mathbf{C}^N - E}{(\mathbf{C}^\times)^n} \right],$$

where E is some exceptional set, and the weight of the i th vector in \mathbf{C}^N under the a th \mathbf{C}^\times is denoted by Q_i^a , then Batyrev’s conjecture is that the quantum cohomology ring is of the form $\mathbf{C}[\sigma_1, \dots, \sigma_n]$ modulo the relations

$$\prod_{i=1}^N \left(\sum_{b=1}^n Q_i^b \sigma_b \right)^{Q_i^a} = q_a$$

for a set of constants q_a .

For example, the quantum cohomology ring of a \mathbf{Z}_k -gerbe over \mathbf{P}^N with characteristic class $-n \bmod k$ (realized as a \mathbf{C}^\times quotient of a principal \mathbf{C}^\times bundle over \mathbf{P}^N) is given by

$$\mathbf{C}[x, y]/(y^k - q_2, x^{N+1} - y^n q_1)$$

This is k copies of the quantum cohomology ring of \mathbf{P}^N , indexed by the value of y .

As an aside, note that physically, sending the gauge coupling to zero not only reduces the quantum cohomology ring to the ordinary cohomology ring, but also reduces the gerbe to the underlying space: the distinction between a gerbe and a space is only visible via nonperturbative effects in physics, so if those nonperturbative effects are suppressed, then physics is unable to distinguish the gerbe from the underlying space.

More generally, a gerbe structure is indicated from the \mathbf{C}^\times quotient description whenever \mathbf{C}^\times charges are nonminimal. In such a case, from our generalization of Batyrev’s conjecture, at least one relation in the quantum cohomology ring will have the form $p^k = q$, where p is a relation in the quantum cohomology of the underlying toric variety, and k is the greatest common divisor of the nonminimal charges. One can rewrite this in the same

form as in the example above of a gerbe on a projective space, and so in this fashion we can see our decomposition conjecture inside our generalization of Batyrev’s conjecture for quantum cohomology to toric stacks.

12.1.2 Gromov–Witten predictions

The fact that the CFT of a string on a gerbe matches the CFT of a disjoint union of spaces makes a prediction for Gromov–Witten theory: Gromov–Witten invariants of a gerbe [17] should be computable in terms of Gromov–Witten invariants of a disjoint union of spaces.

We shall not give a general proof of that claim for Gromov–Witten invariants here, but rather will summarize the results of some calculations in a few special cases.

The first example is $X \times B\mathbf{Z}_k$, or, equivalently, $[X/\mathbf{Z}_k]$ where the \mathbf{Z}_k acts trivially globally. In this case of a trivial \mathbf{Z}_k -gerbe, our decomposition conjecture says

$$\text{CFT}([X/\mathbf{Z}_k]) = \text{CFT} \left(\prod_1^k X \right).$$

We can understand correlation functions as follows. Let Υ denote the generator of the $B\mathbf{Z}_k$ twist fields, and f a product of twist-field-independent correlators. In the case of no twist field insertions, the conjecture predicts

$$\langle f \rangle_{X \times B\mathbf{Z}_k, g} = A^{2g-2} k \langle f \rangle_{X, g},$$

where g is the genus of the worldsheet and A is a convention-dependent factor that can be absorbed into a dilaton shift. Since the twist fields decouple from the rest of the correlators, we can write a slightly more general set of correlation functions as

$$\langle f \prod_i \Upsilon^{n_i} \rangle_{X \times B\mathbf{Z}_k, g} = \begin{cases} A^{2g-2} k \langle f \rangle_{X, g} & \sum_i n_i \equiv 0 \pmod k, \\ 0 & \text{else.} \end{cases}$$

If we specialize to $k = 2$, then the projectors onto each component of the CFT are given by

$$\Pi_1 = \frac{1}{2}(1 + \Upsilon), \quad \Pi_2 = \frac{1}{2}(1 - \Upsilon),$$

which obey $\Pi_1^2 = \Pi_1$, $\Pi_2^2 = \Pi_2$, $\Pi_1 \Pi_2 = 0$. Thus, for example,

$$\langle f \Pi_1 \Pi_2 \rangle_{X \times B\mathbf{Z}_2, g} = 0$$

These results seem consistent with Gromov–Witten calculations for $B\mathbf{Z}_k$ [18].

Another example is the banded \mathbf{Z}_2 -gerbe defined by $[\mathbf{C}^3/D_4]$, where the \mathbf{Z}_2 center of D_4 acts trivially. We discussed this example in Section 5.2 and saw explicitly that the CFT of this noneffective orbifold coincides with the CFT of the disjoint union of two copies of $[\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2]$, one with discrete torsion, the other without.

We would like to thank J. Bryan for supplying specific Gromov–Witten calculations in this example. Let $\{a, b, c\}$ and $\{z, a, b, c\}$ denote nontrivial conjugacy classes in $\mathbf{Z}_2 \times \mathbf{Z}_2$ and D_4 , respectively, then the basic relationship between genus g invariants of the gerbe and genus g invariants of $[\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2]$ is

$$\langle z^n a^i b^j c^k \rangle_{e_g} = \begin{cases} 64^{g-1} 2^{1+2(i+j+k)} \langle a^i b^j c^k \rangle_g & n \text{ even,} \\ 0 & n \text{ odd.} \end{cases}$$

In terms of the genus g potential functions F'_g (for the gerbe) and F_g (for $[\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2]$), one gets the relation

$$F'_g(z, a, b, c) = 64^{g-1} 2 \cosh(z) F_g(4a, 4b, 4c).$$

We can see the decomposition conjecture in the results above. From the decomposition conjecture, a correlation function on the gerbe should decompose as a sum over correlation functions in the underlying effective orbifolds. These particular correlation functions are insensitive to discrete torsion, and so should be identical between the two copies. The projection operators are $\Pi_{\pm} = \frac{1}{2}(1 \pm z)$, so for example in the notation above one should find

$$\langle a^i b^j c^k \rangle_{[\mathbf{C}^3/D_4],g} \propto 2 \langle a^i b^j c^k \rangle_{[\mathbf{C}^3/\mathbf{Z}_2 \times \mathbf{Z}_2],g}$$

up to convention-dependent normalization factors. The factor of 64^{g-1} is a physically meaningless convention-dependent dilaton shift (see Appendix A.), and the factors of $2^{2(i+j+k)}$ can be absorbed into the definition of the vertex operators. (Those factors almost certainly arise because the computation in the D_4 orbifold was performed with vertex operators realizing sums over representatives of conjugacy classes rather than averages; in any event, these also are convention dependent.) Stripping out convention-dependent factors leaves only a factor of 2, exactly as predicted by the decomposition conjecture.

In passing, these ideas also seem consistent with general aspects of the Gromov–Witten–Donaldson–Thomas correspondence. After all, we are saying that Gromov–Witten invariants of gerbes should decompose into Gromov–Witten invariants of spaces; on the other hand, it is a standard result that sheaves on gerbes decompose into (twisted) sheaves on spaces, hence Donaldson–Thomas invariants on gerbes should possess the same

decomposition as that we are claiming for Gromov–Witten invariants of gerbes.

12.2 Gauged linear sigma model analyses

In this section we shall analyze a gauged linear sigma model, in order to show that “gerby” effects are common even in descriptions of ordinary spaces, and so an understanding of gerbes is important to understand generic gauged linear sigma models for ordinary spaces.

Consider a gauged linear sigma model describing the complete intersection of four degree-two hypersurfaces in \mathbf{P}^7 at large radius. This gauged linear sigma model has a total of twelve chiral superfields, eight $(\phi_i, i \in \{1, \dots, 8\})$ of charge 1 corresponding to homogeneous coordinates on \mathbf{P}^7 , and four $(p_a, a \in \{1, \dots, 4\})$ of charge -2 corresponding to the four hypersurfaces.

The D-term for this gauged linear sigma model reads

$$\sum_i |\phi_i|^2 - 2 \sum_a |p_a|^2 = r.$$

When $r \gg 0$, we see that not all the ϕ_i can vanish, corresponding to their interpretation as homogeneous coordinates on \mathbf{P}^7 . More generally, for $r \gg 0$ we recover the geometric interpretation of this gauged linear sigma model as a complete intersection.

For $r \ll 0$, we find a different story. There, the D-term constraint says that not all the p_a s can vanish; in fact, the p_a s act as homogeneous coordinates on a \mathbf{P}^3 , except that these homogeneous coordinates have charge 2 rather than charge 1. That is precisely how one describes, in gauged linear sigma model language, the banded \mathbf{Z}_2 -gerbe over \mathbf{P}^3 whose characteristic class is $-1 \bmod 2$.

Ordinarily, we would identify this phase with a “hybrid” Landau–Ginzburg phase, in which one has a family of ordinary Landau–Ginzburg models fibered over the base. In this case, however, because the hypersurfaces are degree-two, we get a series of mass terms for the ϕ_i , so that at generic points on the gerbe of p_a s, the ϕ_i are all massive and can be integrated out.

If one then works locally in the large $-r$ phase, since we have a \mathbf{Z}_2 -gerbe structure locally, one might loosely argue that we should really be thinking in terms of a 2-fold cover of \mathbf{P}^3 .

Globally we should be more careful, and take into account the subvariety where some of the ϕ_i become massless. We can determine that subvariety as follows. Write the superpotential in the form $v^T A v$, where $v = [\phi_0, \dots, \phi_7]^T$ is the vector of homogeneous coordinates on \mathbf{P}^7 , and the 8×8 matrix A has entries that are linear in the p_{as} . Whenever a set of ϕ s become massless, the matrix A will have a vanishing eigenvalue, and so its determinant will vanish. Thus, the set of points on \mathbf{P}^3 , the manifold of p_{as} , where some ϕ s may become massless is given by $\det A$, a degree-eight polynomial. In particular, thinking in terms of gerbes we should really be considering a double cover of \mathbf{P}^3 branched over a degree-eight hypersurface in \mathbf{P}^3 .

Now, a double cover of \mathbf{P}^3 branched over a degree-eight hypersurface in \mathbf{P}^3 is an example of a Calabi–Yau. We can see this as follows. Let the double cover be denoted by S , with projection map $\pi : S \rightarrow \mathbf{P}^3$, and let a hypersurface class in \mathbf{P}^3 be denoted by H . Assume S is branched over a degree- d hypersurface in \mathbf{P}^3 . Then

$$K_S = \pi^* K_{\mathbf{P}^3} + \tilde{D} = \pi^*(-4H) + \tilde{D},$$

where \tilde{D} is the divisor in S over the branch locus, so $2\tilde{D} = \pi^*(dH)$. Thus,

$$2K_S = \pi^*((-8 + d)H);$$

hence S is Calabi–Yau if $d = 8$. For a closely related discussion in the context of a different example, see [19, Chapter 4.4, p. 548].

We should emphasize, however, that the theory in the large $-r$ phase of this gauged linear sigma model is not the Calabi–Yau 3-fold S , although it is closely related [21].

Double covers of \mathbf{P}^3 branched over a degree-eight hypersurface in \mathbf{P}^3 are known as octic double solids, and are described in greater detail in, e.g., [22, 23].

12.3 Geometric Langlands

The T-duality between gerbes and disconnected spaces that we have discussed in this paper also implicitly appears in [24, 25]. Section 7.1 of [24] discusses how the moduli space of G -Higgs bundles on a curve C has several components, and the universal bundle over each component is potentially a twisted bundle, with twisting determined by an element of $H^2(\mathcal{M}, Z(G))$ which is described as the obstruction to lifting a universal G_{ad} bundle to a $G_{\text{s.c.}}$ -bundle, where $G_{\text{s.c.}}$ denoted the simply connected cover of G . Section 7.2 describes how that twisting enters the physics of the sigma model on the Higgs moduli space: for each e_0 , corresponding to an irreducible

representation of $Z(G)$, there is at least effectively a flat B field on the corresponding component determined by the image of the element of $H^2(\mathcal{M}, Z(G))$ under e_0 .

Using the ideas in this paper, this picture can be equivalently rewritten as a sigma model on a banded $Z(G)$ -gerbe over the moduli space, whose characteristic class is the element of $H^2(\mathcal{M}, Z(G))$ determined in [24, Section 7.1]. As described elsewhere in this paper, a string on a banded $Z(G)$ -gerbe is equivalent to a string on a disjoint union of spaces, one copy for each irreducible representation of $Z(G)$, with a flat B field determined by the characteristic class of the gerbe via the map precisely described in [24, Section 7.2].

In [25], this part of geometric Langlands was rewritten in terms of a moduli stack, which had the structure of a $Z(G)$ -gerbe over the moduli space. We now see that the mathematical manipulations of [25] can be given a direct explicit physical meaning: instead of talking about sigma models on Higgs moduli *spaces*, we could just as well work with sigma models on the Higgs moduli *stacks* discussed in [25].

This gerbiness also has an analogue in the four-dimensional gauge theory of [24]. The gerbiness appears in two-dimensions when $Z(G)$ is nontrivial, and corresponds to a four-dimensional gauge theory in which part of the gauge group is acting ineffectively. In [24], the matter content of the four-dimensional theory transforms in the adjoint representation of the group. If the gauge group is the simply connected cover $G_{\text{s.c.}}$ rather than the centerless adjoint group G_{ad} , then the center acts trivially on the matter, giving a four-dimensional analogue of the two-dimensional gauge theories we have discussed here and in [1–3]. For example, the Higgs moduli space can now be interpreted as a $Z(G)$ -gerbe. Dimensionally reducing this gerby picture of the four-dimensional theory also leads directly to a picture of sigma models on Higgs moduli stacks.

In passing, we should mention that nonperturbative distinctions between four-dimensional gauge theories with noneffectively acting gauge groups and their effectively acting counterparts have been studied by M. Strassler in the context of Seiberg duality; see, for example, [26].

12.4 Speculations on nonsupersymmetric orbifolds

In nonsupersymmetric orbifolds, it has been suggested [27, 28] that the RG endpoint of tachyon condensation involves breaking a connected space into a disjoint union of spaces. In this section we will speculate on a physical mechanism by which this could occur, based on the results of this paper.

The tachyons whose condensation leads to RG flow arise as twist fields, and are localized at orbifold points. Now, in terms of stacks, the orbifold “point” is replaced by a gerbe over a point — this is part of why a quotient stack can be smooth even when the corresponding quotient space is singular. Naively, if one imagines a wavefront expanding outward from such a singularity describing a dynamical process in which a twisted sector tachyon gets a vev, then one might imagine that to be consistent the gerbe structure should be carried along, expanding to cover the entire space. Perhaps the endpoint of tachyon condensation and RG flow should be understood in terms of a gerbe structure covering the original space. From the observations of this paper, a string on a gerbe is equivalent to a disjoint union of spaces, so perhaps the suggestion of [27, 28] that disconnected spaces arise at the endpoint is just the T-dual picture.

At this point we should mention that the analogous idea does not work in supersymmetric cases. For example, consider the quotient stack $[\mathbf{C}^2/\mathbf{Z}_2]$ where the \mathbf{Z}_2 acts by sign flips on the coordinates of \mathbf{C}^2 . The corresponding quotient space has a singularity at the origin, whereas the stack has a gerbe at the origin. Naive blowups of this stack at the origin result in stacks over $\widetilde{\mathbf{C}^2/\mathbf{Z}_2}$ with a \mathbf{Z}_2 -gerbe over the exceptional divisor. It was originally suggested in [29] that this could be a mechanism for understanding old lore concerning “ B fields at orbifold points” [30, 31]. After all, if a string on a gerbe is related to strings on underlying spaces with B fields, then perhaps a string on a stack that looks like a \mathbf{Z}_2 -gerbe over the exceptional divisor should have the same CFT as a string on the underlying space with a flat B field. This original proposal, unfortunately, suffers from the problem that the stack in question, the blowup of $[\mathbf{C}^2/\mathbf{Z}_2]$, is not Calabi–Yau, and so should not arise when deforming along flat directions in the supersymmetric theory.

In the nonsupersymmetric case, the constraint on supersymmetric flat directions is weakened somewhat, so perhaps some version of this picture should apply.

We will not discuss this further here, but thought it appropriate to mention that this is a natural speculation considering the topic of this paper.

13 Conclusions

In this paper, we have described T-duality between strings propagating on gerbes and strings propagating on disconnected sums of spaces. This duality solves a basic problem with the notion of strings propagating on gerbes, namely that the massless spectrum violates cluster decomposition, one of

the foundational axioms of quantum field theory. A sigma model on a disjoint union of spaces also violates cluster decomposition, but in the mildest possible way, hence a T-duality between gerbes and disjoint unions of spaces means that the CFTs associated to gerbes must also be consistent, despite violating cluster decomposition.

We presented a detailed description of how gerbes should decompose. The disjoint union will, in general, consist of different spaces, not a sum of copies of the same space, and there will be different B fields on each component. After presenting a conjecture valid for all cases, we began giving evidence that this conjecture was correct. In examples presented as global quotients by finite groups, we computed partition functions (at arbitrary genus) and massless spectra in a number of examples, and checked that this conjecture held true in each case. For gerbes presented as global quotients by nonfinite groups, we also discussed how this conjecture could be seen directly in the structure of nonperturbative effects in the theories. We also discussed open strings in this context. Because of the relation between D-branes and K -theory, this conjecture makes a prediction for equivariant K -theory, which we checked in Appendix C. For D-branes in the open string B model, which can be described by sheaves and derived categories thereof, our decomposition conjecture corresponds to a known decomposition result for sheaves on gerbes. We also discussed how other aspects of open strings in the B model are consistent with this conjecture. We discussed mirror symmetry in this context, showing how the results on mirror symmetry for stacks presented in [3] are consistent with this decomposition conjecture, and also discussed a noncommutative-geometry-based argument for this conjecture. We discussed the effect of discrete torsion along noneffective directions and also discussed why this decomposition can be understood as a T-duality.

Finally, we discussed applications of these ideas. One important set of applications is to curve-counting; in effect, we are making predictions for quantum cohomology and, more generally, Gromov–Witten invariants. In previous work [3] we presented an analogue of Batyrev’s conjecture for quantum cohomology of toric stacks, and the decomposition conjecture can be seen in that conjecture. We also discuss how the predictions for Gromov–Witten theory work out in a few simple examples.

We also discuss how these ideas are relevant to analyses of ordinary gauged linear sigma models, and in fact their appearance there suggests that these ideas are much more widely relevant to gauged linear sigma models than one might have naively supposed.

We also discuss how these ideas can be used to give a minor improvement in the physical understanding of the geometric Langlands program, by giving

a direct physical relationship between different descriptions of geometric Langlands in terms of disconnected spaces and in terms of gerbes.

Finally, we speculate on how these ideas may also give an alternative understanding of some aspects of tachyon condensation in nonsupersymmetric orbifolds.

14 Acknowledgments

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Appendix A. Partition functions for disconnected targets

What is the partition function for a nonlinear sigma model on the disjoint union of k copies of a manifold X ?

There are two clear possible answers, and strong arguments supporting both.

One possibility is that at any worldsheet genus, the partition function for $\coprod_k X$ should be k times the partition function for X . We can justify this by thinking about the path integral measure: a sum over maps from a connected worldsheet into a disconnected target can be decomposed into a sum over target components, with a sum over maps into each component. Thus, at any genus g , the partition function for a nonlinear sigma model on $\coprod_i X_i$ should have the form

$$Z_g \left(\coprod_i X_i \right) = \sum_i Z_g(X_i)$$

and so, in particular, the genus g partition function for a nonlinear sigma model on the disjoint union of k copies of the same space X should be

given by

$$Z_g\left(\coprod_i X\right) = kZ_g(X).$$

An alternative argument gives a different result. Consider the disjoint union of two copies of X . If we orbifold the nonlinear sigma model with that target by a \mathbf{Z}_2 which exchanges the two copies, since the group acts freely, the orbifold must be the nonlinear sigma model whose target is one copy of X . At genus g , the partition function of the orbifold theory is easily computed to be

$$Z_g(\mathbf{Z}_2) = \frac{1}{|\mathbf{Z}_2|^g} \sum_{\text{t.s.}} Z_{\text{t.s.}}(X \sqcup X),$$

where t.s. denotes the twisted sector boundary conditions that we must sum over. Now, since the \mathbf{Z}_2 exchanges the two copies of X , and the worldsheet is connected, any $Z_{\text{t.s.}}$ with a nontrivial element of \mathbf{Z}_2 at any boundary must vanish, as there are no continuous maps from a connected worldsheet that touch both copies of X in the target. Thus,

$$\sum_{\text{t.s.}} Z_{\text{t.s.}}(X \sqcup X) = Z_g(X \sqcup X),$$

and so we recover the algebraic identity that

$$Z_g(X \sqcup X) = 2^g Z_g(X).$$

This result does agree with the intuition that on a genus g worldsheet, since the nonlinear sigma model on the disjoint union of k copies of X has k times as many states as a nonlinear sigma model on X , the genus g partition function should have a factor of k^g , reflecting the fact that one expects a sum over states along each handle of the worldsheet.

Unfortunately, these two perspectives conflict — these two ways of thinking about the partition function on a disjoint union of copies of X are giving us different results. The first argument said that the numerical factor should be independent of genus, the second said it is not independent of genus. Only at genus one do the two arguments give the same result.

Now, since the partition functions only differ by a numerical factor, the reader might think this is not important. After all, in ordinary quantum field theory such numerical factors are irrelevant. However, worldsheet string theory is a theory coupled to worldsheet gravity, and in a theory coupled to gravity, factors in front of partition functions are important, and reflect state degeneracies, as discussed in more detail in [1]. We cannot ignore differing numerical factors, as that would be tantamount to ignoring the cosmological constant.

The resolution of this puzzle is that these two ways of thinking about the genus g partition function of a theory on a disjoint union differ by a dilaton shift. If we shift the vev of the dilaton by a constant, say A , then the g -loop partition function shifts by a factor as

$$Z_g \mapsto (\exp(A))^{2g-2} Z_g,$$

since in the two-dimensional action the dilaton multiplies the worldsheet Ricci scalar, and so shifting the dilaton by a constant will multiply the path integral by a factor raised to the power of the Euler characteristic of the worldsheet.

In any event, such dilaton shifts are physically trivial, and so any two partition functions differing by factors raised to the $2g - 2$ power are physically equivalent.

Appendix B. Miscellaneous group theory

This paper uses an unusually large amount of group theory which may not be at the reader's fingertips, so for reference in this appendix we summarize a few relevant standard finite group theory results.

The number of elements of a group is the sum of the squares of the dimensions of the irreducible representations [32, equation (1.74)]:

$$|G| = \sum_i (\dim \rho_i)^2.$$

An irreducible representation ρ appears in the regular representation $\dim \rho$ times [33, cor 2.18].

The number of irreducible representations of G is equal to the number of conjugacy classes of G [33, Prop. 2.30].

For example, consider the eight-element dihedral group D_4 . The elements of this group are

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\},$$

where z generates the center, $z^2 = 1 = a^2, b^2 = z$. This group has five conjugacy classes, given by

$$1, z, \{a, az\}, \{b, bz\}, \{ab, ba\}$$

First, let us count the one-dimensional representations of D_4 . These are homomorphisms $f : D_4 \rightarrow \mathbf{C}^\times$. Thus, $f(1) = 1$, and since the projection

$D_4 \rightarrow \mathbf{Z}_2 \times \mathbf{Z}_2$, $f(a) = f(az)$, hence $f(z) = 1$. Define

$$\begin{aligned}\alpha &= f(a), \\ \beta &= f(b),\end{aligned}$$

then $\alpha^2 = \beta^2 = 1$, and so we have four possible one-dimensional representations, indexed by the values of α and β . In addition, there is a two-dimensional representation given by

$$z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Since there are five conjugacy classes, this must be all of the irreducible representations. Also, note that the sum of the squares of the dimensions of the irreducible representations equals the order of D_4 .

Appendix C. Group extensions and twisted K -theory (written by M. Ando)

The conjecture in the paper suggests the following result about K -theory. Let

$$1 \longrightarrow G \longrightarrow H \longrightarrow Q \longrightarrow 1$$

be an extension of compact Lie groups. Let X be a Q -space. Then $K_H(X)$ is isomorphic to the twisted Q -equivariant K -theory of X , where the twist is the one described in the main body of the paper (and below).

This problem has been extensively studied, particularly from the C^* -algebra point of view; see, for example, [37]. M.A. is grateful to Chris Phillips and Marc Rieffel for pointing this out, and for several helpful conversations.

In this appendix we sketch a proof of this result, at least when the groups are finite, and the extension is a semi-direct product. Details and generalizations will appear later.

M.A. first learned some of the ideas used here in the course of joint work in preparation with John Greenlees.

C.1 The category of irreducible representations

Let G be a compact Lie group. Let $\text{Irr}(G)$ be the category of irreducible complex representations of G : an object of $\text{Irr}(G)$ is an irreducible representation

$$\alpha : G \longrightarrow \text{GL}(V),$$

where V is a complex vector space. A morphism is an isomorphism

$$f : V \longrightarrow V'$$

which intertwines the action of G on V and V' . This category is a groupoid. By Schur's Lemma, for every object $\alpha : G \rightarrow \text{GL}(V)$ of $\text{Irr}(G)$, the canonical map

$$\mathbf{C}^\times \longrightarrow \text{Aut}(V)$$

is an isomorphism. (Also, it commutes with any morphism f .) In practice, it is useful to set this up so that the objects of $\text{Irr}(G)$ comprise a set: we can and will do this by choosing a set G^\vee of representatives for the irreducible representations of $\pi_0 \text{Irr}(G)$.

When it is convenient, we shall write V_α for the vector space associated to a representation α .

C.2 Irreducible representations and equivariant vector bundles

For example, if V is a G -equivariant vector bundle over a G -fixed space, then we have a canonical map of equivariant vector bundles

$$\bigoplus_{\alpha \in G^\vee} \alpha \otimes \text{hom}(\alpha, V) \xrightarrow{\cong} V, \tag{C.1}$$

which is an isomorphism by Schur's Lemma. On the left, α is pulled back along

$$X \longrightarrow \text{pt},$$

while G acts trivially on $\text{hom}(\alpha, V)$. The same result, with the same formula, applies whether V is a genuine or virtual G -vector bundle.

C.3 Group extensions and equivariant projective bundles

Suppose that, as in § C.2, we have a G -vector bundle or K -theory object V over a G -fixed space X . In addition we suppose that G participates in a

short exact sequence

$$1 \longrightarrow G \longrightarrow H \longrightarrow Q \longrightarrow 1 \tag{C.2}$$

of compact Lie groups, and Q acts on X . (We use Q rather than K as in the main text to denote the quotient, to avoid talking about K -equivariant K -theory)

In this situation we can ask whether H can be made to act on V , in such a way that the restriction of the action to G is the given one; indeed we can ask for the space of solutions to this problem. It is simplest to analyze the situation in the case that all the groups in sight are finite, and H is a semidirect product. We treat that case.

Let $k \in Q$. Pulling back the isomorphism (C.1) along

$$k : X \longrightarrow X$$

gives

$$\bigoplus_{\alpha \in G^\vee} k^* \alpha \otimes k^* \text{hom}(\alpha, V) \xrightarrow{\cong} k^* V. \tag{C.3}$$

To make Q act on V compatibly with its G -action, we must give an isomorphism of G -objects over X

$$\phi_k : \bigoplus_{\alpha \in G^\vee} \alpha \otimes \text{hom}(\alpha, V) \xrightarrow{\cong} \bigoplus_{\alpha \in G^\vee} k^* \alpha \otimes k^* \text{hom}(\alpha, V).$$

As k varies these isomorphisms must satisfy the cocycle condition

$$\ell^* \phi_k \phi_\ell = \phi_{k\ell}.$$

The subtlety arises from the fact that $k^* \alpha$ is isomorphic to an element of G^\vee , but not canonically.

For any representation

$$\rho : G \longrightarrow \text{GL}(V),$$

let $k^* \rho$ be the representation

$$G \xrightarrow{c_k} G \xrightarrow{\alpha} \text{GL}(V),$$

where c_k is conjugation by Q in the semidirect product. For concreteness, let

$$k^* \rho(g) = \rho(k^{-1} g k),$$

so Q acts on the left:

$$(k\ell)^* \rho = k^*(\ell^* \rho).$$

Clearly ρ is an irreducible representation of G if and only if $k^* \rho$ is, so if $\alpha \in G^\vee$, then $k^* \alpha$ is isomorphic to an element $k(\alpha) \in G^\vee$. Thus the action

of Q on G determines an action of Q on the left of G^\vee . Note that these data do not determine an isomorphism

$$\beta : k^* \alpha \cong k(\alpha).$$

However, any two isomorphisms β, β' differ by an element of

$$\text{Aut}(k(\alpha)) \cong \mathbf{C}^\times.$$

Definition C.1. Let

$$\mathcal{W} \longrightarrow G^\vee$$

be the bundle of projective spaces over G^\vee whose fiber at the representation

$$\alpha : G \longrightarrow \text{GL}(V_\alpha)$$

is $\mathbf{P}(V_\alpha)$. The extension (C.2) gives to \mathcal{W} the structure of an Q -equivariant bundle of projective spaces over G^\vee .

It is illuminating to give a cohomological description of the bundle \mathcal{W} . For each $k \in Q$ and $\alpha \in G^\vee$, let us choose an isomorphism

$$\beta(k, \alpha) : k^* \alpha \longrightarrow k(\alpha).$$

That is, if α and $k(\alpha)$ are homomorphisms

$$\alpha : G \longrightarrow \text{GL}(V)$$

and

$$k(\alpha) : G \longrightarrow \text{GL}(W),$$

then $\beta(k, \alpha)$ is just vector a space isomorphism

$$\beta(k, \alpha) : V \longrightarrow W$$

such that

$$\beta(k, \alpha)(\alpha(k^{-1}gk)v) = k(\alpha)\beta(k, \alpha)(v).$$

We then have two isomorphisms

$$(k\ell)^* \alpha \cong k\ell(\alpha),$$

namely

$$\beta(k\ell, \alpha)$$

and

$$\beta(k, \ell(\alpha))k^* \beta(\ell, \alpha).$$

Here $k^* \beta(\ell, \alpha)$ is the same map of vector spaces as $\beta(\ell, \alpha)$, but now intertwining $k^* \ell^* \alpha$ and $k^* \ell(\alpha)$.

Their ratio

$$w(k, \ell, \alpha) \equiv \beta(k, \ell(\alpha))k^* \beta(\ell, \alpha)\beta(k\ell, \alpha)^{-1}$$

is an element of $\text{Aut}(\alpha) \cong \mathbf{C}^\times$.

Lemma C.2. The function

$$w : Q^2 \times G^\vee \longrightarrow \mathbf{C}^\times$$

is a 2-cocycle for the action of Q on G^\vee . Its cohomology class in

$$H_Q^2(G^\vee; \mathbf{C}^\times) = H^2(EQ \times_Q G^\vee; \mathbf{C}^\times) \longrightarrow H_Q^3(G^\vee; \mathbf{Z})$$

depends only on the action of Q on G .

Remark C.3. We recall how to calculate this sort of Borel cohomology in § C.5.

Proof. The 2-cocycle condition is

$$w(k, \ell, \alpha)w(jk, \ell, \alpha)^{-1}w(j, k\ell, \alpha)w(j, k, \ell(\alpha))^{-1} = 1,$$

which is easily checked. The dependence of w on β is as follows. Any two choices of isomorphisms β and β' differ by

$$\beta'(k, \alpha) = \delta(k, \alpha)\beta(k, \alpha),$$

where $\delta(k, \alpha) \in \mathbf{C}^\times \cong \text{Aut}(k(\alpha))$. This is a function

$$\delta : Q \times G^\vee \longrightarrow \mathbf{C}^\times,$$

and then it is easy to see that

$$w' = wd\delta,$$

so w' and w are cohomologous. □

C.4 Equivariant K -theory

For any compact Lie group H , equivariant complex K -theory has a classifying space \mathbf{K}_H . (We also write \mathbf{K} for a space representing non-equivariant complex K -theory.) Atiyah and Segal give a number of models for \mathbf{K} in [36]. For example, one can stabilize the Grassmannian of finite-dimensional subspaces of a Hilbert space representation \mathcal{U} of H , in which each irreducible representation occurs with infinite multiplicity.

In our situation, if G acts trivially on X , then clearly any H -map

$$X \longrightarrow \mathbf{K}_H$$

factors through

$$\mathbf{K}_H^G \longrightarrow \mathbf{K}_H,$$

and indeed

$$\text{map}_H(X, \mathbf{K}_H) = \text{map}_Q(X, \mathbf{K}_H^G).$$

Thus to understand H -equivariant K -theory of G -fixed spaces amounts to understanding the space \mathbf{K}_H^G , with its Q -action.

The first point is that, if we remember only the G -action on \mathbf{K}_H , then \mathbf{K}_H is a representing space for G -equivariant K -theory. It follows that, if we forget the Q -action, then \mathbf{K}_H^G represents G -equivariant K -theory over G -fixed spaces. The decomposition (C.1) using Schur’s Lemma then implies the following.

Proposition C.4.

$$\mathbf{K}_H^G \simeq \text{map}(G^\vee, \mathbf{K}). \tag{C.4}$$

Remark C.5. Of course this is just a formulation of the isomorphism

$$K_G(X) \cong R(G) \otimes K(X)$$

for G -fixed spaces X [38].

We have written \simeq to indicate homotopy equivalence, but in practice the decomposition will arise from a geometric decomposition of our chosen representing space for K_H . For example, the Hilbert space \mathcal{U} will decompose according to the irreducible representations of G . The right-hand side of (C.4) can profitably be thought of as the sections of a fiber bundle \mathbf{K}_\bullet over G^\vee . The fiber over α is a space \mathbf{K}_α with a stable vector bundle ξ_α over it. The group G acts on ξ_α in such a way that the natural map

$$\alpha \otimes \text{hom}(\alpha, \xi_\alpha) \longrightarrow \xi_\alpha$$

is an isomorphism. If we also forget the G -action, then $\xi_\alpha/\mathbf{K}_\alpha$ represents nonequivariant complex K -theory. The problem remains of describing the Q -action on \mathbf{K}_H^G .

Atiyah and Segal [36] show that $H_Q^3(X; \mathbf{Z})$ classifies Q -equivariant *stable* projective bundles over X : these are bundles P for which $P \cong P \otimes L^2(Q)$. To such a bundle P they attach a Q -equivariant bundle $\mathbf{K}(P)$. Over each point $x \in X$, the fiber $\mathbf{K}(P)_x$ is a representing space for K -theory, but the Q -action is twisted from the usual representing space \mathbf{K}_Q by the equivariant projective bundle P . They define

$$K_{Q,P}^0(X) = \pi_0 \Gamma(X, \mathbf{K}(P))^Q,$$

the homotopy classes of Q -equivariant sections of $\mathbf{K}(P)$.

In particular we can form $\mathbf{K}(w)$, the equivariant Q -bundle over G^\vee associated to the 2-cocycle w . We call this $\mathbf{K}(\mathcal{W})$, because ultimately it comes from this bundle. The discussion in § C.3 (and in the main text of the paper) shows that

Proposition C.6. As Q -equivariant bundles over G^\vee ,

$$\mathbf{K}_\bullet \cong \mathbf{K}(\mathcal{W}),$$

and so

$$\Gamma(G^\vee, \mathbf{K}(\mathcal{W}))$$

represents the Q -space \mathbf{K}_H^G .

Thus if G acts trivially on X ,

$$K_H(X) \cong [X, \Gamma(G^\vee, \mathbf{K}(\mathcal{W}))]^Q. \tag{C.5}$$

The usual adjunctions based on the formula

$$\text{map}(X \times Y, Z) \cong \text{map}(X, \text{map}(Y, Z))$$

show that

$$[X, \Gamma(G^\vee, \mathbf{K}(\mathcal{W}))]^Q \cong \pi_0 \Gamma(X \times G^\vee, \mathbf{K}(\mathcal{W}))^Q. \tag{C.6}$$

The right-hand side is just the twisted Q -equivariant K -theory of $X \times G^\vee$, where the twist is the one obtained by pulling back \mathcal{W} along

$$X \times G^\vee \longrightarrow G^\vee.$$

Putting together (C.5) and (C.6) gives

$$K_H(X) \cong K_{Q, \mathcal{W}}(X),$$

which is the result indicated in the main body of the paper.

C.5 Borel cohomology

Let Z be a space with an action of a topological group Q . Then we can form the simplicial space \mathcal{X}_\bullet whose n space for $n \geq 0$ is

$$\mathcal{X}_n = Q^n \times Z,$$

and whose face maps

$$d_i : \mathcal{X}_{n+1} \longrightarrow \mathcal{X}_n$$

are given by

$$\begin{aligned} d_0(k_0, \dots, k_n, z) &= (k_1, \dots, k_n, z) \\ d_i(k_0, \dots, k_n, z) &= (k_0, \dots, k_{i-1}k_i, \dots, k_n, z) \quad 1 \leq i \leq n \\ d_{n+1}(k_0, \dots, k_n, z) &= (k_0, \dots, k_{n-1}, k_n z). \end{aligned}$$

The geometric realization of \mathcal{X}_\bullet is the Borel construction,

$$|\mathcal{X}_\bullet| = EQ \times_Q Z.$$

This gives a variety of spectral sequences for Borel cohomology. For example, if Q and Z are discrete, then we find that

$$H^2(EQ \times_Q Z; \mathbf{C}^\times)$$

is described by precisely the sorts of cocycles modulo coboundaries we used in Lemma C.2 above.

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